

# Mindlin 板条中弹性波传播问题的分析\*

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**摘要:** 基于 Mindlin 弹性厚板理论, 采用 Hamilton 状态空间求解方法, 研究了板条横向两侧为自由边界条件下的弹性波传播问题, 给出了问题的分析解, 以及各种振动模态和波传播模的存在性。采用本征函数展开法给出了条形板中导波模的频散方程, 并与直接根据 Mindlin 提出的解获得的频散关系进行了比较。通过比较发现: 在短波时, 频散曲线相差较大。即前几阶低频模式有一定差别。根据 Hamilton 体系得到的截止频率高一些。长波时频散曲线几乎相同, 即高频模式无差别。

**关键词:** Mindlin 厚板; 结构导波; Hamilton 体系; 频散关系; 行波控制; 扩展模与局部振动模

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## 引 言

经典平板理论在求解自由边界条件下的动力学问题时, 会遇到不可克服的困难, 有时难以满足角点条件。20 世纪中期 Reissner 提出了考虑横向剪切变形影响的平板静力学方程<sup>[1, 2]</sup>, 而后 Mindlin 则对板作了比较系统的动力学研究<sup>[3]</sup>。Mindlin 板振动理论考虑了横向剪切变形与转动惯量的影响, 弥补了经典薄板理论的不足, 它可在很宽的频率范围内适用, 其结果更接近实际<sup>[4, 5]</sup>。

在工程中常采用复合材料结构, 复合材料平板横向剪切模量都比较小, 在动力学分析中应采用厚板理论。在航天器结构动力学与控制中常用板条结构, 并且将其简化成梁结构进行分析<sup>[6, 7]</sup>。因此, 采用弹性厚板理论, 分析研究板条的振动与弹性行波问题, 具有理论意义和工程应用前景。求解平板弯曲振动过去一直采用半逆解法, 此解法有局限性, 难以分析复杂的边值问题。采用 Hamilton 体系可求解一些经典方法不能解决的边值问题<sup>[8-10]</sup>。

本文将基于 Mindlin 弹性厚板理论, 采用 Hamilton 状态空间求解方法, 研究板条两侧为自由边界条件下的弹性振动与导波问题的解。研究振动模与导波模的存在条件, 给出条形板中导波模的频散方程, 并对结果进行分析讨论。

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## 1 弹性厚板弯曲波动的全状态向量

根据 Mindlin 板理论,在直角坐标系下位移分量  $u_x, u_y, u_z$  的表达式为<sup>[3, 10]</sup>

$$u_x = -z \varphi_x(x, y, t), \quad u_y = -z \varphi_y(x, y, t), \quad u_z = w(x, y, t), \quad (1)$$

式中,  $w$  表示板的法向位移, 而  $\varphi_x, \varphi_y$  分别表示法线在  $xz$  和  $yz$  平面内绕  $y$  和  $x$  轴上的转动。这样, 板内弯矩和剪力可描述为

$$M_x = \int_{-h/2}^{h/2} z \alpha_x dz = -D \left[ \frac{\partial \varphi_x}{\partial x} + \nu \frac{\partial \varphi_y}{\partial y} \right], \quad (2a)$$

$$M_y = \int_{-h/2}^{h/2} z \alpha_y dz = -D \left[ \frac{\partial \varphi_y}{\partial y} + \nu \frac{\partial \varphi_x}{\partial x} \right], \quad (2b)$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} z \alpha_{xy} dz = -\frac{(1-\nu)}{2} D \left[ \frac{\partial \varphi_y}{\partial x} + \frac{\partial \varphi_x}{\partial y} \right], \quad (2c)$$

$$Q_x = \int_{-h/2}^{h/2} \alpha_{xz} dz = C \left[ \frac{\partial w}{\partial x} - \varphi_x \right], \quad (2d)$$

$$Q_y = \int_{-h/2}^{h/2} \alpha_{yz} dz = C \left[ \frac{\partial w}{\partial y} - \varphi_y \right], \quad (2e)$$

