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椭圆板的大挠度问题

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摘　要

本文在圆薄板大挠度问题摄动解法(1948)[1]，(1954)[2]的基础上，求得了椭圆板大挠度问题的摄动解．本文的公式推导是1957年以前完成的，由于某些原因，长期未得发表．1959年见到Nash-Cooley[3]以摘要形式发表的类似工作，但只有$\lambda=a/b=2$的数值结果.这里将原先推导的正确至二级近似的分析公式以及计算结果发表．其中包括泊桑比$\nu=0.25$，0.30，0.35，椭圆半径比$\lambda=1$，2，3，4，5的全部计算结果，以备工程设计计算之用．

关键词　椭圆板　大挠度　摄动

一、引　　言

薄板大挠度问题曾受到应用数学界的长期重视．求解该问题的方程是卡门非线性方程(1910)[4]，(1940)[5]．由于非线性问题求解困难，只有极少数问题曾被计算过．最初韦(S. Way)(1934)[6]曾用幂级数解研究了在均布载荷下边缘固定的圆薄板问题；后来李斐(S. Levy)(1942)[7]用重三角级数法得到了均布载荷下简支长方板的数值解．这两种方法都非常冗繁，以致在一些比较重要的情况下便不能使用．钱伟长在(1948)[1]，(1954)[2]用摄动法重新处理了均布载荷下和各种不同边界条件下的圆薄板的大挠度问题，得到非常满意的结果，有关中心挠度和边缘屈服条件和实验值非常接近(McPherson，Ramburg，Levy，1942[8])．此后，钱伟长和叶开沅曾试图用摄动法处理四边固定的矩形板在均布载荷下的大挠度问题(1956)[9]．稍后，Nash和Cooley(1959)[3]曾试图用摄动法处理边界固定的椭圆板在均布载荷下的大挠度问题，但只给出了$\lambda=2$，$\nu=0.30$的数值结果．本文试图研究固定的椭圆板在均布载荷下的摄动解.不论一级近似和二级近似都不采用纯数值解法而用分析解法．我们计算了各种泊桑比（$\nu=0.25$，0.30，0.35）和各种椭圆半径比（$\lambda=1$，2，3，4，5）的椭圆板解．

二、椭圆板在均布载荷下的大挠度方程

让我们考虑长轴为$2a$、短轴为$2b$、厚为h的椭圆薄板在横向均布载荷q的作用下，所发生的大挠度问题（图1）．设板内中面各点$P(x, y)$处的横向挠度和x，y轴向的拉伸位移分别为

w，u，v，薄膜应力各分量可以写成：

$$\sigma_x=\frac{E}{1-\nu^2}\left\{\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\nu\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\right\} \tag{2.1a}$$

$$\sigma_y=\frac{E}{1-\nu^2}\left\{\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\nu\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\} \tag{2.1b}$$

$$\tau_{xy}=\frac{E}{2(1+\nu)}\left\{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\} \tag{2.1c}$$

其中E为杨氏模量．ν为材料的泊桑比，它在以后计算中取值$0.25<\nu<0.35$．

von Kármán 大挠度方程相当于下列平衡方程

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\qquad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{2.2a,b}$$

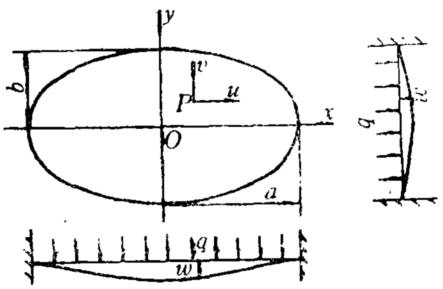

图1　椭圆板的坐标$(x,\ y)$和位移$(u,\ v,\ w)$

$$D\nabla^2\nabla^2 w=q+h\left(\sigma_x\frac{\partial^2 w}{\partial x^2}+\sigma_y\frac{\partial^2 w}{\partial y^2}+2\tau_{xy}\frac{\partial^2 w}{\partial x\partial y}\right) \tag{2.2c}$$

