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定常的磁流体动力学问题的 Galerkin-Petrov 最小二乘混合元方法*

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摘要: 提出了定常的磁流体动力学方程的一种 Galerkin-Petrov 最小二乘混合元法, 并导出 Galerkin-Petrov 最小二乘混合元解的存在性和误差估计. 通过引入 Galerkin-Petrov 最小二乘混合有限元方法使得该方法的混合元空间之间的组合无需满足离散的 Babuska-Brezzi 稳定性条件, 从而使得它们的混合有限元空间可以任意选取, 并得到误差估计最优阶.

关 键 词: 磁流体力学方程; 混合元方法; Galerkin-Petrov 最小二乘法; 误差估计

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引 言

磁流体问题是由电磁场与导电液体或导电气体组成的复杂的非线性耦合系统(参见文献[1]). 本文讨论的定常不可压磁流问题是用来描述带有粘滞性、不可压缩的导电流的定常方程组. 这个方程组已经被广泛地用于熔接技术、潜艇推进设计和核反应堆的冷却设备磁泵中的金属体的建模研究, 因此对该方程组的蜓究更具有很重要的实际意义. 文献[2]先对定常的磁流体动力学问题作了初步的分析, 并利用 Bernardi-Raugel 元给出了一种一阶估计格式. 文献[3]改进了文献[2]的方法并给出了一般性的混合有限元方法, 同时也给出了一些较好的混合有限元格式. 然而这些方法都要求有限元空间之间的组合满足 Babuška-Brezzi 稳定性条件[4-5]. 在解定常的 Navier-Stokes 方程的混合有限元法中, 为了摆脱这种约束, CBB 方法[6]和稳定的有限元方法[7-10]已由 SD(或 SUPG) 方法[11-12]派生出来. 本文的目的是将 Galerkin-Petrov 最小二乘混合元方法应用于处理定常的磁流体动力学问题. 据我们所了解, 目前还没有用 Galerkin-Petrov 最小二乘混合元方法处理定常的磁流体动力学问题的报道.

本文的安排如下: 第 1 节引入问题的 Galerkin-Petrov 最小二乘混合有限元格式; 第 2 节讨论 Galerkin-Petrov 最小二乘混合有限元解的存在唯一性; 第 3 节导出 Galerkin-Petrov 最小二乘混合有限元解的收敛性和误差估计.

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1 Galerkin-Petrov 最小二乘混合有限元格式

设 $\Omega \subset R^3$ 是边界为 $\Gamma = \partial\Omega$ 的有界凸区域. 考虑定常的磁流体动力学问题:

问题(Ⅰ) 求 $\boldsymbol{u}, \boldsymbol{B}, p$ 满足

$$\begin{cases} -\dfrac{1}{M^2}\Delta\boldsymbol{u} + \dfrac{1}{N}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \dfrac{1}{R_m}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{f}, & \text{在 } \Omega \text{ 中}, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{在 } \Omega \text{ 中}, \\ \dfrac{1}{R_m}\nabla\times(\nabla\times\boldsymbol{B}) - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = 0, & \text{在 } \Omega \text{ 中}, \\ \nabla\cdot\boldsymbol{B} = 0, & \text{在 } \Omega \text{ 中}, \\ \boldsymbol{u} = \boldsymbol{g}, & \text{在 } \Gamma \text{ 上}, \\ \boldsymbol{B}\cdot\boldsymbol{n} = q, & \text{在 } \Gamma \text{ 上}, \\ \dfrac{1}{R_m}[(\nabla\times\boldsymbol{B})\times\boldsymbol{n}] - [(\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{n}] = \boldsymbol{k}, & \text{在 } \Gamma \text{ 上}, \end{cases} \tag{1}$$

其中 $\boldsymbol{u}$ 表示速度向量, $\boldsymbol{B}$ 表示磁场, p 表示压力, 所有变量都是无量纲的; 而 M, N, 和 R_m 分别是Hartmann 数, 交互参数和磁场 Reynolds 数, 及 $\boldsymbol{f} \in H^{-1}(\Omega)^3$ 均为已知. 为了满足(1)式解的正则性和相容性条件, 我们需要限定边值满足

$$\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)^3, \quad \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}s = 0, \tag{2}$$

$$q \in H^{1/2}, \quad \int_\Gamma q\,\mathrm{d}s = 0, \tag{3}$$

$$\boldsymbol{k} \in \boldsymbol{H}^{1/2}(\Gamma)^3, \quad \boldsymbol{k}\cdot\boldsymbol{n} = 0, \quad \langle\boldsymbol{k}, \boldsymbol{l}\rangle_\Gamma = \boldsymbol{0}, \quad \langle\boldsymbol{k}, \nabla\phi\rangle_\Gamma = \boldsymbol{0}, \qquad \forall\phi \in H^2(\Omega). \tag{4}$$

问题(Ⅰ)的变分形式可写为:

问题(Ⅰ*) 求 $(\boldsymbol{u}, \boldsymbol{B}) \in \mathscr{W}_{\boldsymbol{g}q}(\Omega)$, $p \in L_0^2(\Omega)$ 满足:

$$\begin{cases} a((\boldsymbol{u}, \boldsymbol{B}), \boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{v}, \Psi)) + b((\boldsymbol{v}, \Psi), p) = F(\boldsymbol{v}, \Psi), \\ \qquad\qquad \forall(\boldsymbol{v}, \Psi) \in \mathscr{W}_{0\boldsymbol{n}}(\Omega), \\ b((\boldsymbol{u}, \boldsymbol{B}), \chi) = \boldsymbol{0}, \qquad \forall\chi \in L_0^2(\Omega), \end{cases} \tag{5}$$

其中

$$\mathscr{W}_{0\boldsymbol{n}}(\Omega) := H_0^1(\Omega)^3 \times H_{\boldsymbol{n}}^1(\Omega), \quad L_0^2(\Omega) := \left\{q \in L^2(\Omega): \int_\Omega q\,\mathrm{d}x = 0\right\},$$

$$H_{\boldsymbol{n}}^1(\Omega) := \left\{\Psi \in H^1(\Omega)^3: (\Psi, \boldsymbol{n})\mid_\Gamma = 0\right\}.$$

$$\mathscr{W}_{\boldsymbol{g}q}(\Omega) := \left\{\boldsymbol{v} \in H^1(\Omega)^3: \boldsymbol{v}\mid_\Gamma = \boldsymbol{g}\right\} \times \left\{\Psi \in H^1(\Omega)^3: \Psi\cdot\boldsymbol{n}\mid_\Gamma = q\right\} := \boldsymbol{X} \times \boldsymbol{Y},$$