式中,  $D$  为平板的抗弯刚度,  $D = Eh^3 / (12(1-\nu^2))$ ,  $C = \kappa Gh$ ,  $G = Eh / (2(1+\nu))$ ,  $\kappa$  是剪切折算系数,  $\kappa = \pi^2 / 12$ 。

为采用 Hamilton 分析求解体系, 将  $x$  坐标模拟为时间变量。这样, 在状态空间中, 可设广义位移变量为  $\mathbf{q} = (w, \varphi_x, \varphi_y)^T$ , 则广义速度为  $\boldsymbol{\varphi} = \partial \mathbf{q} / \partial t = (\dot{w}, \dot{\varphi}_x, \dot{\varphi}_y)^T$ 。

利用平板结构的应变能和动能的表达式, Lagrange 密度函数为

$$L(\mathbf{q}, \boldsymbol{\varphi}) = \frac{1}{2} D \left[ \dot{\varphi}_x^2 + \left( \frac{\partial \varphi_y}{\partial y} \right)^2 + 2\nu \dot{\varphi}_x \frac{\partial \varphi_y}{\partial y} \right] + \frac{Gh^3}{24} \left[ \frac{\partial \varphi_x}{\partial y} + \dot{\varphi}_y \right]^2 + \frac{1}{2} C \left[ (\dot{w} - \varphi_x)^2 + \left( \frac{\partial w}{\partial y} - \dot{\varphi}_y \right)^2 \right] + \frac{1}{2} \rho h \omega^2 w^2 + \frac{1}{2} \rho J \omega^2 \varphi_x^2 + \frac{1}{2} \rho J \omega^2 \varphi_y^2, \quad (3)$$

式中,  $J$  为平板的转动惯量,  $J = h^3 / 12$ 。

在相空间中, 广义位移变量与广义动量分别为  $\mathbf{q} = (w, \varphi_x, \varphi_y)^T$ ,  $\mathbf{p} = (p_w, p_{\varphi_x}, p_{\varphi_y})^T$ 。广义动量各分量的表达式为

$$p_w = \frac{\partial L(\mathbf{q}, \boldsymbol{\varphi})}{\partial \dot{w}} = Q_x, \quad (4a)$$

$$p_{\varphi_x} = \frac{\partial L(\mathbf{q}, \boldsymbol{\varphi})}{\partial \dot{\varphi}_x} = -M_x, \quad (4b)$$

$$p_{\varphi_y} = \frac{\partial L(\mathbf{q}, \boldsymbol{\varphi})}{\partial \dot{\varphi}_y} = -M_{xy}. \quad (4c)$$

用广义位移和广义动量表示广义速度, 由方程(4)可得

$$\dot{w} = \frac{1}{C} Q_x + \varphi_x, \quad (5a)$$

$$\dot{\varphi}_x = -\frac{1}{D} M_x - \nu \frac{\partial \varphi_y}{\partial y}, \quad (5b)$$

$$\dot{\varphi}_y = -\frac{2}{(1-\nu)D} M_{xy} - \frac{\partial \varphi_x}{\partial y}. \quad (5c)$$

平板弹性弯曲波动的 Hamilton 密度函数表达式为<sup>[7]</sup>

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^T \boldsymbol{\varphi} - L(\mathbf{q}, \boldsymbol{\varphi}) = \frac{1}{2} \frac{1}{C} Q_x^2 + Q_x \varphi_x + \frac{1}{2} \frac{1}{D} M_x^2 + M_x \frac{\partial \varphi_y}{\partial y} + \frac{1}{(1-\nu)D} M_{xy}^2 + M_{xy} \frac{\partial \varphi_x}{\partial y} -$$

$$\begin{aligned} & \frac{1}{2}D(1-\nu^2)\left(\frac{\partial\varphi_y}{\partial y}\right)^2 - \frac{1}{2}C\left(\frac{\partial w}{\partial y} - \varphi_y\right)^2 - \\ & \frac{1}{2}\rho_h\omega^2 w^2 - \frac{1}{2}\rho_J\omega^2\varphi_x^2 - \frac{1}{2}\rho_J\omega^2\varphi_y^2. \end{aligned} \quad (6)$$

