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具有对角线化的一致质量矩阵的动力有限元和弹塑性撞击计算

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摘　要

在 EPIC[1,2]、NONSAP[3] 等弹塑性撞击计算的有限元程序中，都有一些共同的弱点．所有这些程序，都采用静力学问题中常用的简单线性形状函数来描写各位移分量．在这样的有限元法中，应变和应力分量在每一有限元中都是常量．但在运动方程中，应力分量都是以它们的空间导数的形式出现的．于是，在采用了线性形状函数来表达的位移分量以后，应力分量对运动方程的贡献必恒等于零．克服这种困难的一般方法是通过虚位移原理，把运动方程化为能量关系的变分形式，从而建立既作用在结点上而又在每一有限元内自相平衡的人为内力平衡系统．把施加在某一结点上的所有相邻有限元的人为内力的作用叠加在一起，就能计算这一结点的加速度．但是从虚位移原理化为能量关系的变分形式时，要求位移和应力在积分域内处处连续．也就是说，要求位移和应力有限元都是协调的．我们很易看到，线性形状函数所描述的位移有限元是连续协调的，但其有关的应力分量在有限元界面上，则并不连续．所以，这样的有限元处理，是否收敛并无把握，即使从近似角度看，也是难以令人满意的．而且，为了计算结点的加速度，我们还应该有建立质量矩阵的计算规则．目前有两种计算方法：一种是集总（lumped）质量法，另一种是一致（consistent）质量法[4]．一致质量矩阵是通过正规的有限元计算求得的，它和所用形状函数相一致．不过，这样求得的一致质量矩阵一般不是对角线化的，这就给数值计算带来不便．在大多数计算程序中，人们采用集总质量矩阵，也即是说，有限元的质量是按一定的比例分配给该有限元的各个结点的．集总质量矩阵是对角线化的，对角线项就是结点分配到的质量．当然，这种集总质量的假定还缺乏证明．其实，所有这些困难都是从采用线性形状函数（静力的）所引起的．

本文采用了二次式形状函数来处理弹塑性撞击问题．这种二次式形状函数不仅给出了对角线化的一致质量矩阵，而且运动方程中的应力的总效应也不等于零．所以，在采用了二次形状函数后，上述的所有困难都迎刃而解了．

一、引　　论

设有四面体有限元，如图１．

结点i顺着x轴向的位移分量为$u_i=x_i-x_i^0$，顺着y和z轴向的位移分量分别为$v_i=y_i-y_i^0$，$w_i=z_i-z_i^0$．典型的四面体有限元中任意点的位移分量$u^{(e)}$，$v^{(e)}$，$w^{(e)}$可以用形状函数

$N_k = L_k^0$（它们是未变形前的几何的函数）和结点位移分量 u_k, v_k, w_k 来表示如下：

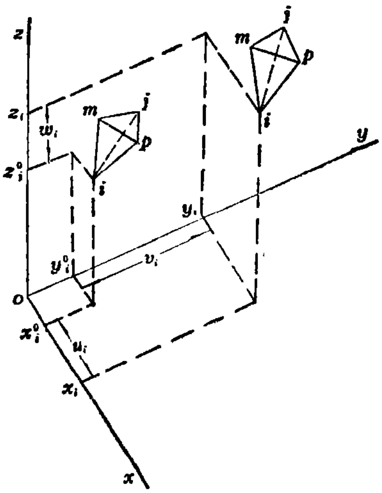


图 1 典型的四面体有限元

$$
\left.
\begin{aligned}
u^{(e)} &= L_i^0 u_i + L_j^0 u_j + L_m^0 u_m + L_p^0 u_p \\
v^{(e)} &= L_i^0 v_i + L_j^0 v_j + L_m^0 v_m + L_p^0 v_p \\
w^{(e)} &= L_i^0 w_i + L_j^0 w_j + L_m^0 w_m + L_p^0 w_p
\end{aligned}
\right\} \tag{1.1}
$$

在程序 EPIC-3 所采用的线性四面体有限元中，L_i^0，L_j^0，L_m^0，L_p^0 为

$$
\left.
\begin{aligned}
L_i^0 &= \frac{1}{6V^0}(a_i^0 + b_i^0 x + c_i^0 y + d_i^0 z) \\
L_j^0 &= \frac{1}{6V^0}(a_j^0 + b_j^0 x + c_j^0 y + d_j^0 z) \\
L_m^0 &= \frac{1}{6V^0}(a_m^0 + b_m^0 x + c_m^0 y + d_m^0 z) \\
L_p^0 &= \frac{1}{6V^0}(a_p^0 + b_p^0 x + c_p^0 y + d_p^0 z)
\end{aligned}
\right\} \tag{1.2}
$$

其中 V^0 为四面体有限元在变形前的原来的体积．

$$
V^0 = \frac{1}{6}
\begin{vmatrix}
1 & x_i^0 & y_i^0 & z_i^0 \\
1 & x_j^0 & y_j^0 & z_j^0 \\
1 & x_m^0 & y_m^0 & z_m^0 \\
1 & x_p^0 & y_p^0 & z_p^0
\end{vmatrix} \tag{1.3}
$$

其它各系数也和有限元在变形前的原来几何有关．它们是

$$
\left.
\begin{aligned}
a_i^0 &= \begin{vmatrix} x_j^0 & y_j^0 & z_j^0 \\ x_m^0 & y_m^0 & z_m^0 \\ x_p^0 & y_p^0 & z_p^0 \end{vmatrix} \qquad
b_i^0 = (-1)\begin{vmatrix} 1 & y_j^0 & z_j^0 \\ 1 & y_m^0 & z_m^0 \\ 1 & y_p^0 & z_p^0 \end{vmatrix} \\
c_i^0 &= \begin{vmatrix} 1 & x_j^0 & z_j^0 \\ 1 & x_m^0 & z_m^0 \\ 1 & x_p^0 & z_p^0 \end{vmatrix} \qquad
d_i^0 = (-1)\begin{vmatrix} 1 & x_j^0 & y_j^0 \\ 1 & x_m^0 & y_m^0 \\ 1 & x_p^0 & y_p^0 \end{vmatrix}
\end{aligned}
\right\} \tag{1.4}
$$

其它和未变形前的几何有关的常数(a_j^0，b_j^0，c_j^0，d_j^0；a_m^0，b_m^0，c_m^0，d_m^0；a_p^0，b_p^0，c_p^0，d_p^0)可以通过轮换下标和正负号求得．

在采用了上述线性四面体有限元的形状函数后，在这一元素中的所有各点上，应变分量和应力分量都是常数．于是，在任一元素中，运动方程

$$\left.\begin{aligned}\rho\ddot{u}&=\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)\\ \rho\ddot{v}&=\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right)\\ \rho\ddot{w}&=\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right)\end{aligned}\right\}\tag{1.5}$$

的右侧各项都恒等于零．因此，计算到此失败．

在程序 EPIC-3 中，这种困难是通过引入虚功原理和假设集总质量来克服的．

设我们研究的材料空间域为D,在D中(1.5)适用,在 D 的表面δD_1上受外力 $(q_x,q_y,q_z)dS$ 作用．表面 δD_1 的外法线矢量为 (n_x,n_y,n_z)，于是在外力作用的表面上，应有

$$\left.\begin{aligned}\sigma_x n_x+\tau_{xy}n_y+\tau_{xz}n_z&=q_x\\ \tau_{xy}n_x+\sigma_y n_y+\tau_{yz}n_z&=q_y\\ \tau_{xz}n_x+\tau_{yz}n_y+\sigma_z n_z&=q_z\end{aligned}\right\}\quad \text{在 } \delta D_1\text{上}\tag{1.6}$$