其范数为 $\|\boldsymbol{v}, \Psi\|_{\mathscr{W}} = (\|\boldsymbol{v}\|_{1,\Omega}^2 + \|\Psi\|_{1,\Omega}^2)^{1/2}$. $(\cdot,\cdot)$ 表示内积, 而且

$$a((\boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{v}, \Psi), (\boldsymbol{w}, \Phi)) := a_0((\boldsymbol{v}, \Psi), (\boldsymbol{w}, \Phi)) + a_1((\boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{v}, \Psi), (\boldsymbol{w}, \Phi)),$$

$$a_0((\boldsymbol{v}, \Psi), (\boldsymbol{w}, \Phi)) := \frac{1}{M^2}\int_\Omega \nabla\boldsymbol{v}\cdot\nabla\boldsymbol{w}\,\mathrm{d}x + \frac{1}{R_m^2}\int_\Omega\left\{(\nabla\times\Psi)\cdot(\nabla\times\Phi) + (\nabla\cdot\Psi)(\nabla\cdot\Phi)\right\}\mathrm{d}x,$$

$$a_1((\boldsymbol{u}, \boldsymbol{B}), (\boldsymbol{v}, \Psi), (\boldsymbol{w}, \Phi)) := \frac{1}{N}b_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) - \frac{1}{R_m}\int_\Omega\left[(\nabla\times\Psi)\times\boldsymbol{B}\cdot\boldsymbol{w} - (\nabla\times\Phi)\times\boldsymbol{B}\cdot\boldsymbol{v}\right]\mathrm{d}x,$$

$$b_1(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) := \frac{1}{2}\int_\Omega\sum_{i,j=1}^{3}\left[u_i\frac{\partial v_j}{\partial x_i}w_j - u_i\frac{\partial w_j}{\partial x_i}v_j\right]\mathrm{d}x, \qquad \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega)^3,$$

$$b((\boldsymbol{v},\ \Psi),\ \chi):=-\int_\Omega(\nabla\cdot\boldsymbol{v})\,\chi\mathrm{d}x,\quad F(\boldsymbol{v},\ \Psi):=\int_\Omega\boldsymbol{f}\boldsymbol{v}\,\mathrm{d}x+\frac{1}{R_m}\int_{\partial\Omega}\boldsymbol{k}\,\Psi\mathrm{d}s.$$

三线性型 $b_1(\cdot;\cdot,\cdot)$ 和 $a_1(\cdot;\cdot,\cdot)$ 有下列性质:

$$\begin{cases} b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{v})=0, \\ b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=-b_1(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}),\ a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v},\ \Psi),(\boldsymbol{v},\ \Psi))=0, \\ a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v},\ \Psi),(\boldsymbol{w},\ \Phi))=-a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w},\ \Phi),(\boldsymbol{v},\ \Psi)), \\ \qquad\qquad \forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{X},\ \forall\boldsymbol{B},\ \Phi,\ \Psi\in\boldsymbol{Y}. \end{cases}\tag{6}$$

定义

$$N_0:=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X}\frac{b_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})}{|\boldsymbol{u}|_1\cdot|\boldsymbol{v}|_1\cdot|\boldsymbol{w}|_1},\quad \|\boldsymbol{f}\|_*=\sup_{x\in X}\frac{(\boldsymbol{f},\boldsymbol{v})}{\|\boldsymbol{v}\|_1}.$$

文献[2]已证明了问题(Ⅰ*)存在唯一的解,并有

$$[\|\nabla\boldsymbol{u}\|_0^2+\|\nabla\times\boldsymbol{B}\|_0^2]^{1/2}\leqslant\mu^{-1}\|\boldsymbol{f}\|_*,\tag{7}$$

其中 $\mu=\min\left\{\dfrac{1}{M^2},\dfrac{1}{R_m^2}\right\}$.

下列假定是熟知的(参见文献[4]和文献[13]).

(A_1) 存在仅与 Ω 有关的常数 c_1 使得

$$\begin{cases} \|\boldsymbol{v}\|_0\leqslant c_1\|\nabla\boldsymbol{v}\|_0, & \forall\boldsymbol{v}\in H_0^1(\Omega)^3, \\ \|\phi\|_{0,\partial\Omega}\leqslant c_1\|\phi\|_1, & \forall\phi\in H^1(\Omega). \end{cases}\tag{8}$$

设 $\{\mathscr{T}_h\}$ 为 Ω 的拟一致三角形剖分(可参见文献[13]或文献[14]),即令 $h:=\max\limits_{K\in\mathscr{T}_h}\{h_K;\ h_K:=\mathrm{diam}(K)\}$,则存在一个与 h 无关的常数 C 使得 $\forall K\in\mathscr{T}_h$ 都有 $h\leqslant Ch_K$.

设 $\boldsymbol{X}^h\subset H^1(\Omega)^3$ 为分块 k 次多项式, $S_0^h\subset L_0^2(\Omega)$ 为分块 l 次多项式, $\boldsymbol{Y}^h\subset H^1(\Omega)^3$ 为分块 m 次多项式. 定义下列空间:

$$\boldsymbol{X}_0^h(\Omega):=\boldsymbol{X}^h(\Omega)\cap H_0^1(\Omega)^3,\quad \boldsymbol{Y}_n^h(\Omega):=\boldsymbol{Y}^h(\Omega)\cap H_n^1(\Omega),\tag{9}$$

$$\mathscr{W}^h(\Omega):=\boldsymbol{X}^h(\Omega)\times\boldsymbol{Y}^h(\Omega),\quad \mathscr{W}_{0n}^h(\Omega):=\boldsymbol{X}_0^h(\Omega)\times\boldsymbol{Y}_n^h(\Omega).\tag{10}$$

并定义 $\boldsymbol{X}_0^h$ 的子空间 $\boldsymbol{Z}^h$ 如下:

$$\boldsymbol{Z}^h(\Omega):=\left\{\boldsymbol{w}_h\in\boldsymbol{X}_0^h:\int_\Omega(\nabla\cdot\boldsymbol{w}_h)\,\chi_h\mathrm{d}x=0,\ \chi_h\in S_0^h\right\}.\tag{11}$$