设状态变量  $\mathbf{v} = (\mathbf{q}, \mathbf{p})^T = (w, \varphi_x, \varphi_y, Q_x, -M_x, -M_{xy})^T$ .

$$p_w = -\frac{\partial H}{\partial w} = -C\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial\varphi_y}{\partial y}\right) + \rho_h\omega^2 w, \quad (7a)$$

$$p_{\varphi_x} = -\frac{\partial H}{\partial\varphi_x} = -Q_x + \frac{\partial M_{xy}}{\partial y} + \rho_J\omega^2\varphi_x, \quad (7b)$$

$$p_{\varphi_y} = -\frac{\partial H}{\partial\varphi_y} = -C\frac{\partial w}{\partial y} - D(1-\nu^2)\left(\frac{\partial^2\varphi_y}{\partial y^2}\right) + C\varphi_y + \rho_J\omega^2\varphi_y + \nu\frac{\partial M_x}{\partial y}. \quad (7c)$$

在状态空间中, 根据方程(5)和(7)可得平板单频波动的方程

$$\mathbf{p} = \mathbf{H}\mathbf{v} = \mu\mathbf{v}, \quad (8)$$

式中

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{C} & 0 & 0 \\ 0 & 0 & -\nu\frac{\partial}{\partial y} & 0 & \frac{1}{D} & 0 \\ 0 & -\frac{\partial}{\partial y} & 0 & 0 & 0 & \frac{2}{(1-\nu)D} \\ -C\frac{\partial^2}{\partial y^2} - \rho_h\omega^2 & 0 & C\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & -\rho_J\omega^2 & 0 & -1 & 0 & -\frac{\partial}{\partial y} \\ -C\frac{\partial}{\partial y} & 0 & -(1-\nu^2)D\frac{\partial^2}{\partial y^2} + C - \rho_J\omega^2 & 0 & -\nu\frac{\partial}{\partial y} & 0 \end{bmatrix}.$$

横向本征值  $\lambda$  与纵向本征值  $\mu$  的关系为

$$(\mu^2 + \lambda^2 + k_1^2)(\mu^2 + \lambda^2 + k_2^2)(\mu^2 + \lambda^2 + k_3^2) = 0, \quad (9a)$$

即有  $\lambda_n^2 = -\mu^2 - k_n^2 \quad (n = 1, 2, 3)$ . (9b)

其中

$$k_{1,2}^4 - \frac{D\rho_h\omega^2 + C\rho_J\omega^2}{CD}k_{1,2}^2 + \frac{\rho_h\omega^2(\rho_J\omega^2 - C)}{CD} = 0, \quad (10a)$$

$$k_3^2 = \frac{2(\rho_J\omega^2 - C)}{D(1-\nu)}. \quad (10b)$$

### 1.1 零本征解的确定

零本征解在弹性力学中具有重要性. 对于矩形域弹性动力学问题, 当两侧边是自由边界条件时, 就会有零本征值对应的本征解. 此时方程为

$$\mathbf{H}\Phi^{(0)} = \mathbf{0}, \quad \mathbf{p} = \mathbf{H}\mathbf{v}, \quad \mathbf{v} = [w, \varphi_x, \varphi_y, Q_x, -M_x, -M_{xy}]^T. \quad (11)$$

当两侧边是自由边值条件时, 其边界条件的表达式为

$$D(1-\nu^2)\frac{\partial\varphi_y}{\partial y} - M_x = D(1-\nu^2)\frac{\partial q_3}{\partial y} + \psi_2 = 0, \quad (12a)$$

$$\frac{\partial w}{\partial y} - \varphi_y = \frac{\partial q_1}{\partial y} - q_3 = 0, \quad (12b)$$