并设在另一部份表面 δD_2 上，位移已知：

$$u=\bar{u},\quad v=\bar{v},\quad w=\bar{w}\quad \text{在 } \delta D_2 \text{ 上}\tag{1.7}$$

于是在体力$(\rho\ddot{u},\rho\ddot{v},\rho\ddot{w})$和面力$(q_x,q_y,q_z)$作用下的物体的虚位移原理可以写成

$$\begin{aligned}&\iiint_D\left\{\left[\rho\ddot{u}-\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)\right]\delta u+\left[\rho\ddot{v}-\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right)\right]\delta v\right.\\ &\qquad\left.+\left[\rho\ddot{w}-\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right)\right]\delta w\right\}dxdydz\\ &\qquad-\iint_{\delta D_1}\{[q_x-(\sigma_x n_x+\tau_{xy}n_y+\tau_{xz}n_z)]\delta u+[q_y-(\tau_{xy}n_x+\sigma_y n_y+\tau_{yz}n_z)]\delta v\\ &\qquad+[q_z-(\tau_{xz}n_x+\tau_{yz}n_y+\sigma_z n_z)]\delta w\}dS=0\end{aligned}\tag{1.8}$$

如果在D内，$\sigma_x,\sigma_y\cdots$和 $\delta u,\delta v,\delta w$ 都是连续的，则利用格林定理，我们可以证明：

$$\begin{aligned}&\iiint_D\left\{\rho\ddot{u}\delta u+\rho\ddot{v}\delta v+\rho\ddot{w}\delta w+\sigma_x\frac{\partial\delta u}{\partial x}+\sigma_y\frac{\partial\delta v}{\partial y}+\sigma_z\frac{\partial\delta w}{\partial z}+\tau_{xy}\left(\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}\right)\right.\\ &\qquad\left.+\tau_{yz}\left(\frac{\partial\delta w}{\partial y}+\frac{\partial\delta v}{\partial z}\right)+\tau_{zx}\left(\frac{\partial\delta u}{\partial z}+\frac{\partial\delta w}{\partial x}\right)\right\}dxdydz-\iint_{\delta D_1}\{q_x\delta u+q_y\delta v+q_z\delta w\}dS\\ &\qquad-\iint_{\delta D_2}\{(\sigma_x n_x+\tau_{xy}n_y+\tau_{xz}n_z)\delta u+(\tau_{xy}n_x+\sigma_y n_y+\tau_{yz}n_z)\delta v\\ &\qquad+(\tau_{xz}n_x+\tau_{yz}n_y+\sigma_z n_z)\delta w\}dS=0\end{aligned}\tag{1.9}$$

在利用了位移边界条件(1.7)后，上式简化为

$$\begin{aligned}&\iiint_D\left\{\left(\rho\ddot{u}\delta u+\sigma_x\frac{\partial\delta u}{\partial x}+\tau_{xy}\frac{\partial\delta u}{\partial y}+\tau_{xz}\frac{\partial\delta u}{\partial z}\right)+\left(\rho\ddot{v}\delta v+\tau_{xy}\frac{\partial\delta v}{\partial x}+\sigma_y\frac{\partial\delta v}{\partial y}+\tau_{zy}\frac{\partial\delta v}{\partial z}\right)\right.\\ &\qquad\left.+\left(\rho\ddot{w}\delta w+\tau_{xz}\frac{\partial\delta w}{\partial x}+\tau_{yz}\frac{\partial\delta w}{\partial y}+\sigma_z\frac{\partial\delta w}{\partial z}\right)\right\}dxdydz-\iint_{\delta D_1}\{q_x\delta u+q_y\delta v+q_z\delta w\}dS=0\end{aligned}\tag{1.10}$$

这就是最小位能原理的变分形式．

从此，我们引进有限元计算．EPIC-3 用了下述两个假定：

（1） 集总质量假设：设每一有限元的质量 $V\rho$（其中 V 为有限元在变形后的容积）平均分配在四个结点上，我们有

$$M_i=M_j=M_m=M_p=\frac{1}{4}V^0\rho^0=\frac{1}{4}V\rho \tag{1.11}$$

其中 V^0,ρ^0 为元素在变形前的原有体积和密度．而 V,ρ 为元素在变形后的体积和密度．

（2） 用线性形状函数（按变形后的几何计算的）来表示位移分量，即

$$\left.\begin{aligned}u&=L_iu_i+L_ju_j+L_mu_m+L_pu_p\\v&=L_iv_i+L_jv_j+L_mv_m+L_pv_p\\w&=L_iw_i+L_jw_j+L_mw_m+L_pw_p\end{aligned}\right\} \tag{1.12}$$

其中 L_i,L_j,L_m,L_p 为 (1.2) 式，但按变形后的有限元几何计算，u_i,v_i,w_i 为 i 结点的位移分量．从(1.12)式计算所得的应力分量在有限元中是常量，把(1.10)式离散为若干有限元．典型有限元的积分可以用(1.11)，(1.12)求得：如

$$\iiint_{(e)}\rho\ddot{u}\delta u\,dxdydz=\frac{1}{4}V\rho(\ddot{u}_i\delta u_i+\ddot{u}_j\delta u_j+\ddot{u}_m\delta u_m+\ddot{u}_p\delta u_p) \tag{1.13}$$

$$\begin{aligned}&\iiint_{(e)}\left(\sigma_x\frac{\partial\delta u}{\partial x}+\tau_{xy}\frac{\partial\delta u}{\partial y}+\tau_{xz}\frac{\partial\delta u}{\partial z}\right)dxdydz\\&=\frac{1}{6}\{(\sigma_xb_i+\tau_{xy}c_i+\tau_{xz}d_i)\delta u_i+(\sigma_xb_j+\tau_{xy}c_j+\tau_{xz}d_j)\delta u_j\\&\quad+(\sigma_xb_m+\tau_{xy}c_m+\tau_{xz}d_m)\delta u_m+(\sigma_xb_p+\tau_{xy}c_p+\tau_{xz}d_p)\delta u_p\}\end{aligned} \tag{1.14}$$

其它各式依此类推．由于 $\delta u_i,\delta u_m,\delta u_j,\delta u_p$ 等都是独立变分，所以，我们从 (1.10) 式的离散有限元的积分，得典型有限元的特微方程

$$\frac{1}{4}V\rho\ddot{u}_i=f_{xi},\quad \frac{1}{4}V\rho\ddot{u}_j=f_{xj},\quad \frac{1}{4}V\rho\ddot{u}_m=f_{xm},\quad \frac{1}{4}V\rho\ddot{u}_p=f_{xp} \tag{1.15}$$

其中

$$\left.\begin{aligned}f_{xi}&=-\frac{1}{6}(b_i\sigma_x+c_i\tau_{xy}+d_i\tau_{xz}),\qquad f_{xj}=-\frac{1}{6}(b_j\sigma_x+c_j\tau_{xy}+d_j\tau_{xz})\\f_{xm}&=-\frac{1}{6}(b_m\sigma_x+c_m\tau_{xy}+d_m\tau_{xz}),\qquad f_{xp}=-\frac{1}{6}(b_p\sigma_x+c_p\tau_{xy}+d_p\tau_{xz})\end{aligned}\right\} \tag{1.16}$$

在 y 轴向和 z 轴向，我们也有相类似的计算公式．例如：

$$\frac{1}{4}V\rho\ddot{v}_i=f_{yi},\qquad \frac{1}{4}V\rho\ddot{w}_i=f_{zi} \tag{1.17}$$

其中

$$f_{yi}=-\frac{1}{6}(b_i\tau_{xy}+c_i\sigma_y+d_i\tau_{yz}),\quad f_{zi}=-\frac{1}{6}(b_i\tau_{xz}+c_i\tau_{yz}+d_i\sigma_z) \tag{1.18}$$

我们很易证明，$f_{xi},f_{xj},f_{xm},f_{xp}$在有限元内是平衡的，即

$$f_{xi}+f_{xj}+f_{xm}+f_{xp}=-\frac{1}{6}(b_i+b_j+b_m+b_p)\sigma_x$$

$$-\frac{1}{6}(c_i+c_j+c_m+c_p)\tau_{xy}-\frac{1}{6}(d_i+d_j+d_m+d_p)\tau_{xz}=0 \tag{1.19}$$

从(1.15)式，我们可以把f_{xi}看作为有限元的内力作用在结点i上的集中力(在x轴向).