问题(Ⅰ*)的 Galerkin-Petrov 最小二乘有限元格式如下:

问题(Ⅰ$_h$) 求 $((\boldsymbol{u}_h,\boldsymbol{B}_h),p_h)\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega)$ 使得 $\boldsymbol{u}_h|_{\partial\Omega}=g_h$, $(\boldsymbol{B}_h\cdot\boldsymbol{n})|_{\partial\Omega}=q_h$, 满足:

$$\begin{aligned} & a_0((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v},\ \Psi))+a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{v},\ \Psi))+ \\ &\quad b((\boldsymbol{v},\ \Psi),p_h)-b((\boldsymbol{v},\ \Psi),\chi)+\sum_{K\in\mathscr{T}_h}\delta_K\Bigg(-\frac{1}{M^2}\Delta\boldsymbol{u}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h+\nabla p_h- \\ &\quad \frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{B}_h,\ -\frac{1}{M^2}\Delta\boldsymbol{v}+\frac{1}{N}(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}+\nabla\chi-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\Psi\Bigg)_K= \\ &\quad F((\boldsymbol{v},\ \Psi))+\sum_{K\in\mathscr{T}_h}\delta_K\Bigg(\boldsymbol{f},\ -\frac{1}{M^2}\Delta\boldsymbol{v}+\frac{1}{N}(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}+\nabla\chi- \\ &\quad \frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\Psi\Bigg)_K,\qquad \forall((\boldsymbol{v},\ \Psi),\chi)\in\mathscr{W}_{0\boldsymbol{n}}^h(\Omega)\times S_0^h(\Omega), \end{aligned}\tag{12}$$

其中 $\delta_K=\alpha h_K^2$, $\alpha>0$ 是任意常数.

对于 $(\hat{\boldsymbol{v}},\hat{\boldsymbol{A}})=((\boldsymbol{v},\boldsymbol{A}),\chi)$, $(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}})=((\boldsymbol{w},\boldsymbol{D}),r)$ 定义:

$$B_\delta((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{v}},\hat{\boldsymbol{A}}),(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}}))=$$
$$a_0((\boldsymbol{v},\boldsymbol{A}),(\boldsymbol{w},\boldsymbol{D}))+a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{v},\boldsymbol{A}),(\boldsymbol{w},\boldsymbol{D}))+$$
$$b((\boldsymbol{v},\boldsymbol{A}),\chi)-b((\boldsymbol{v},\boldsymbol{A}),r)+$$
$$\sum_{K\in\mathscr{T}_h}\delta_K\left(-\frac{1}{M^2}\Delta\boldsymbol{v}+\frac{1}{N}\boldsymbol{u}\cdot\nabla\boldsymbol{v}+\nabla\chi-\frac{1}{R_m}(\nabla\times\boldsymbol{B})\times\boldsymbol{A},\right.$$
$$\left.-\frac{1}{M^2}\Delta\boldsymbol{w}+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}+\nabla r-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{D}\right)_K,$$

$$L_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}}))=F((\boldsymbol{w},\boldsymbol{D}))+\sum_{K\in\mathscr{T}_h}\delta_K\left(\boldsymbol{f},-\frac{1}{M^2}\Delta\boldsymbol{w}+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}+\right.$$
$$\left.\nabla r-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{D}\right)_K,$$

$$B^*_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}}),\boldsymbol{f})=$$
$$B_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}}))-L_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}})),$$

其中 δ 为分段常函数, $\delta|_K=\delta_K,\ \forall K\in\mathscr{T}_h$. 则问题$(\mathrm{I}_h)$ 可重写为:

求 $\hat{\boldsymbol{u}}_h=((\boldsymbol{u}_h,\boldsymbol{B}_h),p_h)\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega)$ 满足:

$$B^*_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}}),\boldsymbol{f})=0,$$
$$\forall(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}})\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega).\tag{13}$$

2 Galerkin-Petrov 最小二乘混合元解的存在唯一性

下面给出问题(I_h)解的存在唯一性证明,其中不需要 Babuška-Brezzi 稳定性条件.

定理 1 假定(A_1) 成立, 则问题(I_h) 至少存在一个解$((\boldsymbol{u}_h,\boldsymbol{B}_h),p_h)\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega)$.

证明 用 Brouwer 不动点定理证明. 对于任意的$(\boldsymbol{v}_h,\boldsymbol{A}_h)\in\mathscr{W}^h(\Omega)$, $\|(\boldsymbol{v}_h,\boldsymbol{A}_h)\|_{\mathscr{W}}\leqslant P$, 其中 $P=2(\mu^{-1}+\delta^{1/2})(\|\boldsymbol{f}\|_*+(c_1/R_m)\|\boldsymbol{k}\|_{-1/2,\partial\Omega})$ 为一正常数.

下面问题:

$$B^*_\delta((\boldsymbol{v}_h,\boldsymbol{A}_h),(\boldsymbol{v}_h,\boldsymbol{A}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}}),\boldsymbol{f})=0,$$
$$\forall(\hat{\boldsymbol{w}},\hat{\boldsymbol{D}})\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega),\tag{14}$$

存在唯一的解$(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h)=((\boldsymbol{u}_h,\boldsymbol{B}_h),p_h)\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega)$(可参见文献[2]和文献[15]). 即$(\boldsymbol{v}_h,\boldsymbol{A}_h)$ 通过(14) 式可唯一的对应一个$(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h)$, 我们把这个对应关系记为 $G:(\hat{\boldsymbol{v}}_h,\hat{\boldsymbol{A}}_h)\to(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h)=G((\hat{\boldsymbol{v}}_h,\hat{\boldsymbol{A}}_h))$, 其中$(\hat{\boldsymbol{v}}_h,\hat{\boldsymbol{A}}_h)=((\boldsymbol{v}_h,\boldsymbol{A}_h),p_h)$.