$$-M_{xy} = p_3 = 0. \quad (12c)$$

通过分析可知, 线性无关的基础本征解可取为

$$\Phi_0^{(1)} = (w, 0, \varphi_y, 0, -M_x, 0)^T, \quad (13a)$$

$$\phi_0^{(2)} = (0, \varphi_x, 0, Q_x, 0, -M_{xy})^T, \quad (13b)$$

其中

$$w = w_0^{(1)} = \left[ \frac{D(1-\nu)k_3^2 - 2Dk_1^2}{2Ck_1} \right] \cos(k_1y) + \left[ \frac{D(1-\nu)k_3^2 - 2Dk_2^2}{2Ck_2} \right] \delta \cos(k_2y),$$

$$\varphi_x = \varphi_{x0}^{(2)} = \cos(k_3y), \quad \varphi_y = \varphi_{y0}^{(1)} = \sin(k_1y) + \delta \sin(k_2y),$$

$$\delta = \delta(k_1, k_2) = -\frac{k_1 \cos(k_1a)}{k_2 \cos(k_2a)}, \quad Q_x = Q_{x0}^{(2)} = -C \cos(k_3y),$$

$$M_x = M_{x0}^{(1)} = -\mathcal{D}k_1 \cos(k_1y) - \mathcal{D}k_2 \delta \cos(k_2y),$$

$$M_{xy} = M_{xy0}^{(2)} = \frac{(1-\nu)D}{2} k_3 \sin(k_3y).$$

为满足两侧边界条件必须  $k_3a = n\pi$ , 即平板存在此种振动的频率是

$$\omega^2 = \frac{1}{\rho J} \left[ (1-\nu)D \frac{n^2 \pi^2}{2a^2} + C \right] \quad (n = 0, 1, 2, \dots) \quad (14)$$

根据平板两侧边界条件, 板内存在与  $k_{1,2}$  有关的振动时, 其频率满足

$$\omega^2 = \frac{D}{\rho J} \frac{k_1^2 k_2 \tan(k_1a) - k_1 k_2^2 \tan(k_2a)}{k_2 \tan(k_1a) - k_1 \tan(k_2a)}. \quad (15)$$

这样, Jordan 型零阶次本征向量为

$$\mathbf{v}_0^{(1)} = \phi_0^{(1)}, \quad \mathbf{v}_0^{(2)} = \phi_0^{(2)}. \quad (16)$$

$\mathbf{v}_0^{(1)}$ 、 $\mathbf{v}_0^{(2)}$  的物理意义是表示一种  $x$  方向没有传播的振动模态, 即沿  $x$  方向是均匀的,  $y$  方向是振荡的驻波。 $\phi_0^{(1)}$  的横向位移为  $w$ , 相应的转角为  $\varphi_y$ , 弯矩为  $M_x$ ; 而  $\phi_0^{(2)}$  的转角为  $\varphi_x$ , 剪力为  $Q_x$ , 弯矩为  $M_{xy}$ 。

求一阶次零本征向量  $\phi_1^{(1)}$

$$\mathbf{H}\phi_1^{(1)} = \phi_0^{(1)}, \quad (17)$$

$$\phi_1^{(1)} = (0, \varphi_x, 0, Q_x, 0, -M_{xy})^T, \quad (18)$$

其中

$$\varphi_x = \varphi_{x1}^{(1)} = -\frac{1}{k_1} \cos(k_1y) - \delta \frac{1}{k_2} \cos(k_2y),$$

$$M_x = M_{xy1}^{(1)} = -D(1-\nu) \sin(k_1y) - \delta D(1-\nu) \sin(k_2y),$$

$$Q_x = Q_{x1}^{(1)} = \frac{D(1-\nu)k_3^2 - 2Dk_1^2 - 2C}{2k_1} \cos(k_1y) + \delta \frac{D(1-\nu)k_3^2 - 2Dk_2^2 - 2C}{2k_2} \cos(k_2y).$$

原方程(11)的解为

$$\mathbf{v}_1^{(1)} = \phi_1^{(1)} + x\phi_0^{(1)}, \quad (19a)$$

$$\text{即有} \quad \mathbf{v}_1^{(1)} = [xw_0^{(1)}, \phi_x^{(1)}, x\phi_{y0}^{(1)}, Q_{x1}^{(1)}, -xM_{xy0}^{(1)}, -M_{xy1}^{(1)}]^T. \quad (19b)$$