结点i上所受力的合力$(\Sigma f_{xi},\Sigma f_{yi},\Sigma f_{zi})$为结点上各相邻有限元中的集中力的总和.假如$\Sigma\frac{1}{4}V\rho=\Sigma M_i$为各相邻有限元在结点$i$上的集总质量的总和，则结点$i$的$x$轴向加速度分量（在$t$时）为

$$\ddot{u}_i(t)=\frac{\Sigma f_{xi}}{\Sigma M_i} \tag{1.20}$$

同样有

$$\ddot{v}_i(t)=\frac{\Sigma f_{yi}}{\Sigma M_i},\quad \ddot{w}_i(t)=\frac{\Sigma f_{zi}}{\Sigma M_i} \tag{1.21}$$

我们必须指出，上述集总质量的假设，有很大的随意性；同时，我们在(1.10)的有限元积分中，采用了线性形状函数来表示近似的位移函数，它在有限元边界上是连续的，但在这样的近似之下，应力分量在每一有限元中都是常量；在有限元之间，应力分量并不连续，所以格林定理不适用，而且(1.10)式根本不成立．这从根本上动摇了EPIC-3的计算基础.

在下节，我们用二次式的形状函数来表示位移分量，从而导出了对角线化的一致质量矩阵，这样，上述诸假设可以完全避免.

二、二次式的位移形状函数和对角线化的一致质量矩阵

设每一有限元（四面体）的位移分量可以用下式表示:

$$u^{(e)}=\mathbf{N}\mathbf{u},\quad v^{(e)}=\mathbf{N}\mathbf{v},\quad w^{(e)}=\mathbf{N}\mathbf{w} \tag{2.1a,b,c}$$

其中

$$\mathbf{N}=\lfloor N_i,N_j,N_m,N_p\rfloor \tag{2.2a}$$

$$\mathbf{u}^T=\lfloor u_i,u_j,u_m,u_p\rfloor \tag{2.2b}$$

$$\mathbf{v}^T=\lfloor v_i,v_j,v_m,v_p\rfloor \tag{2.2c}$$

$$\mathbf{w}^T=\lfloor w_i,w_j,w_m,w_p\rfloor \tag{2.2d}$$

这个四面体有限元有四个结点，其中，i,j,m,p为四面体的四个角点上的结点(图2)．形状函数N_i,N_j,N_m,N_p为四面体的体积坐标L_i,L_j,L_m,L_p的函数，而这些体积坐标则是根据变形后的几何按(1.2)式计算的.

设λ为待定常数．N_i,N_j,N_m,N_p可以写成

$$\left.\begin{aligned}
N_i&=L_i+\lambda[3L_i^2-4L_i+1-(L_j^2+L_m^2+L_p^2)]\\
N_j&=L_j+\lambda[3L_j^2-4L_j+1-(L_i^2+L_m^2+L_p^2)]\\
N_m&=L_m+\lambda[3L_m^2-4L_m+1-(L_i^2+L_j^2+L_p^2)]\\
N_p&=L_p+\lambda[3L_p^2-4L_p+1-(L_i^2+L_j^2+L_m^2)]
\end{aligned}\right\} \tag{2.3}$$

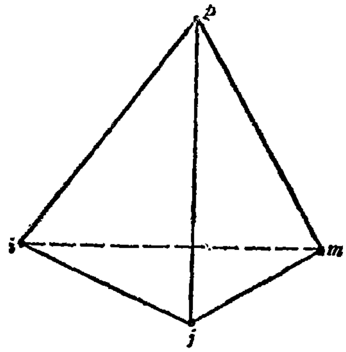


图 2 四结点的四面体有限元

结点	L_i	L_j	L_m	L_p
i	1	0	0	0
j	0	1	0	0
m	0	0	1	0
p	0	0	0	1

很易看到 N_i, N_j, N_m, N_p 满足下列诸关系式

$$N_i+N_j+N_m+N_p=1 \quad (\text{在有限元内}) \tag{2.4a}$$

$$N_i=1\text{，在结点}i\text{上，}N_i=0\text{ 在其它结点上} \tag{2.4b}$$

$$N_j=1\text{，在结点}j\text{上，}N_j=0\text{ 在其它结点上} \tag{2.4c}$$

$$N_m=1\text{，在结点}m\text{上，}N_m=0\text{ 在其它结点上} \tag{2.4d}$$

$$N_p=1\text{，在结点}p\text{上，}N_p=0\text{ 在其它结点上} \tag{2.4e}$$

现在让我们用下列对角线化的条件求 λ:

$$\iiint_{(e)} N_i N_j dxdydz=0 \tag{2.5}$$

还有 N_jN_m, N_mN_p, N_iN_m, N_iN_p, N_jN_p 等的条件，实质上是和(2.5)式相同的．把(2.3)式的N_i, N_j代入(2.5)式，用积分关系式

$$\iiint_{(e)} L_i^\alpha L_j^\beta L_m^\gamma L_p^\delta \, dV = \frac{\alpha!\beta!\gamma!\delta!}{(\alpha+\beta+\gamma+\delta+3)!}6V^{(e)} \tag{2.6}$$

从 (2.5) 得

$$12\lambda^2-14\lambda-21=0 \tag{2.7}$$

其解有

$$\lambda_1=\frac{7}{12}-\frac{1}{12}\sqrt{301}=-0.862446,\quad \lambda_2=\frac{7}{12}+\frac{1}{12}\sqrt{301}=2.029113 \tag{2.8}$$

以N_i为例，取λ_1或λ_2分别为

$$\left.\begin{aligned} N_i^{(-)}&=L_i-0.862446[3L_i^2-4L_i+1-(L_j^2+L_m^2+L_p^2)]\\ N_i^{(+)}&=L_i+2.029113[3L_i^2-4L_i+1-(L_j^2+L_m^2+L_p^2)]\end{aligned}\right\} \tag{2.9}$$

在四面体的ij棱边上，$L_p=L_m=0$，$L_j=1-L_i$，于是(2.9)可以简化为

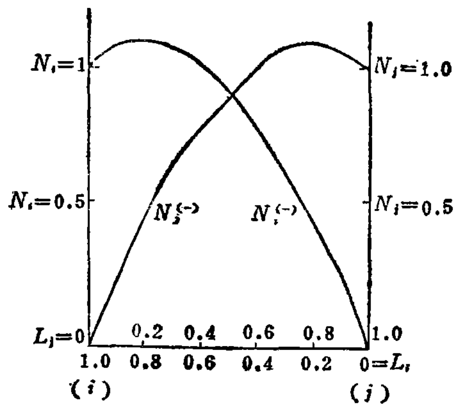
$$\left.\begin{aligned} N_i^{(-)}&=L_i+0.862446L_i(1-L_i)\\ N_i^{(+)}&=L_i-2.029113L_i(1-L_i)\end{aligned}\right\} \tag{2.10}$$

同样，在ij棱边上，$N_j^{(\pm)}$值为

$$\left.\begin{aligned} N_j^{(-)}&=L_j+0.862446L_j(1-L_j)\\ N_j^{(+)}&=L_j-2.029113L_j(1-L_j)\end{aligned}\right\} \tag{2.11}$$