其中$D=Eh^3/12(1-\nu^2)$为薄板抗弯刚度，把(2.1)代入(2.2)，得

$$2\frac{\partial^2 u}{\partial x^2}+(1-\nu)\frac{\partial^2 u}{\partial y^2}+(1+\nu)\frac{\partial^2 v}{\partial x\partial y}=-(1-\nu)\frac{\partial w}{\partial x}\nabla^2 w-\frac{1}{2}(1+\nu)\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] \tag{2.3a}$$

$$2\frac{\partial v^2}{\partial y^2}+(1-\nu)\frac{\partial^2 v}{\partial x^2}+(1+\nu)\frac{\partial^2 u}{\partial x\partial y}=-(1-\nu)\frac{\partial w}{\partial y}\nabla^2 w-\frac{1}{2}(1+\nu)\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] \tag{2.3b}$$

$$\begin{aligned}D\nabla^2\nabla^2 w=&q+\frac{Eh}{1-\nu^2}\left\{\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)\frac{\partial^2 w}{\partial y^2}+(1-\nu)\right.\\&\left.\cdot\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 w}{\partial x\partial y}\right\}+\frac{Eh}{2(1-\nu^2)}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2+\nu\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial^2 w}{\partial x^2}+\left[\left(\frac{\partial w}{\partial y}\right)^2\right.\right.\\&\left.\left.+\nu\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial y^2}+2(1-\nu)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}\right\}\end{aligned} \tag{2.3c}$$

为了无量纲化，我们引进下列无量纲量：

$$\left.\begin{aligned}&\lambda=\frac{a}{b},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{b},\quad Q=(1-\nu^2)\frac{a^4 q}{h^4 E}\\&U=\frac{au}{h^2},\quad V=\frac{bv}{h^2}=\frac{av}{\lambda h^2},\quad W=\frac{w}{h}\end{aligned}\right\} \tag{2.4}$$

(2.3)式可以写成

$$2\frac{\partial^2 U}{\partial \xi^2}+(1-\nu)\lambda^2\frac{\partial^2 U}{\partial \eta^2}+(1+\nu)\lambda^2\frac{\partial^2 V}{\partial \xi\partial \eta}$$

$$=-(1-\nu)\frac{\partial W}{\partial \xi}\left(\frac{\partial^2 W}{\partial \xi^2}+\lambda^2\frac{\partial^2 W}{\partial \eta^2}\right)-\frac{1}{2}(1+\nu)\frac{\partial}{\partial \xi}\left[\left(\frac{\partial W}{\partial \xi}\right)^2+\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2\right] \tag{2.5a}$$

$$2\lambda^2\frac{\partial^2 V}{\partial \eta^2}+(1-\nu)\frac{\partial^2 V}{\partial \xi^2}+(1+\nu)\frac{\partial^2 U}{\partial \xi\partial \eta}$$

$$=-(1-\nu)\frac{\partial W}{\partial \eta}\left(\frac{\partial^2 W}{\partial \xi^2}+\lambda^2\frac{\partial^2 W}{\partial \eta^2}\right)-\frac{1}{2}(1+\nu)\frac{\partial}{\partial \eta}\left[\left(\frac{\partial W}{\partial \xi}\right)^2+\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2\right] \tag{2.5b}$$

$$\frac{\partial^4 W}{\partial \xi^4}+2\lambda^2\frac{\partial^4 W}{\partial \xi^2\partial \eta^2}+\lambda^4\frac{\partial^4 W}{\partial \eta^4}-12Q=12\left\{\left(\frac{\partial U}{\partial \xi}+\nu\lambda^2\frac{\partial V}{\partial \eta}\right)\frac{\partial^2 W}{\partial \xi^2}\right.$$