$\mathbf{v}_1^{(1)}$  是一种非传播振动模态, 它的物理意义是表示  $xoz$  平面内的刚体旋转。经分析可知, 其它的次本征解链直到  $\phi_2^{(1)}$  断绝。

求一阶次零本征向量  $\phi_1^{(2)}$

$$\mathbf{H}\phi_1^{(2)} = \phi_0^{(2)}, \quad (20)$$

$$\phi_1^{(2)} = (w, 0, \varphi_y, 0, -M_x, 0)^T, \quad (21)$$

其中

$$w = w_1^{(2)} = \frac{1}{(k_s^2 - k_3^2)} \left\{ 1 + \frac{2C - (1 + \nu)D(k_3^2 - k_s^2)}{2D\nu(k_3^2 - k_s^2) - 2C} \right\} \cos(k_3 y),$$

$$\varphi_y = \varphi_{y1}^{(2)} = \frac{2C - (1 + \nu)D(k_3^2 - k_s^2)}{[2D\nu(k_3^2 - k_s^2) - 2C]k_3} \sin(k_3 y),$$

$$M_x = M_{x1}^{(2)} = -D \left[ 1 + \nu \frac{2C - (1 + \nu)D(k_3^2 - k_s^2)}{2D\nu(k_3^2 - k_s^2) - 2C} \right] \cos(k_3 y).$$

原方程(11)的解为

$$v_1^{(2)} = \phi_1^{(2)} + x\phi_0^{(2)}, \quad (22a)$$

$$\text{即有 } v_1^{(2)} = [w_1^{(2)}, x\varphi_{x0}^{(2)}, \varphi_{y1}^{(2)}, xQ_{x0}^{(2)}, -M_{xy1}^{(2)}, -xM_{xy0}^{(2)}]^T. \quad (22b)$$

$v_1^{(2)}$ 也是一种非传播振动模态,它的物理意义是表示  $xoz$  平面内的刚体旋转。经分析可知,次本征解链到  $\phi_2^{(2)}$  断绝。对于零本征值,其所对应的动力学模式相当于振动力学中的固有频率的概念,例如工程中常把结构的第一阶模态振动频率看成是结构的固有频率,只不过此时所有力学量沿  $y$  的积分等于零。

## 1.2 非零本征解的确定

研究对称情况,板弯曲振动对称问题的本征解  $\phi_n = (q_n, p_n)^T$  可描述为

$$w = A_{11}\cosh(\lambda_1 y) + A_{12}\cosh(\lambda_2 y) + A_{13}\cosh(\lambda_3 y), \quad (23a)$$

$$\varphi_x = A_{21}\cosh(\lambda_1 y) + A_{22}\cosh(\lambda_2 y) + A_{23}\cosh(\lambda_3 y), \quad (23b)$$

$$\varphi_y = A_{31}\sinh(\lambda_1 y) + A_{32}\sinh(\lambda_2 y) + A_{33}\sinh(\lambda_3 y), \quad (23c)$$

$$Q_x = A_{41}\cosh(\lambda_1 y) + A_{42}\cosh(\lambda_2 y) + A_{43}\cosh(\lambda_3 y), \quad (23d)$$

$$-M_x = -A_{51}\cosh(\lambda_1 y) - A_{52}\cosh(\lambda_2 y) - A_{53}\cosh(\lambda_3 y), \quad (23e)$$

$$-M_{xy} = -A_{61}\sinh(\lambda_1 y) - A_{62}\sinh(\lambda_2 y) - A_{63}\sinh(\lambda_3 y), \quad (23f)$$

式中,  $\lambda_n^2 = -\mu^2 - k_n^2$ ,  $A_{mn}$  ( $m = 1, 2, \dots, 6; n = 1, 2, 3$ ) 是模式系数,它们并不是独立的,经分析可知,只有3个是独立的。