图 3 为形状函数 N_i, N_j在ij稜边上的分布．$N_i^{(-)}$, $N_j^{(-)}$ 都是正的，$N_i^{(+)}$, $N_j^{(+)}$ 有正有负．以后计算中建议用 $\lambda^{(-)}$．

现在让我们计算对角线项．



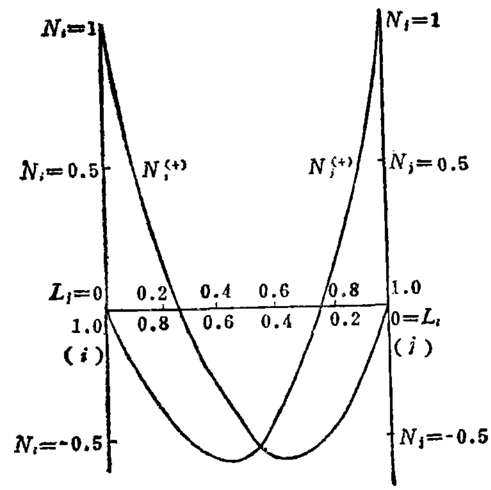


图 3　形状函数N_i,N_j在棱边ij上的分布

$$\iiint_{(e)} N_i N_j \, dxdydz = \iiint_{(e)} \{L_i+\lambda[3L_i^2-4L_i+1-(L_j^2+L_m^2+L_p^2)]\}^2 dxdydz$$

$$=\frac{1}{140}(14-14\lambda+12\lambda^2)\,V^{(0)}=\frac{1}{3}V^0 \qquad (2.12)$$

同样，有限元中任意点上的加速度可以用诸结点上的值表示

$$\ddot{u}^{(e)}=\mathbf{N}\ddot{\mathbf{u}},\quad \ddot{v}^{(e)}=\mathbf{N}\ddot{\mathbf{v}},\quad \ddot{w}^{(e)}=\mathbf{N}\ddot{\mathbf{w}} \qquad (2.13)$$

其中$\mathbf{N}$见(2.2a)，$\ddot{\mathbf{u}}$、$\ddot{\mathbf{v}}$、$\ddot{\mathbf{w}}$为

$$\left.\begin{aligned}\ddot{\mathbf{u}}^T&=\lfloor \ddot{u}_i,\ddot{u}_j,\ddot{u}_m,\ddot{u}_p\rfloor\\ \ddot{\mathbf{v}}^T&=\lfloor \ddot{v}_i,\ddot{v}_j,\ddot{v}_m,\ddot{v}_p\rfloor\\ \ddot{\mathbf{w}}^T&=\lfloor \ddot{w}_i,\ddot{w}_j,\ddot{w}_m,\ddot{w}_p\rfloor\end{aligned}\right\} \qquad (2.13\text{a,b,c})$$

还有任意有限元的应力分量的表达式

$$\sigma_x^{(e)}=\mathbf{N}\boldsymbol{\sigma}_x,\quad \sigma_y^{(e)}=\mathbf{N}\boldsymbol{\sigma}_y,\quad \sigma_z^{(e)}=\mathbf{N}\boldsymbol{\sigma}_z \qquad (2.14\text{a,b,c})$$

$$\tau_{yz}^{(e)}=\mathbf{N}\boldsymbol{\tau}_{yz},\quad \tau_{zx}^{(e)}=\mathbf{N}\boldsymbol{\tau}_{zx},\quad \tau_{xy}^{(e)}=\mathbf{N}\boldsymbol{\tau}_{xy} \qquad (2.14\text{d,e,f})$$

其中

$$\boldsymbol{\sigma}_x^T=\lfloor \sigma_{xi},\sigma_{xj},\sigma_{xm},\sigma_{xp}\rfloor \qquad (2.15\text{a})$$

$$\boldsymbol{\sigma}_y^T=\lfloor \sigma_{yi},\sigma_{yj},\sigma_{ym},\sigma_{yp}\rfloor \qquad (2.15\text{b})$$

$$\boldsymbol{\sigma}_z^T=\lfloor \sigma_{zi},\sigma_{zj},\sigma_{zm},\sigma_{zp}\rfloor \qquad (2.15\text{c})$$

$$\boldsymbol{\tau}_{yz}^T=\lfloor \tau_{yzi},\tau_{yzj},\tau_{yzm},\tau_{yzp}\rfloor \qquad (2.15\text{d})$$

$$\boldsymbol{\tau}_{zx}^T=\lfloor \tau_{zxi},\tau_{zxj},\tau_{zxm},\tau_{zxp}\rfloor \qquad (2.15\text{e})$$

$$\boldsymbol{\tau}_{xy}^T=\lfloor \tau_{xyi},\tau_{xyj},\tau_{xym},\tau_{xyp}\rfloor \qquad (2.15\text{f})$$

现在让我们研究第一个运动方程(1.5a)．在(1.5a)上乘δu，并在域内积分，得

$$\iiint_D \rho\ddot{u}\delta u \, dxdydz=\iiint_D\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)\delta u dxdydz \qquad (2.16)$$

把D分为若干个有限元，典型有限元的特征方程为

$$\iiint_{(e)} \rho \ddot{u} \delta u \, dxdydz = \iiint_{(e)} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta u dxdydz \tag{2.17}$$

把(2.13)，(2.1)，(2.14)代入上式，得

$$\iiint_{(e)} \rho \mathbf{N} \ddot{\mathbf{u}} \, \mathbf{N} \delta \mathbf{u} \, dxdydz = \iiint_{(e)} \left(\frac{\partial \mathbf{N}}{\partial x} \boldsymbol{\sigma}_x + \frac{\partial \mathbf{N}}{\partial y} \boldsymbol{\tau}_{xy} + \frac{\partial \mathbf{N}}{\partial z} \boldsymbol{\tau}_{xz} \right) \mathbf{N} \delta \mathbf{u} dxdydz \tag{2.18}$$

由于$\delta u_i, \delta u_j, \delta u_m, \delta u_p$都是独立的变分，同时设在有限元内，$\rho$近似地为常数，在用了(2.5)和(2.12)后，(2.18)式可以分为下列四个特征方程

$$\left.\begin{aligned}
\alpha \rho V \ddot{u}_i &= \iiint_{(e)} \left(\frac{\partial \mathbf{N}}{\partial x} \boldsymbol{\sigma}_x + \frac{\partial \mathbf{N}}{\partial y} \boldsymbol{\tau}_{xy} + \frac{\partial \mathbf{N}}{\partial z} \boldsymbol{\tau}_{xz} \right) N_i \, dxdydz \\
\alpha \rho V \ddot{u}_j &= \iiint_{(e)} \left(\frac{\partial \mathbf{N}}{\partial x} \boldsymbol{\sigma}_x + \frac{\partial \mathbf{N}}{\partial y} \boldsymbol{\tau}_{xy} + \frac{\partial \mathbf{N}}{\partial z} \boldsymbol{\tau}_{xz} \right) N_j dxdydz \\
\alpha \rho V \ddot{u}_m &= \iiint_{(e)} \left(\frac{\partial \mathbf{N}}{\partial x} \boldsymbol{\sigma}_x + \frac{\partial \mathbf{N}}{\partial y} \boldsymbol{\tau}_{xy} + \frac{\partial \mathbf{N}}{\partial z} \boldsymbol{\tau}_{xz} \right) N_m dxdydz \\
\alpha \rho V \ddot{u}_p &= \iiint_{(e)} \left(\frac{\partial \mathbf{N}}{\partial x} \boldsymbol{\sigma}_x + \frac{\partial \mathbf{N}}{\partial y} \boldsymbol{\tau}_{xy} + \frac{\partial \mathbf{N}}{\partial z} \boldsymbol{\tau}_{xz} \right) N_p dxdydz
\end{aligned}\right\} \tag{2.19}$$

