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压电梁的多项式解(II) ——典型问题解析解*

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(我刊编委 丁皓江来稿)

摘要: 从正交各向异性压电介质平面问题, 对于材料 3 个特征根互不相等情况下, 以 3 个拟调和位移函数表达位移、电势、应力和电位移的通解出发, 利用调和多项式的显式表达式, 结合试凑法, 给出了平面压电梁的若干典型问题的解析解, 包括悬臂压电梁自由端作用横向集中力和点电荷, 悬臂压电梁表面作用线性电势和均布载荷, 以及两端简支压电梁作用均布载荷等的解析解•

关 键 词: 压电梁; 平面问题; 调和多项式; 解析解
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引 言

本文是文献[1]的继续, 在前文我们利用压电平面问题通解和调和多项式的显式表达式, 结合试凑法, 给出了一系列的精确解• 有了这些最简单、最基本的精确解之后, 可以继续采用文献[1]所使用的方法, 并进一步使用叠加原理, 就能比较方便地给出两端简支压电梁和悬臂压电梁在一些典型载荷和电势作用下的解析解• 从这些典型的解析解中可以清楚地看到位移、电势、应力和电位移沿梁高度和长度的分布规律, 它不仅为发展实用的工程压电梁理论提供某种借鉴, 同时和文献[1]的精确解一起, 为各种数值计算方法提供校核和验证的算例•

1 悬臂梁自由端同时作用横向集中力 P 和点电荷 Q

选取文献[1]附录中式(B2)中的 $\varphi_2^1(x,z)$ 和 $\varphi_4^1(x,z)$, 可构造如下位移函数:

$$\psi_j = B_{2j}xz_j + B_{4j}(x^3z_j - xz_j^3) \qquad (j = 1, \cdots, 3), \tag{1}$$

式中, B_{2j} 和 B_{4j} 均为待定未知常数•

将式(1)代入文献[1]式(3), 并叠加上文献[1]中式(11)所示的 z 向刚体平动和均匀电势解有:

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$$
\begin{cases}
u = \sum_{j=1}^{3} B_{2j} z_j + \sum_{j=1}^{3} B_{4j} (3x^2 z_j - z_j^3), \\
w = w_0 + \sum_{j=1}^{3} s_j k_{1j} B_{2j} x + \sum_{j=1}^{3} s_j k_{1j} B_{4j} (x^3 - 3xz_j^3), \\
\Phi = \Phi_0 + \sum_{j=1}^{3} s_j k_{2j} B_{2j} x + \sum_{j=1}^{3} s_j k_{2j} B_{4j} (x^3 - 3xz_j^2), \\
\sigma_x = -6 \sum_{j=1}^{3} \omega_{3j} B_{4j} xz_j, \quad \sigma_z = -6 \sum_{j=1}^{3} \omega_{1j} B_{4j} xz_j, \\
\tau_{xz} = \sum_{j=1}^{3} s_j \omega_{1j} B_{2j} + \sum_{j=1}^{3} s_j \omega_{1j} B_{4j} (3x^2 - 3z_j^2), \\
D_x = \sum_{j=1}^{3} s_j \omega_{2j} B_{2j} + \sum_{j=1}^{3} s_j \omega_{2j} B_{4j} (3x^2 - 3z_j^2), \quad D_z = -6 \sum_{j=1}^{3} \omega_{2j} B_{4j} xz_j .
\end{cases} \tag{2}
$$

压电悬臂梁自由端$(x = 0)$同时作用横向集中力P和点电荷Q情况下的边界条件为:

$$z = \pm h/2:\ \sigma_z = 0,\ \tau_{xz} = 0,\ D_z = 0, \tag{3a}$$

$$x = 0:\ \sigma_x = 0,\ \int_{-h/2}^{+h/2} \tau_{xz}\,\mathrm{d}z = Q_1,\ \int_{-h/2}^{+h/2} D_x\,\mathrm{d}z = Q_2, \tag{3b}$$

$$x = L,\ z = 0:\ u = 0,\ w = 0,\ \frac{\partial w}{\partial x} = 0, \tag{3c}$$

式中, $Q_1 = -P$, $Q_2 = Q$.