由 $B_\delta(\cdot,\cdot;\cdot,\cdot)$ 的定义, 有

$$B_\delta((\boldsymbol{v}_h,\boldsymbol{A}_h),(\boldsymbol{v}_h,\boldsymbol{A}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h))=$$
$$\frac{1}{M^2}|\boldsymbol{u}_h|_1^2+\frac{1}{R_m^2}(|\nabla\times\boldsymbol{B}_h|_1^2+|\nabla\cdot\boldsymbol{B}_h|_1^2)+$$
$$\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h+\frac{1}{N}\boldsymbol{v}_h\cdot\nabla\boldsymbol{u}_h+\nabla p_h-\frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{B}_h\right)\right\|_{0,h}^2,\tag{15}$$

其中 $\|\cdot\|_{0,h}=\sum_{K\in\mathscr{T}_h}\|\cdot\|_{0,K}^2$. 由 Hölder 不等式和$(\mathrm{A}_1)$有,

$$B_\delta((\boldsymbol{v}_h,\boldsymbol{A}_h),(\boldsymbol{v}_h,\boldsymbol{A}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h))-$$
$$B^*_\delta((\boldsymbol{v}_h,\boldsymbol{A}_h),(\boldsymbol{v}_h,\boldsymbol{A}_h);(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{u}}_h,\hat{\boldsymbol{B}}_h),\boldsymbol{f})=$$

$$F((\boldsymbol{u}_h, \boldsymbol{B}_h)) + \sum_{K\in\mathscr{T}_h}\delta_K\left(\boldsymbol{f}, -\frac{1}{M^2}\Delta\boldsymbol{u}_h + \frac{1}{N}\boldsymbol{v}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{B}_h\right)_K \leqslant \mu^{-1}\left\{\|\boldsymbol{f}\|_* + \frac{c_1}{R_m}\|\boldsymbol{k}\|_{-1/2,\partial\Omega}\right\}\|(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{B}}_h)\|_{\mathscr{W}} + \delta^{1/2}\left\{\|\boldsymbol{f}\|_* + \frac{c_1}{R_m}\|\boldsymbol{k}\|_{-1/2,\partial\Omega}\right\}\times\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h + \frac{1}{N}\boldsymbol{v}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{B}_h\right)\right\|_{0,h} \leqslant$$
$$2(\mu^{-1} + \delta^{1/2})\left(\|\boldsymbol{f}\|_* + \frac{c_1}{R_m}\|\boldsymbol{k}\|_{-1/2,\partial\Omega}\right)\times\left(\frac{1}{M^2}|\boldsymbol{u}_h|_1^2 + \frac{1}{R_m^2}(|\nabla\times\boldsymbol{B}_h|_1^2 + |\nabla\cdot\boldsymbol{B}_h|_1^2) + \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h + \frac{1}{N}\boldsymbol{v}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{B}_h\right)\right\|_{0,h}^2\right)^{1/2}. \quad (16)$$

合并(15)式和(16)式,得

$$\left(\frac{1}{M^2}|\boldsymbol{u}_h|_1^2 + \frac{1}{R_m^2}(|\nabla\times\boldsymbol{B}_h|_1^2 + |\nabla\cdot\boldsymbol{B}_h|_1^2) + \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h + \frac{1}{N}\boldsymbol{v}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{B}_h\right)\right\|_{0,h}^2\right)^{1/2} \leqslant 2(\mu^{-1} + \delta^{1/2})\left(\|\boldsymbol{f}\|_* + \frac{c_1}{R_m}\|\boldsymbol{k}\|_{-1/2,\partial\Omega}\right). \quad (17)$$

由(17)式得

$$\left(\frac{1}{M^2}|\boldsymbol{u}_h|_1^2 + \frac{1}{R_m^2}(|\nabla\times\boldsymbol{B}_h|_1^2 + |\nabla\cdot\boldsymbol{B}_h|_1^2) + \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h + \frac{1}{N}\boldsymbol{v}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{B}_h\right)\right\|_{0,h}^2\right)^{1/2} \leqslant P. \quad (18)$$

定义 $B_P = \left\{(\hat{\boldsymbol{v}}_h, \hat{\boldsymbol{A}}_h) = ((\boldsymbol{v}_h, \boldsymbol{A}_h), \chi_h) \in \mathscr{W}^h(\Omega);\ \|(\boldsymbol{v}_h, \boldsymbol{A}_h)\|_{\mathscr{W}} \leqslant P\right\}$, (18) 式表明 G 是从 B_P 到 B_P 的映射, 剩下只需证明 G 是连续的. 对任意的 $(\hat{\boldsymbol{v}}_h^i, \hat{\boldsymbol{A}}_h^i)$, $\|(\boldsymbol{v}_h^i, \boldsymbol{A}_h^i)\|_{\mathscr{W}} \leqslant P\ (i = 1, 2)$, 通过(14)式确定 $(\hat{\boldsymbol{u}}_h^i, \hat{\boldsymbol{B}}_h^i) = ((\boldsymbol{u}_h^i, \boldsymbol{B}_h^i), p_h^i)$, 令 $(\hat{\boldsymbol{v}}_h^i, \hat{\boldsymbol{A}}_h^i) = ((\boldsymbol{v}_h^i, \boldsymbol{A}_h^i), p_h^i)$ 有 $(\hat{\boldsymbol{v}}_h^i, \hat{\boldsymbol{A}}_h^i) \in B_P$, $(\hat{\boldsymbol{u}}_h^i, \hat{\boldsymbol{B}}_h^i) = G((\hat{\boldsymbol{v}}_h^i, \hat{\boldsymbol{A}}_h^i))$ 及

$$B_\delta^*((\boldsymbol{v}_h^i, \boldsymbol{A}_h^i), (\boldsymbol{v}_h^i, \boldsymbol{A}_h^i); (\hat{\boldsymbol{u}}_h^i, \hat{\boldsymbol{B}}_h^i), (\hat{\boldsymbol{w}}, \hat{\boldsymbol{D}}), f) = 0,\quad \forall(\hat{\boldsymbol{w}}, \hat{\boldsymbol{D}}) \in \mathscr{W}^h(\Omega)\times S_0^h(\Omega). \quad (19)$$

由(18)式有

$$\left(\frac{1}{M^2}|\boldsymbol{u}_h^i|_1^2 + \frac{1}{R_m^2}(|\nabla\times\boldsymbol{B}_h^i|_1^2 + |\nabla\cdot\boldsymbol{B}_h^i|_1^2) + \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h^i + \frac{1}{N}\boldsymbol{v}_h^i\cdot\nabla\boldsymbol{u}_h^i + \nabla p_h^i - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h^i)\times\boldsymbol{B}_h^i\right)\right\|_{0,h}^2\right)^{1/2} \leqslant P. \quad (20)$$