(2.19)也可以写成

$$\left.\begin{aligned}
\alpha \rho V \ddot{u}_i &= \mathbf{B}_i \boldsymbol{\sigma}_x + \mathbf{C}_i \boldsymbol{\tau}_{xy} + \mathbf{D}_i \boldsymbol{\tau}_{xz} = f_{xi} \\
\alpha \rho V \ddot{u}_j &= \mathbf{B}_j \boldsymbol{\sigma}_x + \mathbf{C}_j \boldsymbol{\tau}_{xy} + \mathbf{D}_j \boldsymbol{\tau}_{xz} = f_{xj} \\
\alpha \rho V \ddot{u}_m &= \mathbf{B}_m \boldsymbol{\sigma}_x + \mathbf{C}_m \boldsymbol{\tau}_{xy} + \mathbf{D}_m \boldsymbol{\tau}_{xz} = f_{xm} \\
\alpha \rho V \ddot{u}_p &= \mathbf{B}_p \boldsymbol{\sigma}_x + \mathbf{C}_p \boldsymbol{\tau}_{xy} + \mathbf{D}_p \boldsymbol{\tau}_{xz} = f_{xp}
\end{aligned}\right\} \tag{2.20}$$

其中α见(2.12)，可以写成

$$\alpha = \frac{1}{140}(14 - 14\lambda + 12\lambda^2) \tag{2.21}$$

$\mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i$ 等为

$$\left.\begin{aligned}
\mathbf{B}_i &= \lfloor B_{ii}, B_{ji}, B_{mi}, B_{pi} \rfloor \\
\mathbf{C}_i &= \lfloor C_{ii}, C_{ji}, C_{mi}, C_{pi} \rfloor \\
\mathbf{D}_i &= \lfloor D_{ii}, D_{ji}, D_{mi}, D_{pi} \rfloor \\
&\text{等}
\end{aligned}\right\} \tag{2.22}$$

其中B_{ij}, C_{ij}, D_{ij} 定义为

$$\left.\begin{aligned}
B_{ij} &= \iiint_{(e)} \frac{\partial N_i}{\partial x} N_j dxdydz \\
C_{ij} &= \iiint_{(e)} \frac{\partial N_i}{\partial y} N_j dxdydz \\
D_{ij} &= \iiint_{(e)} \frac{\partial N_i}{\partial z} N_j dxdydz
\end{aligned}\right\} \tag{2.23}$$

把(2.3) 式代入 (2.23)，积分后得

$$B_{ii} = \frac{1}{120}(5 - 8\lambda - 6\lambda^2)\, b_i - \frac{1}{180}(3 + \lambda)\lambda(b_j + b_m + b_p) \tag{2.24a}$$

$$B_{ij}=-\frac{1}{60}\lambda(2-\lambda)\,b_i+\frac{1}{120}(5-14\lambda+2\lambda^2)\,b_j-\frac{1}{180}(3+\lambda)\lambda(b_m+b_p) \tag{2.24b}$$

$$C_{ii}=\frac{1}{120}(5-8\lambda-6\lambda^2)\,c_i-\frac{1}{180}(3+\lambda)\lambda(c_j+c_m+c_p) \tag{2.25a}$$

$$C_{ij}=-\frac{1}{60}\lambda(2-\lambda)\,c_i+\frac{1}{120}(5-14\lambda+2\lambda^2)\,c_j-\frac{1}{180}(3+\lambda)\lambda(c_m+c_p) \tag{2.25b}$$

$$D_{ii}=\frac{1}{120}(5-8\lambda-6\lambda^2)\,d_i-\frac{1}{180}(3+\lambda)\lambda(d_j+d_m+d_p) \tag{2.26a}$$

$$D_{ij}=-\frac{1}{60}\lambda(2-\lambda)\,d_i+\frac{1}{120}(5-14\lambda+2\lambda^2)\,d_j-\frac{1}{180}(3+\lambda)\lambda(d_m+d_p) \tag{2.26b}$$

其它系数可以轮换下标求得.

在 y,z 轴向，我们也有相类似的计算公式．例如：

$$\left.\begin{aligned}
&\alpha V\rho\ddot{v}_i=f_{yi}=\mathbf{B}_i\tau_{xy}+\mathbf{C}_i\sigma_y+\mathbf{D}_i\tau_{xz}\\
&\alpha V\rho\ddot{v}_j=f_{yj}=\mathbf{B}_j\tau_{xy}+\mathbf{C}_j\sigma_y+\mathbf{D}_j\tau_{xz}\\
&\alpha V\rho\ddot{w}_i=f_{zi}=\mathbf{B}_i\tau_{xz}+\mathbf{C}_i\tau_{yz}+\mathbf{D}_i\sigma_z\\
&\alpha V\rho\ddot{w}_j=f_{zj}=\mathbf{B}_j\tau_{xz}+\mathbf{C}_j\tau_{yz}+\mathbf{D}_j\sigma_z\\
&\text{等}
\end{aligned}\right\} \tag{2.27}$$

把有关的相邻有限元的贡献组合在一起，得时间为 t 时，结点 i 的加速度的三个分量：

$$\ddot{u}_i^t=\frac{\sum^i f_{xi}}{\sum^i \alpha\rho V},\qquad \ddot{v}_i^t=\frac{\sum^i f_{yi}}{\sum^i \alpha\rho V},\qquad \ddot{w}_i^t=\frac{\sum^i f_{zi}}{\sum^i \alpha\rho V} \tag{2.28}$$

其中 $\sum^i$ 为结点 i 相邻元素中求和的符号．还有结点 j,m,p 上的相类的加速度分量的诸表达式.

在下一时间增量 Δt 后，新的速度为

$$\dot{u}_i^{t+\Delta t}=\dot{u}_i^t+\ddot{u}_i^t\Delta t \qquad (\text{还有 } \dot{v}_i^{t+\Delta t},\ \ \dot{w}^{t+\Delta t}) \tag{2.29}$$

其中 $\dot{u}_i^t$, $\dot{v}_i^t$, $\dot{w}_i^t$ 为结点i在前一时间增量结束时的速度分量，Δt 为时间增量．最后，时间 $t+\Delta t$ 时的新的位移分量为

$$u_i^{t+\Delta t}=u_i^t+\dot{u}_i^t\Delta t \qquad (\text{还有 } v_i^{t+\Delta t},\ \ w_i^{t+\Delta t}) \tag{2.30}$$

结点 i 在 $t+\Delta t$ 时坐标为

$$x_i^{t+\Delta t}=x_i^0+u_i^{t+\Delta t} \qquad (\text{还有 } y_i^{t+\Delta t},\ \ z_i^{t+\Delta t}) \tag{2.31}$$

其它结点在时间为 $t+\Delta t$ 时的位移分量，速度分量和坐标位置也可以用相同的方法求得.