$$+\left(\lambda^2\frac{\partial V}{\partial \eta}+\nu\frac{\partial U}{\partial \xi}\right)\lambda^2\frac{\partial^2 W}{\partial \eta^2}+\left.\left(\frac{\partial U}{\partial \eta}+\frac{\partial V}{\partial \xi}\right)(1-\nu)\lambda^2\frac{\partial^2 W}{\partial \eta\partial \xi}\right\}+6\left\{\left[\left(\frac{\partial W}{\partial \xi}\right)^2\right.\right.$$

$$\left.+\nu\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2\right]\frac{\partial^2 W}{\partial \xi^2}+\left[\lambda^2\left(\frac{\partial W}{\partial \eta}\right)^2+\nu\left(\frac{\partial W}{\partial \xi}\right)^2\right]\lambda^2\frac{\partial^2 W}{\partial \eta^2}$$

$$\left.+2(1-\nu)\lambda^2\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\frac{\partial^2 W}{\partial \xi\partial \eta}\right\} \tag{2.5c}$$

求解这些方程的边界条件为

$$W=U=V=\partial W/\partial n=0 \qquad \text{在边界}\ \xi^2+\eta^2=1\ \text{上} \tag{2.6}$$

其中$\partial/\partial n$为椭圆边界上的外法线方向的偏导数.

三、摄动法求解

按圆薄板大挠度问题摄动法求解的成功先例(1948)[1]，我们将以椭圆板中心挠度的无量纲量W_m为摄动参数.

$$W_m=W(0,\ 0) \tag{3.1}$$

并设(2.5a,b,c)中的U, V, W, Q都可以展开为W_m的幂级数如下：

$$U(\xi,\ \eta)=U_2(\xi,\ \eta)W_m^2+U_4(\xi,\ \eta)W_m^4+\cdots \tag{3.2a}$$

$$V(\xi,\ \eta)=V_2(\xi,\ \eta)W_m^2+V_4(\xi,\ \eta)W_m^4+\cdots \tag{3.2b}$$

$$W(\xi,\ \eta)=W_1(\xi,\ \eta)W_m+W_3(\xi,\ \eta)W_m^3+\cdots \tag{3.2c}$$

$$3Q/2(3+2\lambda^2+3\lambda^4)=\alpha_1 W_m+\alpha_3 W_m^3+\cdots \tag{3.2d}$$

把(3.2a,b,c,d)代入(2.5a,b,c)各式，得W_m的幂级数表达式. 由于各式对任意W_m值都适用。所以，各该式的W_m^n各系数项都必独立满足方程式. 由此导出了求解α_k, $U_k(\xi,\ \eta)$, $V_k(\xi,\ \eta)$和$W_k(\xi,\ \eta)$的各级近似表达式.

一级近似为(2.5c)的W_m项，和(2.5a,b)的W_m^2项，于是得求解$W_1(\xi,\ \eta)$, α_1, $U_2(\xi,\ \eta)$, $V_2(\xi,\ \eta)$的方程式如下：

$$\frac{\partial^4 W_1}{\partial \xi^4}+2\lambda^2\frac{\partial^4 W_1}{\partial \xi^2\partial \eta^2}+\lambda^4\frac{\partial^4 W_1}{\partial \eta^4}=8(3+2\lambda^2+3\lambda^4)\alpha_1 \tag{3.3a}$$

$$2\frac{\partial^2 U_2}{\partial \xi^2}+(1-\nu)\lambda^2\frac{\partial^2 U_2}{\partial \eta^2}+(1+\nu)\lambda^2\frac{\partial^2 V_2}{\partial \xi\partial \eta}=-(1-\nu)\frac{\partial W_1}{\partial \xi}\left(\frac{\partial^2 W_1}{\partial \xi^2}\right.$$