由(19)式有

$$B_\delta((\boldsymbol{v}_h^1, \boldsymbol{A}_h^1), (\boldsymbol{v}_h^1, \boldsymbol{A}_h^1); (\hat{\boldsymbol{u}}_h^1, \hat{\boldsymbol{B}}_h^1), (\hat{\boldsymbol{w}}, \hat{\boldsymbol{D}})) - B_\delta((\boldsymbol{v}_h^2, \boldsymbol{A}_h^2), (\boldsymbol{v}_h^2, \boldsymbol{A}_h^2); (\hat{\boldsymbol{u}}_h^2, \hat{\boldsymbol{B}}_h^2), (\hat{\boldsymbol{w}}, \hat{\boldsymbol{D}})) = \sum_{K\in\mathscr{T}_h}\delta_k\left(\boldsymbol{f}, -\frac{1}{M^2}\Delta\boldsymbol{w} + \frac{1}{N}(\boldsymbol{v}_h^1\cdot\nabla)\boldsymbol{w} + \nabla r - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{D}\right)_K +$$

$$\sum_{K\in\mathcal{T}_h}\delta_k\left(\boldsymbol{f}, \frac{1}{M^2}(\boldsymbol{v}_h^1-\boldsymbol{v}_h^2)\cdot\nabla\boldsymbol{w}\right)_K =: G_1, \qquad \forall(\hat{\boldsymbol{w}}, \boldsymbol{D})\in\mathcal{W}^h. \tag{21}$$

取 $(\hat{\boldsymbol{w}}, \boldsymbol{D}) = ((\hat{\boldsymbol{u}}_h^1-\hat{\boldsymbol{u}}_h^2), (\boldsymbol{B}_h^1-\boldsymbol{B}_h^2))$, 即

$(\hat{\boldsymbol{w}}, \boldsymbol{D}) = ((\boldsymbol{w}, \boldsymbol{D}), r) = (((\boldsymbol{u}_h^1-\boldsymbol{u}_h^2), (\boldsymbol{B}_h^1-\boldsymbol{B}_h^2)), p_h^1-p_h^2)$, 一方面, 有

$$\begin{aligned}&B_\delta((\boldsymbol{v}_h^1, \boldsymbol{A}_h^1), (\boldsymbol{v}_h^1, \boldsymbol{A}_h^1); (\hat{\boldsymbol{w}}, \boldsymbol{D}), (\hat{\boldsymbol{w}}, \boldsymbol{D})) =\\
&\quad \frac{1}{M^2}|\boldsymbol{w}|_1^2 + \frac{1}{R_m^2}(|\nabla\times\boldsymbol{D}|_1^2 + |\nabla\cdot\boldsymbol{D}|_1^2) +\\
&\quad \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{w} + \frac{1}{N}\boldsymbol{v}_h^1\cdot\nabla\boldsymbol{w} + \nabla r - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h^1)\times\boldsymbol{D}\right)\right\|_{0,h}^2,\end{aligned} \tag{22}$$

另一方面, 由(21)式有

$$\begin{aligned}&B_\delta((\boldsymbol{v}_h^1, \boldsymbol{A}_h^1), (\boldsymbol{v}_h^1, \boldsymbol{A}_h^1); (\hat{\boldsymbol{w}}, \boldsymbol{D}), (\hat{\boldsymbol{w}}, \boldsymbol{D})) =\\
&\quad B_\delta((\boldsymbol{v}_h^1, \boldsymbol{A}_h^1), (\boldsymbol{v}_h^1, \boldsymbol{A}_h^1); (\hat{\boldsymbol{u}}_h^1, \boldsymbol{B}_h^1), (\hat{\boldsymbol{w}}, \boldsymbol{D})) -\\
&\quad B_\delta((\boldsymbol{v}_h^1, \boldsymbol{A}_h^1), (\boldsymbol{v}_h^1, \boldsymbol{A}_h^1); (\hat{\boldsymbol{u}}_h^2, \boldsymbol{B}_h^2), (\hat{\boldsymbol{w}}, \boldsymbol{D})) =\\
&\quad B_\delta((\boldsymbol{v}_h^2, \boldsymbol{A}_h^2), (\boldsymbol{v}_h^2, \boldsymbol{A}_h^2); (\hat{\boldsymbol{u}}_h^2, \boldsymbol{B}_h^2), (\hat{\boldsymbol{w}}, \boldsymbol{D})) -\\
&\quad B_\delta((\boldsymbol{v}_h^1, \boldsymbol{A}_h^1), (\boldsymbol{v}_h^1, \boldsymbol{A}_h^1); (\hat{\boldsymbol{u}}_h^2, \boldsymbol{B}_h^2), (\hat{\boldsymbol{w}}, \boldsymbol{D})) + G_1 =\\
&\quad a_1((\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1); (\boldsymbol{u}_h^2, \boldsymbol{B}_h^2), (\boldsymbol{w}, \boldsymbol{D})) +\\
&\quad \sum_{K\in\mathcal{T}_h}\delta_K\left((\boldsymbol{v}_h^2-\boldsymbol{v}_h^1)\cdot\nabla\boldsymbol{u}_h^2, -\frac{1}{M^2}\Delta\boldsymbol{w} + \frac{1}{N}(\boldsymbol{v}_h^1\cdot\nabla)\boldsymbol{w} + \nabla r -\right.\\
&\quad \left.\frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{D}\right)_K + \sum_{K\in\mathcal{T}_h}\delta_K\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h^2 + \frac{1}{N}(\boldsymbol{v}_h^2\cdot\nabla)\boldsymbol{u}_h^2 + \nabla p_h^2 -\right.\\
&\quad \left.\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h^2)\times\boldsymbol{A}_h^2, (\boldsymbol{v}_h^2-\boldsymbol{v}_h^1)\cdot\nabla\boldsymbol{w}\right)_K + G_1 =:\\
&\quad S_1 + S_2 + S_3 + G_1.\end{aligned} \tag{23}$$

由Sobolev空间的嵌入定理及逆不等式(参见文献[14]的推论1.24)可以得到

$$\|\boldsymbol{v}\|_0 \leqslant C\|\boldsymbol{v}\|_{0,\infty} \leqslant C_0 h^{-1/2}|\boldsymbol{v}|_1, \qquad \forall\boldsymbol{v}\in\boldsymbol{X}^h. \tag{24}$$