在积分运动方程时所用的积分增量 Δt 可用下式求得[1],[2]：

$$\Delta t=\frac{4}{3}\left\{\frac{h}{\sqrt{g^2}+\sqrt{g^2+c_s^2}}\right\} \tag{2.32}$$

其中

$$g^2=c_s^2 Q/\rho \tag{2.33}$$

c_s 为材料的声速，Q 为人为粘度，它将留待下节解说．h 为四面体有限元的最小高度．

当结点上的应力分量已给以后，我们可以用上面的计算程序求下一时间增量 Δt 以后的各结点的位移分量．

我们可以看到上述方法求得的质量矩阵为对角线化的，它可以写成

$$\mathbf{M}=\begin{bmatrix} \alpha M & \cdot & \cdot & \cdot \\ \cdot & \alpha M & \cdot & \cdot \\ \cdot & \cdot & \alpha M & \cdot \\ \cdot & \cdot & \cdot & \alpha M \end{bmatrix} \qquad M=\rho V=\rho^0 V^0 \tag{2.34}$$

α 见 (2.21)．

三、结点上应力分量的计算

在上节业已证明，当每一结点的应力分量已给后，元素中各结点所受等效力的分量 f_{xi}，f_{yi}，f_{zi} 等都可以从(2.20)，(2.27)式计算．为了计算所有结点上的应力分量，我们首先应计算弹性区域中的各应变分量，或塑性区域的应变速度分量．

设某一元素中的位移分量 $u^{(e)}$，$v^{(e)}$，$w^{(e)}$ 可以通过形函数 N_i^0，N_j^0，N_m^0，N_p^0 用这些分量的结点值来表示．上述形函数是按原变形前的几何来描述的．它们根据(2.3)，是

$$\left.\begin{aligned} N_i^0 &= L_i^0+\lambda\,[3L_i^{0^2}-4L_i^0+1-(L_j^{0^2}+L_m^{0^2}+L_p^{0^2})] \\ N_j^0 &= L_j^0+\lambda\,[3L_j^{0^2}-4L_j^0+1-(L_i^{0^2}+L_m^{0^2}+L_p^{0^2})] \\ N_m^0 &= L_m^0+\lambda\,[3L_m^{0^2}-4L_m^0+1-(L_i^{0^2}+L_j^{0^2}+L_p^{0^2})] \\ N_p^0 &= L_p^0+\lambda\,[3L_p^{0^2}-4L_p^0+1-(L_i^{0^2}+L_j^{0^2}+L_m^{0^2})] \end{aligned}\right\} \tag{3.1}$$

其中 L_i^0，L_j^0，L_m^0，L_p^0 见(1.2)．

这个元素的位移分量为

$$u^{(e)}=\mathbf{N}^0\mathbf{u} \qquad v^{(e)}=\mathbf{N}^0\mathbf{v} \qquad w^{(e)}=\mathbf{N}^0\mathbf{w} \tag{3.2}$$

其中 $\mathbf{u}$，$\mathbf{v}$，$\mathbf{w}$ 见(2.2b,c,d)．在这个元素中的应变分量，可以从下列非线性应变位移关系中求得[5]．

$$\epsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{3.3a}$$

$$\epsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] \tag{3.3b}$$

$$\epsilon_z=\frac{\partial w}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right] \tag{3.3c}$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{3.3d}$$

$$\gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \tag{3.3e}$$

$$\gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial w}{\partial x} \tag{3.3f}$$

$$\epsilon_v=\frac{V^{(e)}}{V_0}-1 \tag{3.4}$$

其中 $V^{(e)}$ 是从 x'_i， y'_i， z'_i 等值用下式计算的.

$$V=V^{(e)}=\frac{1}{6}\begin{vmatrix} 1 & x'_i & y'_i & z'_i \\ 1 & x'_j & y'_j & z'_j \\ 1 & x'_m & y'_m & z'_m \\ 1 & x'_p & y'_p & z'_p \end{vmatrix} \tag{3.5}$$

把(3.2)代入(3.3)，得

$$\epsilon'_x=\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{u}+\frac{1}{2}\left\{\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{u}\right)^2+\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{v}\right)^2+\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{w}\right)^2\right\} \tag{3.6a}$$

$$\epsilon'_y=\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{v}+\frac{1}{2}\left\{\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{u}\right)^2+\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{v}\right)^2+\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{w}\right)^2\right\} \tag{3.6b}$$

$$\epsilon'_z=\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{w}+\frac{1}{2}\left\{\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{u}\right)^2+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{v}\right)^2+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{w}\right)^2\right\} \tag{3.6c}$$

$$\gamma'_{xy}=\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{u}+\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{v}+\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{u}\right)\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{u}\right)+\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{v}\right)\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{v}\right)+\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{w}\right)\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{w}\right) \tag{3.6d}$$

$$\gamma'_{yz}=\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{v}+\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{w}+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{u}\right)\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{u}\right)+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{v}\right)\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{v}\right)+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{w}\right)\left(\frac{\partial \mathbf{N}^0}{\partial y}\mathbf{w}\right) \tag{3.6e}$$

$$\gamma'_{zx}=\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{w}+\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{u}+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{u}\right)\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{u}\right)+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{v}\right)\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{v}\right)+\left(\frac{\partial \mathbf{N}^0}{\partial z}\mathbf{w}\right)\left(\frac{\partial \mathbf{N}^0}{\partial x}\mathbf{w}\right) \tag{3.6f}$$

在结点 i 上，我们有(L^0_i， L^0_j， L^0_m， L^0_p)=(1,0,0,0)，而

$$\epsilon'_{xi}=\mathbf{H}^0_{xi}\mathbf{u}+\frac{1}{2}\{(\mathbf{H}^0_{xi}\mathbf{u})^2+(\mathbf{H}^0_{xi}\mathbf{v})^2+(\mathbf{H}^0_{xi}\mathbf{w})^2\} \tag{3.7a}$$

$$\epsilon'_{yi}=\mathbf{H}^0_{yi}\mathbf{v}+\frac{1}{2}\{(\mathbf{H}^0_{yi}\mathbf{u})^2+(\mathbf{H}^0_{yi}\mathbf{v})^2+(\mathbf{H}^0_{yi}\mathbf{w})^2\} \tag{3.7b}$$

$$\epsilon'_{zi}=\mathbf{H}^0_{zi}\mathbf{w}+\frac{1}{2}\{(\mathbf{H}^0_{zi}\mathbf{u})^2+(\mathbf{H}^0_{zi}\mathbf{v})^2+(\mathbf{H}^0_{zi}\mathbf{w})^2\} \tag{3.7c}$$

$$\gamma'_{xyi}=\mathbf{H}^0_{yi}\mathbf{u}+\mathbf{H}^0_{xi}\mathbf{v}+(\mathbf{H}^0_{xi}\mathbf{u})(\mathbf{H}^0_{yi}\mathbf{u})+(\mathbf{H}^0_{xi}\mathbf{v})(\mathbf{H}^0_{yi}\mathbf{v})+(\mathbf{H}^0_{xi}\mathbf{w})(\mathbf{H}^0_{yi}\mathbf{w}) \tag{3.7d}$$

$$\gamma'_{yzi}=\mathbf{H}^0_{zi}\mathbf{v}+\mathbf{H}^0_{yi}\mathbf{w}+(\mathbf{H}^0_{zi}\mathbf{u})(\mathbf{H}^0_{yi}\mathbf{u})+(\mathbf{H}^0_{zi}\mathbf{v})(\mathbf{H}^0_{yi}\mathbf{v})+(\mathbf{H}^0_{zi}\mathbf{w})(\mathbf{H}^0_{yi}\mathbf{w}) \tag{3.7e}$$

$$\gamma'_{zxi}=\mathbf{H}^0_{xi}\mathbf{w}+\mathbf{H}^0_{zi}\mathbf{u}+(\mathbf{H}^0_{xi}\mathbf{u})(\mathbf{H}^0_{zi}\mathbf{u})+(\mathbf{H}^0_{xi}\mathbf{v})(\mathbf{H}^0_{zi}\mathbf{v})+(\mathbf{H}^0_{xi}\mathbf{w})(\mathbf{H}^0_{zi}\mathbf{w}) \tag{3.7f}$$