$$\left.+\lambda^2\frac{\partial^2 W_1}{\partial \eta^2}\right)-\frac{1}{2}(1+\nu)\frac{\partial}{\partial \xi}\left[\left(\frac{\partial W_1}{\partial \xi}\right)^2+\lambda^2\left(\frac{\partial W_1}{\partial \eta}\right)^2\right] \tag{3.3b}$$

$$2\lambda^2\frac{\partial^2 V_2}{\partial\eta^2}+(1-\nu)\frac{\partial^2 V_2}{\partial\xi^2}+(1+\nu)\frac{\partial^2 U_2}{\partial\xi\partial\eta}$$

$$=-(1-\nu)\frac{\partial W_1}{\partial\eta}\left(\frac{\partial^2 W_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 W_1}{\partial\eta^2}\right)-\frac{1}{2}(1+\nu)\frac{\partial}{\partial\eta}\left[\left(\frac{\partial W_1}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial W_1}{\partial\eta}\right)^2\right] \quad (3.3c)$$

为了求解(3.3a,b,c)，我们从(3.1)，(2.6)得下列边界条件：

$$W_1(0,\ 0)=1 \quad (3.4a)$$

$$W_1(\xi,\ \eta)=U_2(\xi,\ \eta)=V_2(\xi,\ \eta)=\partial W_1(\xi,\ \eta)/\partial n=0, \qquad \text{在椭圆}\xi^2+\eta^2=1\text{上} \quad (3.4b,c,d,e)$$

二级近似为(2.5c)的W_m^3项，和(2.5a，b)的W_m^4项。于是得求解$W_3(\xi,\ \eta)$，a_3，$U_4(\xi,\ \eta)$，$V_4(\xi,\ \eta)$的方程式如下：

$$\frac{\partial^4 W_3}{\partial\xi^4}+2\lambda^2\frac{\partial^4 W_3}{\partial\xi^2\partial\eta^2}+\lambda^4\frac{\partial^4 W_3}{\partial\eta^4}-8(3+2\lambda^2+3\lambda^4)a_3$$

$$=12\left\{\left(\frac{\partial U_2}{\partial\xi}+\nu\lambda^2\frac{\partial V_2}{\partial\eta}\right)\frac{\partial^2 W_1}{\partial\xi^2}+\left(\lambda^2\frac{\partial V_2}{\partial\eta}+\nu\frac{\partial U_2}{\partial\xi}\right)\lambda^2\frac{\partial^2 W_1}{\partial\eta^2}+\left(\frac{\partial U_2}{\partial\eta}+\frac{\partial V_2}{\partial\xi}\right)\right.$$

$$\left.\cdot(1-\nu)\lambda^2\frac{\partial^2 W_1}{\partial\xi\partial\eta}\right\}+6\left\{\left[\left(\frac{\partial W_1}{\partial\xi}\right)^2+\nu\lambda^2\left(\frac{\partial W_1}{\partial\eta}\right)^2\right]\frac{\partial^2 W_1}{\partial\xi^2}+\left[\lambda^2\left(\frac{\partial W_1}{\partial\eta}\right)^2\right.\right.$$

$$\left.\left.+\nu\left(\frac{\partial W_1}{\partial\xi}\right)^2\right]\lambda^2\frac{\partial^2 W_1}{\partial\eta^2}+2(1-\nu)\lambda^2\frac{\partial W_1}{\partial\xi}\frac{\partial W_1}{\partial\eta}\frac{\partial^2 W_1}{\partial\xi\partial\eta}\right\} \quad (3.5a)$$

$$2\frac{\partial^2 U_4}{\partial\xi^2}+(1-\nu)\lambda^2\frac{\partial^2 U_4}{\partial\eta^2}+(1+\nu)\lambda^2\frac{\partial^2 V_4}{\partial\xi\partial\eta}=-(1-\nu)\left\{\frac{\partial W_1}{\partial\xi}\left(\frac{\partial^2 W_3}{\partial\xi^2}+\lambda^2\frac{\partial^2 W_3}{\partial\eta^2}\right)\right.$$