由(19)式和Cauchy不等式有

$$|S_1| \leqslant C\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathcal{W}}\|(\boldsymbol{w}, \boldsymbol{D})\|_{\mathcal{W}}, \tag{25}$$

$$\begin{aligned}|S_2| \leqslant{}& Ch^{-1/2}\delta_M^{1/2}\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathcal{W}}\times\\
&\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{w} + \frac{1}{N}(\boldsymbol{v}_h^1\cdot\nabla)\boldsymbol{w} + \nabla r - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h^1)\times\boldsymbol{D}\right)\right\|_{0,h},\end{aligned} \tag{26}$$

$$|S_3| \leqslant Ch^{-1/2}\delta_M^{1/2}\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathcal{W}}\|(\boldsymbol{w}, \boldsymbol{D})\|_{\mathcal{W}}, \tag{27}$$

其中 $\delta_M = \max\limits_{K\in\mathcal{T}_h}\delta_K = \alpha h^2$, C 为是与 h 无关的常数(不同位置的 C 可以不同).

$$\begin{aligned}|G_1| ={}& \left|\sum_{K\in\mathcal{T}_h}\delta_K\left(\boldsymbol{f}, -\frac{1}{M^2}\Delta\boldsymbol{w} + \frac{1}{N}(\boldsymbol{v}_h^1\cdot\nabla)\boldsymbol{w} + \nabla r - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{D}\right)_K +\right.\\
&\left.\sum_{K\in\mathcal{T}_h}\delta_K\left(\boldsymbol{f}, \frac{1}{M^2}(\boldsymbol{v}_h^1-\boldsymbol{v}_h^2)\cdot\nabla\boldsymbol{w}\right)_K\right| \leqslant\\
& C\delta_M^{1/2}\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathcal{W}}\times\\
&\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{w} + \frac{1}{N}(\boldsymbol{v}_h^1\cdot\nabla)\boldsymbol{w} + \nabla r - \frac{1}{R_m}(\nabla\times\boldsymbol{A}_h)\times\boldsymbol{D}\right)\right\|_{0,h} +\\
& C\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathcal{W}}\|(\boldsymbol{w}, \boldsymbol{D})\|_{\mathcal{W}}.\end{aligned} \tag{28}$$

合并(25)式至(28)式有

$$|S_1|+|S_2|+|S_3|+|G_1| \leqslant L(R)\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathscr{W}} \times \left\{\frac{1}{M^2}|\boldsymbol{w}|_1^2+\frac{1}{R_m^2}(|\nabla\times\boldsymbol{D}|_1^2+|\nabla\cdot\boldsymbol{D}|_1^2)+\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{w}+\frac{1}{N}\boldsymbol{v}_h^1\cdot\nabla\boldsymbol{w}+\nabla\chi-\frac{1}{R_m}(\nabla\times\boldsymbol{A}_h^1)\times\boldsymbol{D}\right)\right\|_{0,h}^2\right\}^{1/2}. \quad (29)$$

由(27)式、(28)式和(29)式有

$$\left\{\frac{1}{M^2}|\boldsymbol{w}|_1^2+\frac{1}{R_m^2}(|\nabla\times\boldsymbol{D}|_1^2+|\nabla\cdot\boldsymbol{D}|_1^2)+\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{w}+\frac{1}{N}\boldsymbol{v}_h^1\cdot\nabla\boldsymbol{w}+\nabla\chi-\frac{1}{R_m}(\nabla\times\boldsymbol{A}_h^1)\times\boldsymbol{D}\right)\right\|_{0,h}^2\right\}^{1/2} \leqslant L(R)\|(\boldsymbol{v}_h^2-\boldsymbol{v}_h^1, \boldsymbol{A}_h^2-\boldsymbol{A}_h^1)\|_{\mathscr{W}}. \quad (30)$$

注意到(30)式中$(\boldsymbol{w}, \boldsymbol{D})=(\boldsymbol{u}_h^1-\boldsymbol{u}_h^2, \boldsymbol{B}_h^1-\boldsymbol{B}_h^2)$, $L(R)$是不依赖于$(\hat{\boldsymbol{v}}_h^i, \hat{\boldsymbol{A}}_h^i)$和$(\hat{\boldsymbol{u}}_h^i, \hat{\boldsymbol{B}}_h^i)(i=1,2)$的常数, 表明 G 是连续的. 由 Brouwer 不动点定理, 至少存在一个不动点$(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{B}}_h)=G((\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{B}}_h))$, 则问题$(\mathrm{I}_h)$ 至少存在一个解$((\boldsymbol{u}_h, \boldsymbol{B}_h), p_h)\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega)$. 定理 1 证毕.

用类似于文献[14]的定理4.12 的方法可以证得下一定理.

定理 2 假设$f\in L^2(\Omega)^3$, $\mu^{-2}\|\boldsymbol{f}\|_*<1$. 则存在一个常数 h_0, 使得对任意 $h\leqslant h_0$, 问题(I_h) 存在唯一解$((\boldsymbol{u}_h, \boldsymbol{B}_h), p_h)\in\mathscr{W}^h(\Omega)\times S_0^h(\Omega)$, 且满足估计

$$\left\{\frac{1}{M^2}|\boldsymbol{u}_h|_1^2+\frac{1}{R_m^2}(|\nabla\times\boldsymbol{B}_h|_1^2+|\nabla\cdot\boldsymbol{B}_h|_1^2)+\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{u}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h+\nabla p_h-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{B}_h\right)\right\|_{0,h}^2\right\}^{1/2} \leqslant (M^2\|\boldsymbol{f}\|_*^2+\|\delta^{1/2}f\|_0^2)^{1/2}. \quad (31)$$

3 Galerkin-Petrov 最小二乘混合有限元法的收敛性

本节给出问题(I_h)的 Galerkin-Petrov 最小二乘混合有限元法的收敛性和误差估计. 用类似于文献[14]的定理4.13 可以证明得下一定理.

定理 3 在定理 1 和定理 2 的条件下, 问题(I_h) 的解序列$\{((\boldsymbol{u}_h, \boldsymbol{B}_h), p_h)\}$存在一个子列弱收敛于问题$(\mathrm{I}^*)$ 的解$((\boldsymbol{u}, \boldsymbol{B}), p)$.