把式(2)代入式(3), 可得:

$$\sum_{j=1}^{3} s_j \omega_{1j} B_{4j} = 0,\quad \sum_{j=1}^{3} s_j \omega_{1j} B_{2j} - \frac{3h^2}{4} \sum_{j=1}^{3} s_j^3 \omega_{1j} B_{4j} = 0, \tag{4}$$

$$\sum_{j=1}^{3} s_j \omega_{2j} B_{4j} = 0,\quad \sum_{j=1}^{3} s_j k_{1j} B_{2j} + 3L^2 \sum_{j=1}^{3} s_j k_{1j} B_{4j} = 0, \tag{5}$$

$$h \sum_{j=1}^{3} s_j \omega_{1j} B_{2j} - \frac{h^3}{4} \sum_{j=1}^{3} s_j^3 \omega_{1j} B_{4j} = Q_1,\quad h \sum_{j=1}^{3} s_j \omega_{2j} B_{2j} - \frac{h^3}{4} \sum_{j=1}^{3} s_j^3 \omega_{2j} B_{4j} = Q_2, \tag{6}$$

$$w_0 + L \sum_{j=1}^{3} s_j k_{1j} B_{2j} + L^3 \sum_{j=1}^{3} s_j k_{1j} B_{4j} = 0. \tag{7}$$

进而, B_{2j}和$B_{4j}(j = 1, \ldots, 3)$可由式(4)~(6)的6个方程解出. 把B_{2j}、B_{4j}代入式(7), 我们便可导出未知常数w_0. 同时, 根据任一指定点(x', z')处电势条件, 即$\Phi(x', z') = 0$, 就能导出另一未知常数Φ_0. 最终, 我们就得到了相应解析解, 特别地, 当仅承受集中荷载$P(Q = 0)$时, 应力解可简化为:

$$\sigma_x = \frac{12P}{h^3} xz,\quad \tau_{xz} = \frac{6P}{h^3}\left(\frac{h^2}{4} - z^2\right),\quad \sigma_z = 0, \tag{8}$$

可以看出, 此时应力与材料常数无关.

假如我们在式(3c)中以$\partial u/\partial z = 0$替代$\partial w/\partial x = 0$, 那么式(5)第2式将换成另外形式. 同上所述, 我们将得到该问题的另一解. 这两种解均与 Timoshenko 和 Goodier[2] 得到的各向同性弹性材料悬臂梁的解相似.

从式(2)看到电势在横截面上沿高度为二次曲线分布, 而从文献[1]的式(30)也看到同样结论, 这都可成为文献[3]研究压电板工程理论时的一个重要假设的依据.

2 悬臂梁表面作用线性电势

选取文献[1]附录中式(B2)中的 $\varphi_2^1(x,z)$ 和 $\varphi_3^0(x,z)$，可构造如下位移函数：

$$\psi_j = B_{2j}xz_j + A_{3j}(x^3 - 3xz_j^2) \qquad (j = 1, \cdots, 3), \tag{9}$$

式中，B_{2j} 和 A_{3j} 均为待定未知常数。

将式(9)代入文献[1]中式(3)，得：

$$\begin{cases} u = \sum_{j=1}^{3}[z_jB_{2j} + (3x^2 - 3z_j^2)A_{3j}], \\ w_m = \sum_{j=1}^{3}s_jk_{mj}(xB_{2j} - 6xz_jA_{3j}) \qquad (m = 1, 2), \end{cases} \tag{10}$$

$$\sigma_x = -6x\sum_{j=1}^{3}\omega_{3j}A_{3j}, \quad \sigma_z = -6x\sum_{j=1}^{3}\omega_{1j}A_{3j}, \quad D_z = -6x\sum_{j=1}^{3}\omega_{2j}A_{3j}, \tag{11}$$

$$\tau_{xz} = \sum_{j=1}^{3}s_j\omega_{1j}(B_{2j} - 6z_jA_{3j}), \quad D_x = \sum_{j=1}^{3}s_j\omega_{2j}(B_{2j} - 6z_jA_{3j}). \tag{12}$$