其中 $\mathbf{H}^0_{xi}$ ， $\mathbf{H}^0_{yi}$ ， $\mathbf{H}^0_{zi}$ 为

$$\mathbf{H}^0_{xi}=\frac{1}{6V^0}\lfloor(1+2\lambda)b^0_i,-2\lambda b^0_i+(1-4\lambda)b^0_j,-2\lambda b^0_i+(1-4\lambda)b^0_m,-2\lambda b^0_i+(1-4\lambda)b^0_p\rfloor \tag{3.8a}$$

$$\mathbf{H}^0_{yi}=\frac{1}{6V^0}\lfloor(1+2\lambda)c^0_i,-2\lambda c^0_i+(1-4\lambda)c^0_j,-2\lambda c^0_i+(1-4\lambda)c^0_m,-2\lambda c^0_i+(1-4\lambda)c^0_p\rfloor \tag{3.8b}$$

$$\mathbf{H}^0_{zi}=\frac{1}{6V^0}\lfloor(1+2\lambda)d^0_i,-2\lambda d^0_i+(1-4\lambda)d^0_j,-2\lambda d^0_i+(1-4\lambda)d^0_m,-2\lambda d^0_i+(1-4\lambda)d^0_p\rfloor \tag{3.8c}$$

在结点j,m,p上还有各有关应变分量的同类的表达式．

应变速度$\dot\epsilon'_x,\dot\epsilon'_y,\dot\epsilon'_z$，$\dot\gamma'_{xy}$，$\dot\gamma'_{yz}$，$\dot\gamma'_{zx}$可以从下式计算，

$$\left.\begin{aligned}&\dot\epsilon'_x=\frac{\partial\dot u'}{\partial x},\quad \dot\epsilon'_y=\frac{\partial\dot v'}{\partial y},\quad \dot\epsilon'_z=\frac{\partial\dot w'}{\partial z}\\&\dot\gamma'_{yz}=\frac{\partial\dot v'}{\partial z}+\frac{\partial\dot w'}{\partial y},\ \dot\gamma_{zx}=\frac{\partial\dot w'}{\partial x}+\frac{\partial\dot u'}{\partial z},\ \dot\gamma'_{xy}=\frac{\partial\dot u'}{\partial y}+\frac{\partial\dot v'}{\partial x}\end{aligned}\right\} \tag{3.9}$$

我们用变形后的几何描写形函数，即

$$\dot u'=\mathbf{N}\dot{\mathbf{u}}',\quad \dot v'=\mathbf{N}\dot{\mathbf{v}}',\quad \dot w'=\mathbf{N}\dot{\mathbf{w}}' \tag{3.10}$$

所以，我们有

$$\left.\begin{aligned}&\dot\epsilon'_x=\frac{\partial\mathbf{N}}{\partial x}\dot{\mathbf{u}}',\ \dot\epsilon'_y=\frac{\partial\mathbf{N}}{\partial y}\dot{\mathbf{v}}',\ \dot\epsilon'_z=\frac{\partial\mathbf{N}}{\partial z}\dot{\mathbf{w}}'\\&\dot\gamma'_{xy}=\frac{\partial\mathbf{N}}{\partial x}\dot{\mathbf{v}}'+\frac{\partial\mathbf{N}}{\partial y}\dot{\mathbf{u}}',\ \dot\gamma_{yz}=\frac{\partial\mathbf{N}}{\partial y}\dot{\mathbf{w}}'+\frac{\partial\mathbf{N}}{\partial z}\dot{\mathbf{v}}',\ \dot\gamma'_{zx}=\frac{\partial\mathbf{N}}{\partial z}\dot{\mathbf{u}}'+\frac{\partial\mathbf{N}}{\partial x}\dot{\mathbf{w}}'\end{aligned}\right\} \tag{3.11}$$

在结点i上，$(L_i,L_j,L_m,L_p)=(1,0,0,0)$．所以有

$$\left.\begin{aligned}&\dot\epsilon'_{xi}=\mathbf{H}_{xi}\dot{\mathbf{u}}',\ \dot\epsilon'_{yi}=\mathbf{H}_{yi}\dot{\mathbf{v}}',\ \dot\epsilon'_{zi}=\mathbf{H}_{zi}\dot{\mathbf{w}}'\\&\dot\gamma'_{xyi}=\mathbf{H}_{xi}\dot{\mathbf{v}}'+\mathbf{H}_{yi}\dot{\mathbf{u}}',\ \dot\gamma'_{yzi}=\mathbf{H}_{yi}\dot{\mathbf{w}}'+\mathbf{H}_{zi}\dot{\mathbf{v}}',\ \dot\gamma'_{zxi}=\mathbf{H}_{zi}\dot{\mathbf{u}}'+\mathbf{H}_{xi}\dot{\mathbf{w}}'\end{aligned}\right\} \tag{3.12}$$

其中 $\mathbf{H}_{xi},\mathbf{H}_{yi},\mathbf{H}_{zi}$ 见(3.8a,b,c)式，但通过变形后的几何计算．在结点j,m,p上还有相类的应变速度分量的表达式．

弹性应力是通过胡克定律[5]从应变分量直接计算的．

$$\left.\begin{aligned}&\sigma'_x=\bar\lambda\epsilon'_v+2G\epsilon'_x-Q',\qquad \tau'_{yz}=G\gamma'_{yz}\\&\sigma'_y=\bar\lambda\epsilon'_v+2G\epsilon'_y-Q',\qquad \tau'_{zx}=G\gamma'_{zx}\\&\sigma'_z=\bar\lambda\epsilon'_v+2G\epsilon'_z-Q',\qquad \tau'_{xy}=G\gamma'_{xy}\end{aligned}\right\} \tag{3.13}$$

其中，$\bar{\lambda}$、G为拉梅弹性常数，Q^t为人为粘度[8]，

$$Q^t=\begin{cases}C_L\rho c_s h|\dot{\epsilon}_v^t|+C_0^2\rho h^2|\dot{\epsilon}_v^t|^2 & 在\dot{\epsilon}_v^t<0中\\ 0 & 在\dot{\epsilon}_v^t\geqslant 0中\end{cases} \tag{3.14}$$

这里，c_s为材料的声速，h为四面体元素的最小高度．C_L和C_0为无量纲系数

$$C_L=0.5 \qquad C_0^2=4.0 \tag{3.15}$$

从结点i到其它三结点构成的平面的垂直高度为

$$h_i=\frac{6V^{(e)}}{\sqrt{b_i^2+c_i^2+d_i^2}} \tag{3.16}$$

结点i上的弹性应力分量是根据(3.13)式用该点的有关应变分量表示的．它们是

$$\left.\begin{aligned}\sigma_{xi}^t&=\bar{\lambda}\epsilon_v^t+2G\epsilon_{xi}^t-Q^t \qquad \tau_{yzi}^t=G\gamma_{yzi}^t\\ \sigma_{yi}^t&=\bar{\lambda}\epsilon_v^t+2G\epsilon_{yi}^t-Q^t \qquad \tau_{zxi}^t=G\gamma_{zxi}^t\\ \sigma_{zi}^t&=\bar{\lambda}\epsilon_v^t+2G\epsilon_{zi}^t-Q^t \qquad \tau_{xyi}^t=G\gamma_{xyi}^t\end{aligned}\right\} \tag{3.17}$$

其中ϵ_v^t见(3.4)，Q^t见(3.14)，在每一元素中，它们都是常数．我们必须指出，按(3.17)计算所得的在结点上的应力分量，在各有关的相邻有限元中并不保证是等值的，也即是说，在诸有限元间，结点位移是连续的，但应力分布不连续．

在结点i处的弹性应力分量组成一个等效应力，它是

$$\bar{\sigma}_i^t=\sqrt{\frac{1}{2}\left\{(\sigma_{xi}^t-\sigma_{yi}^t)^2+(\sigma_{yi}^t-\sigma_{zi}^t)^2+(\sigma_{zi}^t-\sigma_{xi}^t)^2+6(\tau_{xyi}^{t2}+\tau_{yzi}^{t2}+\tau_{zxi}^{t2})\right\}} \tag{3.18}$$