定理 4 在定理 3 的条件下, 如果问题(I^*)的精确解$((\boldsymbol{u}, \boldsymbol{B}), p)\in\mathscr{W}(\Omega)\times L_0^2(\Omega)$, 则问题$(\mathrm{I}_h)$ 的解$((\boldsymbol{u}_h, \boldsymbol{B}_h), p_h)$, 存在一个常数 $h^*>0$, 使得对 $\forall h\leqslant h^*$ 都有如下估计:

$$\left[\mu\|(\boldsymbol{u}, \boldsymbol{B})-(\boldsymbol{u}^h, \boldsymbol{B}^h)\|_{\mathscr{W}}^2+\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta(\boldsymbol{u}-\boldsymbol{u}_h)+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h)+\nabla(p-p_h)-\frac{1}{R_m}(\nabla\times(\boldsymbol{B}-\boldsymbol{B}_h))\times(\boldsymbol{B}-\boldsymbol{B}_h)\right)\right\|_{0,h}^2\right]^{1/2} \leqslant C(h^k+h^{l+1}+h^m), \quad (32)$$

其中 C 是不依赖于h 的常数, $\mu=\min\{M^{-2}, R_m^{-2}\}$.

证明 由定理 2 知问题(I_h)有唯一确定的解, 取$(\hat{\boldsymbol{w}}_h, \hat{\boldsymbol{D}}_h)=((\boldsymbol{w}_h, \boldsymbol{D}_h), r_h)=((\pi_h^1\boldsymbol{u}-\boldsymbol{u}_h, \pi_h^1\boldsymbol{B}-\boldsymbol{B}_h), \pi_h^2 p-p_h)$. 一方面, 有

$$B_\delta((\boldsymbol{u}_h, \boldsymbol{B}_h), (\boldsymbol{u}_h, \boldsymbol{B}_h); (\hat{\boldsymbol{w}}_h, \hat{\boldsymbol{D}}_h), (\hat{\boldsymbol{w}}_h, \hat{\boldsymbol{D}}_h))=$$

$$\frac{1}{M^2}|\boldsymbol{w}_h|_1^2+\frac{1}{R_m^2}(|\nabla\times\boldsymbol{D}_h|_1^2+|\nabla\cdot\boldsymbol{D}_h|_1^2)+$$
$$\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta\boldsymbol{w}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}_h+\nabla r_h-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h^1)\times\boldsymbol{D}_h\right)\right\|_{0,h}^2=:S_1, \tag{33}$$

另一方面,由(13)式和(15)式有

$$\begin{aligned}S_1=&B_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\pi_h\hat{\boldsymbol{u}}_h,\pi_h\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}}_h,\hat{\boldsymbol{D}}_h))-\\&B_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\pi_h\hat{\boldsymbol{u}}_h,\pi_h\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}}_h,\hat{\boldsymbol{D}}_h))=\\&B_\delta((\boldsymbol{u}_h,\boldsymbol{B}_h),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\pi_h\hat{\boldsymbol{u}}_h,\pi_h\hat{\boldsymbol{B}}_h),(\hat{\boldsymbol{w}}_h,\hat{\boldsymbol{D}}_h))-\\&B_\delta((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u}_h,\boldsymbol{B}_h);(\hat{\boldsymbol{u}},\hat{\boldsymbol{B}}),(\hat{\boldsymbol{w}}_h,\hat{\boldsymbol{D}}_h)),\end{aligned} \tag{34}$$

即 $S_1=S_2+S_3+S_4$, 其中

$$S_2=a_0((\pi_h^1\boldsymbol{u}-\boldsymbol{u},\pi_h^1\boldsymbol{B}-\boldsymbol{B}),(\boldsymbol{w}_h,\boldsymbol{D}_h))-(\pi_h^2p-p,\operatorname{div}\boldsymbol{w}_h),$$
$$\begin{aligned}S_3=&\sum_{K\in\mathcal{T}_h}\delta_K\left(\frac{1}{M^2}\Delta(\pi_h^1\boldsymbol{u}-\boldsymbol{u})+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\pi_h^1\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla(\pi_h^2p-p)-\right.\\&\frac{1}{R_m}(\nabla\times(\pi_h^1\boldsymbol{B}-\boldsymbol{B})\times\boldsymbol{B}),\frac{1}{M^2}\Delta\boldsymbol{w}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}_h+\\&\left.\nabla r_h-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{D}_h\right),\end{aligned}$$
$$\begin{aligned}S_4=&a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\pi_h^1\boldsymbol{u},\pi_h^1\boldsymbol{B}),(\boldsymbol{w}_h,\boldsymbol{D}_h)-\\&a_1((\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w}_h,\boldsymbol{D}_h)+(r_h,\operatorname{div}(\pi_h^1\boldsymbol{u}-\boldsymbol{u})).\end{aligned}$$

由经典的插值误差估计(可参见文献[13]和文献[14])有

$$|S_2|\leqslant C(h^k+h^{l+1}+h^m)\,\|(\boldsymbol{w}_h,\boldsymbol{D}_h)\|_{\mathscr{W}}, \tag{35}$$
$$\begin{aligned}|S_3|\leqslant&\frac{1}{4}\sum_{K\in\mathcal{T}_h}\delta_K\left\|-\frac{1}{M^2}\Delta\boldsymbol{w}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}_h+\nabla r_h-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{D}_h\right\|_{0,K}^2+\\&C\delta_M h^{2l}+C\delta_M\|\boldsymbol{u}_h\Delta(\pi_h^1\boldsymbol{u}-\boldsymbol{u})+(\boldsymbol{u}_h-\boldsymbol{u})\nabla\boldsymbol{u}\|_0^2+\\&C\delta_M\|\nabla\times(\pi_h^1\boldsymbol{B}_h-\boldsymbol{B})\|_0^2\leqslant\\&\frac{1}{4}\delta^{1/2}\left\|-\frac{1}{M^2}\Delta\boldsymbol{w}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}_h+\nabla r_h-\frac{1}{R_m}(\nabla\times\boldsymbol{B}_h)\times\boldsymbol{D}_h\right\|_{0,h}^2+\\&C\delta_M\|(\boldsymbol{w}_h,\boldsymbol{D}_h)\|_{\mathscr{W}}^2+C\delta_M(h^{2l}+h^{2k-1}+h^{2m-1}).\end{aligned} \tag{36}$$

由 Green 公式,有

$$(r_h,\operatorname{div}(\pi_h^1\boldsymbol{u}-\boldsymbol{u}))=(\boldsymbol{u}_h\nabla\boldsymbol{w}_h,\pi_h^1\boldsymbol{u}-\boldsymbol{u})-(\boldsymbol{u}_h\nabla\boldsymbol{w}_h+\nabla r_h,\pi_h^1\boldsymbol{u}-\boldsymbol{u})$$