边界条件为：

$$z = +\frac{h}{2}: \ \sigma_z = 0, \ \tau_{xz} = 0, \ w_2 = \Phi\left(x, +\frac{h}{2}\right) = \alpha_1x, \tag{13a}$$

$$z = -\frac{h}{2}: \ \sigma_z = 0, \ \tau_{xz} = 0, \ w_2 = \Phi\left(x, -\frac{h}{2}\right) = \beta_1x, \tag{13b}$$

$$x = 0: \int_{-h/2}^{+h/2}\sigma_x\mathrm{d}z = 0, \ \int_{-h/2}^{+h/2}\sigma_xz\,\mathrm{d}z = 0, \ \int_{-h/2}^{+h/2}\tau_m\mathrm{d}z = 0 \qquad (m = 1, 2), \tag{14}$$

$$x = L, \ z = 0: \ u = 0, \ w = 0, \ \frac{\partial w}{\partial x} = 0. \tag{15}$$

将式(10)～式(12)代入边界条件(13)式和(15)式第2式，得

$$\sum_{j=1}^{3}\omega_{1j}A_{3j} = 0, \quad \sum_{j=1}^{3}s_j^2\omega_{1j}A_{3j} = 0, \quad 3h\sum_{j=1}^{3}s_j^2k_{2j}A_{3j} = \frac{\beta_1 - \alpha_1}{2}, \tag{16}$$

$$\sum_{j=1}^{3}s_j\omega_{1j}B_{2j} = 0, \quad \sum_{j=1}^{3}s_jk_{1j}B_{2j} = 0, \quad \sum_{j=1}^{3}s_jk_{2j}B_{2j} = \frac{\alpha_1 + \beta_1}{2}. \tag{17}$$

由(16)中3式可解出 A_{3j}，再由(17)中3式解出 B_{2j}。进而，为满足边界条件(14)第3式，需将解(10)～(12)叠加第1节中悬臂梁自由端面受载相应解，并令其中 $Q_1 = 0$ 及

$$Q_2 = -\int_{-h/2}^{+h/2}\tau_2\mathrm{d}z = -h\sum_{j=1}^{3}s_j\omega_{2j}B_{2j}. \tag{18}$$

再叠加如下刚体位移，便能满足边界条件(15)的第1式：

$$u = u_0 = -3L^2\sum_{j=1}^{3}A_{3j}. \tag{19}$$

3 简支梁承受均布荷载

选取文献[1]附录中式(B2)中的 $\varphi_3^1(x,z)$ 和 $\varphi_5^1(x,z)$，可构造如下位移函数：

$$\psi_j = A_{3j}(x^2z_j - z_j^3/3) + A_{5j}(x^4z_j - 2x^2z_j^3 + z_j^5/5) \qquad (j = 1, \cdots, 3), \tag{20}$$

将式(20)代入文献[1]中式(3)，并叠加上文献[1]中式(11)所示的 z 向刚体平动位移解，得：

$$
\begin{cases}
u = \sum_{j=1}^{3} 2A_{3j}xz_j + \sum_{j=1}^{3} 4A_{5j}(x^3 z_j - xz_j^3), \\
w = w_0 + \sum_{j=1}^{3} s_j k_{1j} A_{3j}(x^2 - z_j^2) + \sum_{j=1}^{3} s_j k_{1j} A_{5j}(x^4 - 6x^2 z_j^2 + z_j^4), \\
\Phi = \sum_{j=1}^{3} s_j k_{2j} A_{3j}(x^2 - z_j^2) + \sum_{j=1}^{3} s_j k_{2j} A_{5j}(x^4 - 6x^2 z_j^2 + z_j^4), \\
\sigma_x = -2\sum_{j=1}^{3} \omega_{3j} A_{3j} z_j + \sum_{j=1}^{3} \omega_{3j} A_{5j}(-12x^2 z_j + 4z_j^3), \\
\sigma_z = -2\sum_{j=1}^{3} \omega_{1j} A_{3j} z_j + \sum_{j=1}^{3} \omega_{1j} A_{5j}(-12x^2 z_j + 4z_j^3), \\
D_z = -2\sum_{j=1}^{3} \omega_{2j} A_{3j} z_j + \sum_{j=1}^{3} \omega_{2j} A_{5j}(-12x^2 z_j + 4z_j^3), \\
\tau_{xz} = 2\sum_{j=1}^{3} s_j \omega_{1j} A_{3j} x + \sum_{j=1}^{3} s_j \omega_{1j} A_{5j}(4x^3 - 12xz_j^2), \\
D_x = 2\sum_{j=1}^{3} s_j \omega_{2j} A_{3j} x + \sum_{j=1}^{3} s_j \omega_{2j} A_{5j}(4x^3 - 12xz_j^2),
\end{cases}
\tag{21}
$$