将(23)式代入到(8)式中,可得模式系数的关系式

$$\begin{cases} \frac{A_{2n}}{A_{1n}} = \frac{\mu[2C(Dk_n^2 - \rho\omega^2) - D(1 + \nu)\rho h\omega^2]}{D(1 - \nu)(Dk_n^2 - \rho\omega^2)(k_n^2 - k_3^2)}, \\ \frac{A_{3n}}{A_{1n}} = \frac{\lambda_n[2C(Dk_n^2 - \rho\omega^2) - D(1 + \nu)\rho h\omega^2]}{D(1 - \nu)(Dk_n^2 - \rho\omega^2)(k_n^2 - k_3^2)}, \\ \frac{A_{4n}}{A_{1n}} = \frac{\mu\rho\omega^2[Dh(1 - \nu)(Dk_n^2 - \rho\omega^2) + 2C(D(h - Jk_n^2)) + \rho\omega^2]}{D(1 - \nu)(Dk_n^2 - \rho\omega^2)(k_n^2 - k_3^2)}, \\ \frac{A_{5n}}{A_{1n}} = \left\{ D\mu^2[Dh(1 - \nu^2)\rho\omega^2 - 2CDk_n^2(1 - \nu) + 2CJ(1 - \nu)\rho\omega^2] - \right. \\ \left. D[\nu Dk_n^2(1 - \nu)\rho h\omega^2 - 2\nu C\rho h\omega^2 + 2\nu\rho^2 Jh\omega^4] \right\} / [D(1 - \nu)(Dk_n^2 - \rho\omega^2)(k_n^2 - k_3^2)], \\ \frac{A_{6n}}{A_{1n}} = \frac{\mu\lambda_n D(1 - \nu)[D(1 + \nu)\rho h\omega^2 - 2C(Dk_n^2 - \rho\omega^2)]}{D(1 - \nu)(Dk_n^2 - \rho\omega^2)(k_n^2 - k_3^2)} \quad (n = 1, 2), \\ A_{13} = 0, \frac{A_{33}}{A_{23}} = -\frac{\mu}{\lambda_3}, \frac{A_{43}}{A_{22}} = -C, \frac{A_{53}}{A_{23}} = -\mu D(1 - \nu), \\ \frac{A_{63}}{A_{23}} = \frac{D(1 - \nu)(2\mu^2 + k_3^2)}{2\lambda_3}. \end{cases} \quad (24)$$

(25)

公式化简中用到如下关系

$$\frac{2C(Dk^2 - \rho h \omega^2) - D(1 + \nu)\rho h \omega^2}{Dk^2 - \rho h \omega^2} = \frac{2C^2 - D(1 + \nu)(Ck^2 - \rho h \omega^2)}{C} \tag{26}$$

满足板条两侧面的自由边界条件, 可得本征值应满足的方程

$$\lambda k_1^2 [2k_1^2 - k_s^2(1 - \nu)]^{-1} \left\{ Ck_2^2 k_s^2 + D(k_2^2 - k_s^2)[k_1^2 k_2^2 + (1 - \nu)\mu^2] \right\} \times \\ \left\{ D\mu^2(1 - \nu)[k_s^2(1 - \nu) - 2k_1^2] + 2k_s^2[C - D(k_1^2 - k_s^2)]\nu^2 - 2Dk_1^2 k_2^2(1 - \nu^2) + \right. \\ \left. Dk_2^2 k_s^2(1 - \nu - 4\nu^2 + 2\nu^3) \right\} \coth(\lambda_2 a) - \lambda_2 k_2^2 [2k_2^2 - k_s^2(1 - \nu)]^{-1} \left\{ Ck_1^2 k_s^2 + \right. \\ \left. D(k_1^2 - k_s^2)[k_1^2 k_2^2 + (1 - \nu)\mu^2] \right\} \left\{ D\mu^2(1 - \nu)[k_s^2(1 - \nu) - 2k_2^2] + 2k_s^2[C - \right. \\ \left. D(k_2^2 - k_s^2)]\nu^2 - 2Dk_1^2 k_2^2(1 - \nu^2) + Dk_1^2 k_s^2(1 - \nu - 4\nu^2 + 2\nu^3) \right\} \coth(\lambda_1 a) + \\ D^2(1 - \nu)^2 \lambda_1 \lambda_2 \lambda_3 (k_1^2 - k_2^2) k_s^4 \mu^2 \coth(\lambda_3 a) = 0, \tag{27a}$$

式中  $k_s^2 = \rho h \omega^2 / C$ .