$$\left.+\frac{\partial W_3}{\partial\xi}\left(\frac{\partial^2 W_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 W_1}{\partial\eta^2}\right)\right\}-(1+\nu)\frac{\partial}{\partial\xi}\left[\frac{\partial W_1}{\partial\xi}\frac{\partial W_3}{\partial\xi}+\lambda^2\frac{\partial W_1}{\partial\eta}\frac{\partial W_3}{\partial\eta}\right] \quad (3.5b)$$

$$2\lambda^2\frac{\partial^2 V_4}{\partial\eta^2}+(1-\nu)\frac{\partial^2 V_4}{\partial\xi^2}+(1+\nu)\frac{\partial^2 U_4}{\partial\xi\partial\eta}=-(1-\nu)\left\{\frac{\partial W_1}{\partial\eta}\left(\frac{\partial^2 W_3}{\partial\xi^2}+\lambda^2\frac{\partial^2 W_3}{\partial\eta^2}\right)\right.$$

$$\left.+\frac{\partial W_3}{\partial\eta}\left(\frac{\partial^2 W_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 W_1}{\partial\eta^2}\right)\right\}-(1+\nu)\frac{\partial}{\partial\eta}\left[\frac{\partial W_1}{\partial\xi}\frac{\partial W_3}{\partial\xi}+\lambda^2\frac{\partial W_1}{\partial\eta}\frac{\partial W_3}{\partial\eta}\right] \quad (3.5c)$$

求解(3.5a,b,c)的边界条件为

$$W_3(0,\ 0)=0 \quad (3.6a)$$

$$W_3(\xi,\ \eta)=U_4(\xi,\ \eta)=V_4(\xi,\ \eta)=\partial W_3(\xi,\ \eta)/\partial n=0, \qquad \text{在椭圆}\xi^2+\eta^2=1\text{上} \quad (3.6b,c,d,e)$$

依次类推可以逐级求解，按圆薄板的经验，求得两级近似解就足够了。

四、一级近似解

先求(3.3a)在(3.4a,b,e)条件下的解。满足(3.4b，e)的解可以写成

$$W_1(\xi,\ \eta)=A_1^*(1-\xi^2-\eta^2)^2 \qquad A_1^*=\text{待定常数} \quad (4.1)$$

代入(3.3a)，得

$$A_1^*=a_1 \quad (4.2)$$

但是，把(4.1)用于(3.4a)，即得

$$A_1^*=1 \tag{4.3a}$$

从而，得待定常数α_1:

$$\alpha_1=1 \tag{4.3b}$$

于是，得(3.3a)在(3.4a,b,e)条件下的解（一级近似解）

$$W_1(\xi,\ \eta)=(1-\xi^2-\eta^2)^2,\qquad \alpha_1=1 \tag{4.4}$$

把(4.4)式的$W_1(\xi,\ \eta)$代入(3.3b，c)，求得决定$U_2(\xi,\ \eta)$和$V_2(\xi,\ \eta)$的两个联立方程式：

$$2\frac{1}{\lambda^2}\frac{\partial^2 U_2}{\partial\xi^2}+(1-\nu)\frac{\partial^2 U_2}{\partial\eta^2}+(1+\nu)\frac{\partial^2 V_2}{\partial\xi\partial\eta}$$
$$=16\xi(1-\xi^2-\eta^2)\left\{\left[\frac{6}{\lambda^2}+(1-\nu)\right]\xi^2+\left[\frac{2}{\lambda^2}+5-\nu\right]\eta^2-\frac{2}{\lambda^2}-(1-\nu)\right\} \tag{4.5a}$$

$$2\lambda^2\frac{\partial^2 V_2}{\partial\eta^2}+(1-\nu)\frac{\partial^2 V_2}{\partial\xi^2}+(1+\nu)\frac{\partial^2 U_2}{\partial\xi\partial\eta}$$
$$=16\eta(1-\eta^2-\xi^2)\{[6\lambda^2+(1-\nu)]\eta^2+[2\lambda^2+5-\nu]\xi^2-2\lambda^2-(1-\nu)\} \tag{4.5b}$$