式中, w_0、A_{3j} 和 A_{5j} $(j = 1, \cdots, 3)$ 均为待定未知常数•

上表面受压、下表面受拉的两端简支压电梁, 其边界条件为:

$$
z = \pm \frac{h}{2}:\ \sigma_z = \pm \frac{q}{2},\ \tau_{xz} = 0,\ D_z = 0, \tag{22}
$$

$$
x = \pm \frac{L}{2}:\ \int_{-h/2}^{+h/2} \sigma_x \,\mathrm{d}z = 0,\ \int_{-h/2}^{+h/2} \sigma_x z \,\mathrm{d}z = 0,\ \int_{-h/2}^{+h/2} D_x \,\mathrm{d}z = 0,\ w\,|_{z=0} = 0• \tag{23}
$$

将式(21)代入式(22)和(23), 经简化可得:

$$
\begin{cases}
\sum_{j=1}^{3} s_j \omega_{1j} A_{3j} = -\frac{3q}{4h},\ \sum_{j=1}^{3} s_j^3 \omega_{1j} A_{5j} = -\frac{q}{2h^3},\ \sum_{j=1}^{3} s_j \omega_{1j} A_{5j} = 0, \\
\sum_{j=1}^{3} s_j \omega_{2j} A_{5j} = 0,\ -\sum_{j=1}^{3} s_j \omega_{2j} A_{3j} + \frac{h^2}{2}\sum_{j=1}^{3} s_j^3 \omega_{2j} A_{5j} = 0, \\
-\frac{2}{3}\sum_{j=1}^{3} s_j \omega_{3j} A_{3j} + \frac{h^2}{5}\sum_{j=1}^{3} s_j^3 \omega_{3j} A_{5j} - L^2 \sum_{j=1}^{3} s_j \omega_{3j} A_{5j} = 0, \\
w_0 + \frac{L^2}{4}\sum_{j=1}^{4} s_j k_{1j} A_{3j} + \frac{L^4}{16}\sum_{j=1}^{4} s_j k_{1j} A_{5j} = 0•
\end{cases}
\tag{24}
$$

将由式(24)中 7 个方程解出的 w_0、A_{3j} 和 A_{5j} 回代式(21), 我们即得到了上表面受压、下表面受拉$(z = \pm h/2:\ \sigma_z = \pm q/2)$的两端简支压电梁的解析解• 在上述解析解(21)的基础上, 再叠加上文献[1]中压电梁在 z 轴向均匀拉伸精确解并取 $\sigma_2 = q/2$, 即得到压电简支梁下表面$(z = \pm h/2)$承受均布荷载 q、上表面自由情况下的解析解•

4　悬臂梁承受均布荷载

矩形压电悬臂梁承受均布荷载情况下边界条件:

$$
z = \pm \frac{h}{2}:\ \sigma_z = \pm \frac{q}{2},\ \tau_{xz} = 0,\ D_z = 0, \tag{25}
$$

$$
x = 0:\ \int_{-h/2}^{+h/2} \sigma_x z \,\mathrm{d}z = 0,\ \int_{-h/2}^{+h/2} \sigma_x \,\mathrm{d}z = 0,\ \tau_{xz} = 0,\ D_x = 0, \tag{26}
$$

$$x = L,\ z = 0:\ u = 0,\ w = 0,\ \frac{\partial w}{\partial x} = 0,\quad \Phi = \mathrm{const}. \tag{27}$$

选取文献[1]附录中式(B2)中的 $\varphi_3^1(x, z)$ 和 $\varphi_5^1(x, z)$，可构造如下位移函数:

$$\psi_j = B_{3j}(x^2 z_j - z_j^3/3) + B_{5j}(x^4 z_j - 2x^2 z_j^3 + z_j^5/5) \qquad (j = 1, \cdots, 3), \tag{28}$$

式中，B_{3j} 和 B_{5j} 均为待定未知常数.