当$\bar{\sigma}_i^t$小于材料的拉伸屈服强度时，应力就属于弹性范围的．

当$\bar{\sigma}_i^t$超过材料的拉伸屈服强度时，材料就产生塑性流动．在发生塑性流动后，正应力分量就由塑性应力偏量，静水压强，和人为粘度三项组成，亦即

$$\left.\begin{aligned}\sigma_{xi}^t&=S_{xi}^t-(P_i^t+Q^t)\\ \sigma_{yi}^t&=S_{yi}^t-(P_i^t+Q^t)\\ \sigma_{zi}^t&=S_{zi}^t-(P_i^t+Q^t)\end{aligned}\right\} \tag{3.19}$$

塑性应力偏量代表材料剪力强度特性，采用冯·米西斯塑性增量理论后，应力偏量(S_{xi}^t, S_{yi}^t, S_{zi}^t)和剪应力(τ_{yzi}^t, τ_{zxi}^t, τ_{xyi}^t)为

$$\left.\begin{aligned}S_{xi}^t&=\frac{2}{3}\left(\frac{\dot{e}_{xi}^t}{\dot{\epsilon}_i^t}\right)\bar{S} \qquad \tau_{xyi}^t=\frac{1}{3}\left(\frac{\dot{\gamma}_{xyi}^t}{\dot{\epsilon}_i^t}\right)\bar{S}\\ S_{yi}^t&=\frac{2}{3}\left(\frac{\dot{e}_{yi}^t}{\dot{\epsilon}_i^t}\right)\bar{S} \qquad \tau_{yzi}^t=\frac{1}{3}\left(\frac{\dot{\gamma}_{yzi}^t}{\dot{\epsilon}_i^t}\right)\bar{S}\\ S_{zi}^t&=\frac{2}{3}\left(\frac{\dot{e}_{zi}^t}{\dot{\epsilon}_i^t}\right)\bar{S} \qquad \tau_{zxi}^t=\frac{1}{3}\left(\frac{\dot{\gamma}_{zxi}^t}{\dot{\epsilon}_i^t}\right)\bar{S}\end{aligned}\right\} \tag{3.20}$$

其中$\bar{S}$为材料的等效拉伸强度，而$\bar{\dot{\epsilon}}_i^t$为等效应变速度

$$\overline{\dot{\epsilon}_i^t}=\sqrt{\frac{2}{9}\left\{(\dot{\epsilon}_{xi}^t-\dot{\epsilon}_{yi}^t)^2+(\dot{\epsilon}_{yi}^t-\dot{\epsilon}_{zi}^t)^2+(\dot{\epsilon}_{zi}^t-\dot{\epsilon}_{xi}^t)^2+\frac{3}{2}(\dot{\gamma}_{xyi}^{t2}+\dot{\gamma}_{yzi}^{t2}+\dot{\gamma}_{zxi}^{t2})\right\}} \tag{3.21}$$

$\dot{e}_x^t, \dot{e}_y^t, \dot{e}_z^t$为应变偏量速度.如果把(3.19),(3.20)中的应力分量代入(3.18)，即得结果

$$\bar{\sigma}_i^t=\bar{S} \tag{3.22}$$

等效拉伸强度可以用静力拉伸强度来表示，在这时，$\bar{S}$ 只是 $\bar{\epsilon}_i^t$的函数.$\bar{\epsilon}_i^t$为

$$\bar{\epsilon}_i^t=\sqrt{\frac{2}{9}\left\{(\epsilon_{xi}^t-\epsilon_{yi}^t)^2+(\epsilon_{yi}^t-\epsilon_{zi}^t)^2+(\epsilon_{zi}^t-\epsilon_{xi}^t)^2+\frac{3}{2}(\gamma_{xyi}^{t2}+\gamma_{yzi}^{t2}+\gamma_{zxi}^{t2})\right\}} \tag{3.23}$$

在大多数的撞击问题中，应变总有少量的恢复，而$\bar{\epsilon}_i^t$总是增加的，这样，使(3.23)式成为一个适用的近似式.

静水压强既和体积变化有关，也和有限元内的内能有关[7]. 本文用 Mie-gruneisen 物态方程

$$P_i^t=(K_1\mu+K_2\mu^2+K_3\mu^3)\left(1-\frac{\Gamma\mu}{2}\right)+\Gamma\rho E_i^t \tag{3.24}$$

这里的符号为

$$\left.\begin{array}{l} K_1,K_2,K_3=\text{和材料有关的常数} \\ \Gamma=\text{Grüneisen 常数} \\ \mu=\dfrac{\rho}{\rho^0}-1=\dfrac{V^0}{V^{(e)}}-1 \end{array}\right\} \tag{3.25}$$

而E_i^t为比内能.它是各应力分量在结点i处对有限元每单位质量做的功.它是从下式的积分求得的

$$\frac{d}{dt}(\rho E_i^t)=V^{(e)}\{S_{xi}^t\dot{\epsilon}_{xi}^t+S_{yi}^t\dot{\epsilon}_{yi}^t+S_{zi}^t\dot{\epsilon}_{zi}^t+2\tau_{xyi}^t\dot{\gamma}_{xyi}^t$$
$$+2\tau_{yzi}^t\dot{\gamma}_{yzi}^t+2\tau_{zxi}^t\dot{\gamma}_{zxi}^t\}-(Q^t+P_i^t)\dot{V}^{(e)} \tag{3.26}$$

以上给出了计算撞击的三维问题的全部公式. 根据本文所编出的计算程序和计算例题将另文发表.

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Dynamic Finite Element with Diagonalized Consistent Mass Matrix and Elastic-Plastic Impact Calculation

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Abstract

There are some common difficulties encountered in elastic-plastic impact codes such as EPIC[1],[2], NONSAP[3], etc. Most of these codes use the simple linear functions usually taken from static problems to represent the displacement components. In such finite element formulation, the strain and stress components are constants in every element. In the equations of motion, the stress components in general appear in the form of their space derivatives. Thus, if we use such form functions to represent the displacement components, the effect of internal stresses to the equations of motion vanishes identically. The usual practice to overcome such difficulties is to establish a self-equilibrium system of internal forces acting on various nodal points by means of transforming equations of motion into variational form of energy relation through the application of virtual displacement principle. The nodal acceleration is then calculated from the total forces acting on this node from all the neighbouring elements. The transformation of virtual displacement principle into the variational energy form is performed on the bases of continuity conditions of stress and displacement throughout the integrated space. That is to say, on the interface boundary of finite element, the assumed displacement and stress functions should be conformed. However, it is easily seen that, for linear form function of finite element calculation, the displacement continues everywhere, but not the stress components. Thus, the convergence of such kind of finite element computation is open to question. This kind of treatment has never been justified even in approximation sense. Furthermore, the calculation of acceleration of nodal points needs a rule to calculate the mass matrix. There are two ways to establish mass matrix, namely, lumped mass method and consistent mass method.[4] The consistent mass matrix can be obtained naturally through finite element formulation, which is consistent to the assumed form functions. However, the resulting consistent mass matrix is not in diagonalized form, which is inconvenient for numerical computation. For most codes, the lumped mass matrix is used, and in this case, the element mass is distributed in certain assumed proportions to all the nodal points of this element. The lumped mass matrix is diagonalized with the diagonal terms composed of the nodal masses. However, the lumped mass assumption has never been justified. All these difficulties are originated from the simple linear form functions usually used in static problems.

In this paper, we introduced a new quadratic form function for elastic-plastic impact problems. This quadratic form function possesses diagonalized consistent mass matrix, and non-vanishing effect of internal stress to the equations of motion. Thus with this kind of dynamic finite element, all above-said difficulties can be eliminated.