考虑沿  $x$  轴正方向存在波导情况. 根据式(27a), 令  $\mu = ik$ , 可得Hamilton体系下频散方程, 其中  $k$  是平板中弹性波传播波数.

Mindlin 理论下板条中弯曲波频散方程为

$$\left( k^2 - \frac{k_1^2}{1 - \nu} \right)^2 (k_s^2 - k_1^2) k_2^2 \lambda_1^{-1} \coth(\lambda_1 a) - \left( k^2 - \frac{k_2^2}{1 - \nu} \right)^2 (k_s^2 - k_2^2) k_1^2 \lambda_2^{-1} \coth(\lambda_2 a) + \\ k^2 k_s^2 (k_1^2 - k_2^2) \lambda_3 \coth(\lambda_3 a) = 0 \tag{27b}$$

图 1、3、5 表示 Hamilton 体系下板条中弯曲波频散关系曲线, 图 2、4、6 表示 Mindlin 理论下板条中弯曲波频散关系曲线. 比较两种理论下的弹性波频散关系, 可以看到: 在短波时, 频散曲线相差较大, 即前几阶低频模式有一定差别, 按 Hamilton 体系得到的截止频率高一些; 长波时频散曲线几乎相同, 即高频模式无差别<sup>[11]</sup>.

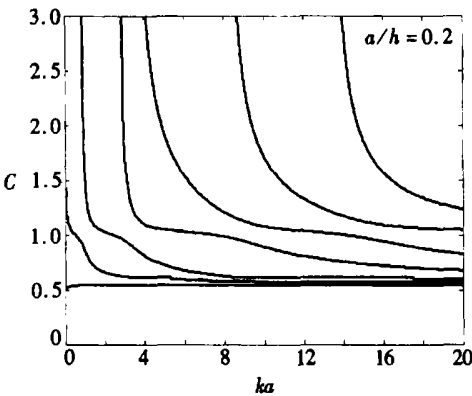


图 1 Hamilton 体系下板条中弯曲波频散关系曲线

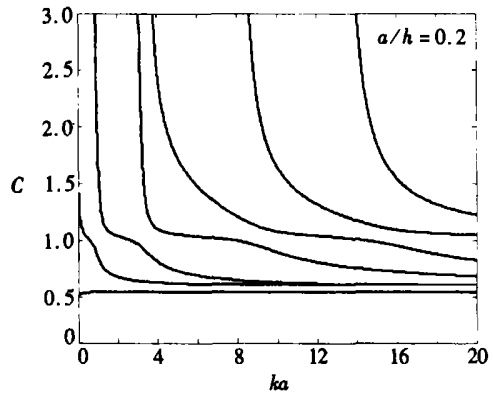


图 2 Mindlin 理论下板条中弯曲波频散关系曲线

## 2 分析与讨论

通过分析可以看到, 在平板动力学问题研究中, 板中并不存在静力学中的刚体平移和旋转等概念, 代之以在板中存在着非传播模, 例如  $x$  方向分布均匀,  $y$  方向振荡的驻波. 采用 Hamilton 列式, 除了可确定出沿  $x$  方向的扩展模以外, 还能得到在板内的局部化振动模式. 可是, 采用 Mindlin 板理论, 只能得到沿  $x$  方向的扩展模, 部分局部化振动模可能丢失.