将式(28)代入文献[1]中式(3)，并叠加上文献[1]中式(11)刚体位移和均匀电势解，得:

$$\begin{cases} u = u_0 + \omega_0 z + 2\sum_{j=1}^{3} B_{3j} x z_j + 4\sum_{j=1}^{3} B_{5j}(x^3 z_j - x z_j^3), \\ w = w_0 - \omega_0 x + \sum_{j=1}^{3} s_j k_{1j} B_{3j}(x^2 - z_j^2) + \sum_{j=1}^{3} s_j k_{1j} B_{5j}(x^4 - 6x^2 z_j^2 + z_j^4), \\ \Phi = \Phi_0 + \sum_{j=1}^{3} s_j k_{2j} B_{3j}(x^2 - z_j^2) + \sum_{j=1}^{3} s_j k_{2j} B_{5j}(x^4 - 6x^2 z_j^2 + z_j^4), \\ \sigma_x = -2\sum_{j=1}^{3} \omega_{3j} B_{3j} z_j + \sum_{j=1}^{3} \omega_{3j} B_{5j}(-12x^2 z_j + 4z_j^3), \\ \sigma_z = -2\sum_{j=1}^{3} \omega_{1j} B_{3j} z_j + \sum_{j=1}^{3} \omega_{1j} B_{5j}(-12x^2 z_j + 4z_j^3), \\ D_z = -2\sum_{j=1}^{3} \omega_{2j} B_{3j} z_j + \sum_{j=1}^{3} \omega_{2j} B_{5j}(-12x^2 z_j + 4z_j^3), \\ \tau_{xz} = 2\sum_{j=1}^{3} s_j \omega_{1j} B_{3j} x + \sum_{j=1}^{3} s_j \omega_{1j} B_{5j}(4x^3 - 12x z_j^2), \\ D_x = 2\sum_{j=1}^{3} s_j \omega_{2j} B_{3j} x + \sum_{j=1}^{3} s_j \omega_{2j} B_{5j}(4x^3 - 12x z_j^2), \end{cases} \tag{29}$$

式中，u_0、w_0、ω_0、Φ_0、B_{3j} 和 B_{5j} $(j = 1, \cdots, 3)$ 均为待定常数.

将式(29)代入式(25)和(26)，并经简化可得到:

$$\begin{cases} \sum_{j=1}^{3} s_j \omega_{1j} B_{3j} = -\dfrac{3q}{4h},\quad \sum_{j=1}^{3} s_j^3 \omega_{1j} B_{5j} = -\dfrac{q}{2h^3},\quad \sum_{j=1}^{3} s_j \omega_{1j} B_{5j} = 0, \\ -\sum_{j=1}^{3} s_j \omega_{2j} B_{3j} + \dfrac{h^2}{2}\sum_{j=1}^{3} s_j^3 \omega_{2j} B_{5j} = 0,\quad \sum_{j=1}^{3} s_j \omega_{2j} B_{5j} = 0, \\ -\dfrac{2}{3}\sum_{j=1}^{3} s_j \omega_{3j} B_{3j} + \dfrac{h^2}{5}\sum_{j=1}^{3} s_j^3 \omega_{3j} B_{5j} = 0. \end{cases} \tag{30}$$

于是，待定常数 B_{3j} 和 B_{5j} $(j = 1, \cdots, 3)$ 可由式(30)中的6个方程解出. 进而，应用式(27)便可解出 u_0、w_0、ω_0 和 Φ_0. 最终，我们就得到了压电悬臂梁在两表面 $(z = \pm h/2)$ 分别承受均布荷载 $\pm(q/2)$ 相应解析解.

在上述解析解(29)基础上，如叠加上文献[1]中压电梁在轴向均匀拉伸精确解(取 $\sigma_2 = q/2$)，并将文献[1]中式(13)中的 x 替代以 $(x - L)$，就可得到压电悬臂梁 $(x = L$ 处固定$)$ 下表面 $(z = \pm h/2)$ 承受均布荷载 q、上表面自由情况下的解析解.

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Polynomial Solutions to Piezoelectric Beams(Ⅱ) ——Analytical Solutions to Typical Problems

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Abstract: For the orthotropic piezoelectric plane problem, a series of piezoelectric beams is solved and the corresponding analytical solutions are obtained with the trial_and_error method on the basis of the general solution in the case of three distinct eigenvalues, in which all displacements, electrical potential, stresses and electrical displacements are expressed by three displacement functions in terms of harmonic polynomials. These problems are cantilever beam with cross force and point charge at free end, cantilever beam and simply_supported beam subjected to uniform loads on the upper and lower surfaces, and cantilever beam subjected to linear electrical potential.

Key words: piezoelectric beam; plane problem; harmonic polynomial; analytical solution