从上式(4.5a，b)中可以看到，如果把(4.5a)式中的U_2，V_2，ξ，η，$1/\lambda^2$分别换成V_2,U_2，η，ξ，λ^2，即得(4.5b)式，也即是说，如果U_2为ξ，η，λ^2，ν的某一函数，如

$$U_2=f(\xi,\ \eta,\ \lambda^2,\ \nu) \tag{4.6a}$$

则V_2必为η，ξ，$1/\lambda^2$，ν的同一函数，即

$$V_2=f(\eta,\ \xi,\ 1/\lambda^2,\ \nu) \tag{4.6b}$$

反之亦然，即

$$V_2=g(\xi,\ \eta,\ \lambda^2,\ \nu),\qquad U_2=g(\eta,\ \xi,\ 1/\lambda^2,\ \nu) \tag{4.7a, b}$$

根据变形位移的边界条件(3.4c，d)，以及对称性，我们可以设(4.5a，b)的解具有下列形式。

$$U_2(\xi,\ \eta)=\xi(1-\xi^2-\eta^2)\{A_1\xi^4+B_1\xi^2\eta^2+C_1\eta^4+D_1\xi^2+E_1\eta^2+F_1\} \tag{4.8a}$$

$$V_2(\xi,\ \eta)=\eta(1-\eta^2-\xi^2)\{A_2\eta^4+B_2\eta^2\xi^2+C_2\xi^4+D_2\eta^2+E_2\xi^2+F_2\} \tag{4.8b}$$

其中待定系数A_1，A_2，B_1，B_2，C_1，C_2，D_1，D_2，E_1，E_2，F_1，F_2都是λ^2，ν的函数。而且根据(4.6a，b)或(4.7a，b)，如果A_1是λ^2，ν的某一函数，则A_2必为$1/\lambda^2$，ν的同一函数，即

$$A_1=A_1(\lambda^2,\ \nu),\qquad A_2=A_1(1/\lambda^2,\ \nu) \tag{4.9a, b}$$

B_1和B_2，C_1和C_2，D_1和D_2，E_1和E_2，F_1和F_2都有相同的对应关系，我们很易看到(4.8a,b)满足边界条件(3.4c，d)。把(4.8a，b)代入(4.5a，b)，得

$$\left[\frac{42}{\lambda^2}A_1+(1-\nu)(A_1+B_1)+3(1+\nu)C_2\right]\xi^4+\left[\frac{20}{\lambda^2}(A_1+B_1)+6(1-\nu)(B_1+C_1)\right.$$
$$\left.+6(1+\nu)(B_2+C_2)\right]\xi^2\eta^2+\left[\frac{6}{\lambda^2}(B_1+C_1)+15(1-\nu)C_1+5(1+\nu)(A_2+B_2)\right]\eta^4$$
$$+\left[\frac{20}{\lambda^2}(D_1-A_1)+(1-\nu)(D_1+E_1-B_1)+2(1+\nu)(E_2-C_2)\right]\xi^2$$
$$+\left[\frac{6}{\lambda^2}(D_1+E_1-B_1)+6(1-\nu)(E_1-C_1)+3(1+\nu)(D_2+E_2-B_2)\right]\eta^2$$

$$+\left[\frac{6}{\lambda^2}(F_1-D_1)+(1-\nu)(F_1-E_1)+(1+\nu)(F_2-E_2)\right]$$

$$=-8(1-\xi^2-\eta^2)\left\{\left[\frac{6}{\lambda^2}+1-\nu\right]\xi^2+\left[\frac{2}{\lambda^2}+5-\nu\right]\eta^2-\frac{2}{\lambda^2}-(1-\nu)\right\} \tag{4.10a}$$