对于零本征值对应的动力学模式, 它相当于振动力学中的固有频率的概念, 例如工程中常

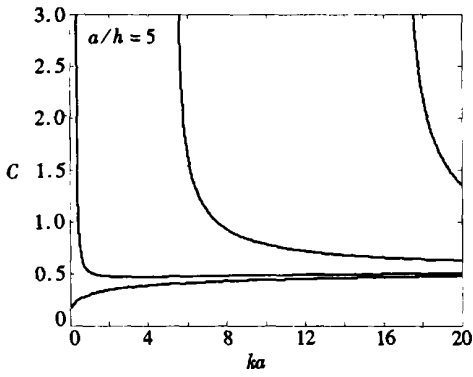


图3 Hamilton体系下板条中弯曲波频散关系曲线

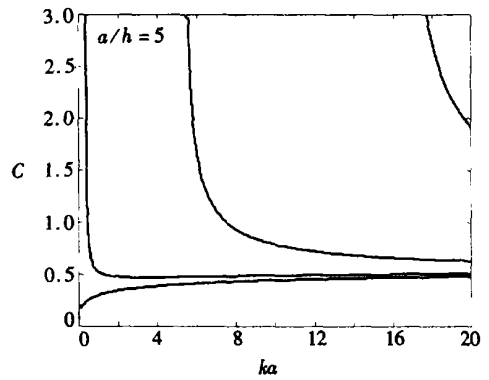


图4 Mindlin理论下板条中弯曲波频散关系曲线

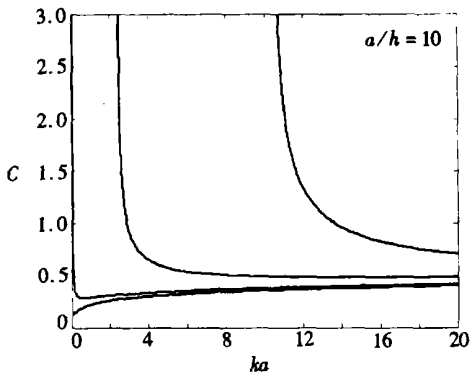


图5 Hamilton体系下板条中弯曲波频散关系曲线

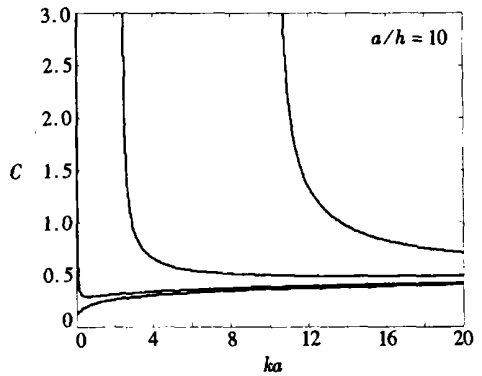


图6 Mindlin理论下板条中弯曲波频散关系曲线

把结构的第一阶模态振动频率看成是结构的固有频率,只不过是在本文问题中,此时所有力学量沿  $y$  的积分等于零。

对于非零本征值,当虚部大于零时,若实部小于零,则表示  $x$  正向衰减传播模,否则表示不稳定共振状态;当虚部小于零时,若实部大于零,则表示沿  $x$  负向衰减传播模,否则表示能量积聚状态。

当实部等于零时,则表示  $x$  正向或负向传播模;当实部小于零时,表示局部化振动;实部大于零时,则表示共振失稳。此时结构的振动频率满足色散关系,即结构的振动频率、入射波波数、结构的几何参数与力学参数应满足一定的关系。

本文分析方法与结果可望能在航天器结构振动控制的分析与设计中起到参考作用。

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## Flexural Wave Propagation in Narrow Mindlin' s Plate

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**Abstract:** Applying Mindlin' s theory of thick plates and Hamilton formulism to propagation of elastic waves under free boundary condition, a solution of the problem was given. Dispersion equations of propagation mode of strip plates were deduced from eigenfunction expansion method. It was compared with the dispersion relation that was gained through solution of thick plate theory proposed by Mindlin. Based on the two kinds of theories, the dispersion curves show great difference in the region of short waves, and the cutoff frequencies are higher in Hamiltonian systems. However, the dispersion curves are almost the same in the region of long waves.

**Key words:** Mindlin' s plate; wave propagation; Hamilton formulism; dispersion relation; control of traveling wave; extended wave and localized vibration