和

$$\begin{aligned}S_4=&a_1((\boldsymbol{u}_h,\boldsymbol{B}_h),(\pi_h^1\boldsymbol{u}-\boldsymbol{u},\pi_h^1\boldsymbol{B}-\boldsymbol{B}),(\boldsymbol{w}_h,\boldsymbol{D}_h))-\\&a_1((\pi_h^1\boldsymbol{u}-\boldsymbol{u},\pi_h^1\boldsymbol{B}-\boldsymbol{B}),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w}_h,\boldsymbol{D}_h))-\\&a_1((\boldsymbol{w}_h,\boldsymbol{D}_h),(\boldsymbol{u},\boldsymbol{B}),(\boldsymbol{w}_h,\boldsymbol{D}_h))+(\boldsymbol{u}_h\nabla\boldsymbol{w}_h,\pi_h^1\boldsymbol{u}-\boldsymbol{u})-\\&(\boldsymbol{u}_h\nabla\boldsymbol{w}_h+\nabla r_h,\pi_h^1\boldsymbol{u}-\boldsymbol{u}).\end{aligned}$$

由(7)式和逆估计定理有

$$\begin{aligned}|S_4|\leqslant&C(h^k+h^m)\,\|(\boldsymbol{w}_h,\boldsymbol{D}_h)\|_{\mathscr{W}}+\mu^{-1}\|f\|_*\,\|(\boldsymbol{w}_h,\boldsymbol{D}_h)\|_{\mathscr{W}}+\\&\left|\sum_{K\in\mathcal{T}_h}\left(\frac{1}{M^2}\Delta\boldsymbol{w}_h+\frac{1}{N}\boldsymbol{u}_h\cdot\nabla\boldsymbol{w}_h+\nabla r_h,\pi_h^1\boldsymbol{u}-\boldsymbol{u}\right)_K\right|+\\&\left|\sum_{K\in\mathcal{T}_h}\left(\frac{1}{M^2}\Delta\boldsymbol{w}_h,\pi_h^1\boldsymbol{u}-\boldsymbol{u}\right)_K\right|\leqslant\end{aligned}$$

$$C(h^k + h^m)\|(w_h, D_h)\|_{\mathscr{W}} + \mu^{-1}\|f\|_*\|(w_h, D_h)\|_{\mathscr{W}} + C\delta_{\min}^{-1}h^{2k+2} +$$
$$\frac{1}{6}\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta w_h + \frac{1}{N}u_h\cdot\nabla w_h + \nabla r_h - \frac{1}{R_m}(\nabla\times B_h^1)\times D_h\right)\right\|_{0,h}^2, \tag{37}$$

其中 $\delta_{\min} = \min\limits_{x\in\Omega}\delta = \inf\limits_{K\in\mathscr{T}_h}\delta_K$. 由(33)式至(37)式, 有

$$\min\left\{\frac{1}{M^2}, \frac{1}{R_m^2}\right\}\|(w_h, D_h)\|_{\mathscr{W}}^2(1-\mu^{-2}\|f\|_* - C\delta_M) +$$
$$\frac{1}{2}\left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta w_h + \frac{1}{N}u_h\cdot\nabla w_h + \nabla r_h - \frac{1}{R_m}(\nabla\times B_h^1)\times D_h\right)\right\|_{0,h}^2 \leqslant$$
$$C(h^k + h^{l+1} + h^m)\|(w_h, D_h)\|_{\mathscr{W}} +$$
$$C\delta_M(h^{2l} + h^{2k-1} + h^{2m-1}) + C\delta_{\min}^{-1}h^{2k+2}. \tag{38}$$

由 $\delta_M = \alpha h^2$ 和 $\mu^{-2}\|f\|_* < 1$ 知, $1-\mu^{-2}\|f\|_* \geqslant \delta_1 > 0$, 存在 $h^* > 0$, 使得对 $\forall h \leqslant h^*$ 有 $C\delta_M \leqslant \delta_1/2$, 则由(38)式得

$$\left\{\mu\|(w_h, D_h)\|_{\mathscr{W}}^2 + \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta(u-u_h) + \frac{1}{N}u_h\cdot\nabla(u-u_h) +\right.\right.\right.$$
$$\left.\left.\left.\nabla(p-p_h) - \frac{1}{R_m}(\nabla\times(B-B_h)\times(B-B_h)\right)\right\|_{0,h}^2\right\}^{1/2} \leqslant$$
$$C(h^k + h^{l+1} + h^m + \delta_{\min}^{-1/2}h^{k+1}).$$

注意到 $\delta_K = \alpha h_K^2$, $h/h_K \leqslant C$, $(w_h, D_h) = (\pi_h^1 u - u_h, \pi_h^1 B - B_h)$. 由三角不等式得

$$\left[\mu\|(u, B) - (u^h, B^h)\|_{\mathscr{W}}^2 + \left\|\delta^{1/2}\left(-\frac{1}{M^2}\Delta(u-u_h) + \frac{1}{N}u_h\cdot\nabla(u-u_h) +\right.\right.\right.$$
$$\left.\left.\left.\nabla(p-p_h) - \frac{1}{R_m}(\nabla\times(B-B_h))\times(B-B_h)\right)\right\|_{0,h}^2\right]^{1/2} \leqslant$$
$$C(h^k + h^{l+1} + h^m).$$

定理 4 证毕.

本文对定常的磁流体动力学问题引入了 Galerkin-Petrov 最小二乘混合有限元方法使得混合元空间之间的组合无需满足离散的 Babuška-Brezzi 稳定性条件, 从而使得它们的有限元空间可以任意选取, 误差估计达到最优阶. 我们在另一文中, 再将非线性 Galerkin 方法与 Petrov 最小二乘方法结合起来, 处理定常的磁流体动力学问题.

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Petrov-Galerkin Least Squares Mixed Element Method for the Stationary Incompressible Magnetohydrodynamics

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Abstract: A Galerkin-Petrov least squares mixed finite element method for the stationary magnetohydrodynamics problems was introduced and the existence and error estimates of the Galerkin-Petrov least squares mixed finite element solution were derived. The combination among mixed finite element spaces of this method dose not demand the discrete Babuska-Brezzi stability conditions so that the mixed finite element spaces could be arbitrarily chosen and the error estimates with optimal order could be obtained.

Key words: equation of magnetohydrodynamics; mixed element method; Galerkin-Petrov-least squares method; error estimate