$$\begin{aligned}&[42\lambda^2A_2+(1-\nu)(A_2+B_2)+3(1+\nu)C_1]\eta^4+[20\lambda^2(A_2+B_2)+6(1-\nu)(B_2+C_2)\\&\quad+6(1+\nu)(B_1+C_1)]\xi^2\eta^2+[6\lambda^2(B_2+C_2)+15(1-\nu)C_2+5(1+\nu)(B_1+A_1)]\xi^4\\&\quad+[20\lambda^2(D_2-A_2)+(1-\nu)(D_2+E_2-B_2)+2(1+\nu)(E_1-C_1)]\eta^2\\&\quad+[6\lambda^2(D_2+E_2-B_2)+6(1-\nu)(E_2-C_2)+3(1+\nu)(D_1+E_1-B_1)]\xi^2\\&\quad+[6\lambda^2(F_2-D_2)+(1-\nu)(F_2-E_2)+(1+\nu)(F_1-E_1)]\\&=-8(1-\xi^2-\eta^2)\{[6\lambda^2+1-\nu]\eta^2+[2\lambda^2+5-\nu]\xi^2-2\lambda^2-(1-\nu)\}\end{aligned} \tag{4.10b}$$

(4.10a,b)两式在$\xi^2+\eta^2\leqslant 1$域内各点上都适用．所以，(4.10a)两边的$\xi^4,\xi^2\eta^2$，η^4，ξ^2，η^2，1各项系数必互等；于是从(4.10a)得

$$\frac{42}{\lambda^2}A_1+(1-\nu)(A_1+B_1)+3(1+\nu)C_2=8\left[\frac{6}{\lambda^2}+1-\nu\right] \tag{4.11a}$$

$$\frac{20}{\lambda^2}(A_1+B_1)+6(1-\nu)(B_1+C_1)+6(1+\nu)(B_2+C_2)=8\left[\frac{8}{\lambda^2}+2(3-\nu)\right] \tag{4.11b}$$

$$\frac{6}{\lambda^2}(B_1+C_1)+15(1-\nu)C_1+5(1+\nu)(A_2+B_2)=8\left[\frac{2}{\lambda^2}+5-\nu\right] \tag{4.11c}$$

$$\frac{20}{\lambda^2}(D_1-A_1)+(1-\nu)(D_1+E_1-B_1)+2(1+\nu)(E_2-C_2)=-8\left[\frac{8}{\lambda^2}+2(1-\nu)\right] \tag{4.11d}$$

$$\frac{6}{\lambda^2}(D_1+E_1-B_1)+6(1-\nu)(E_1-C_1)+3(1+\nu)(D_2+E_2-B_2)$$

$$=-8\left[\frac{4}{\lambda^2}+2(3-\nu)\right] \tag{4.11e}$$

$$\frac{6}{\lambda^2}(F_1-D_1)+(1-\nu)(F_1-E_1)+(1+\nu)(F_2-E_2)=8\left[\frac{2}{\lambda^2}+1-\nu\right] \tag{4.11f}$$

同理(4.10b)两边的η^4，$\eta^2\xi^2$，ξ^4，η^2，ξ^2，1各项系数也必互等，得

$$42\lambda^2A_2+(1-\nu)(A_2+B_2)+3(1+\nu)C_1=8[6\lambda^2+1-\nu] \tag{4.12a}$$

$$20\lambda^2(A_2+B_2)+6(1-\nu)(B_2+C_2)+6(1+\nu)(B_1+C_1)=8[8\lambda^2+2(3-\nu)] \tag{4.12b}$$

$$6\lambda^2(B_2+C_2)+15(1-\nu)C_2+5(1+\nu)(A_1+B_1)=8[2\lambda^2+5-\nu] \tag{4.12c}$$

$$20\lambda^2(D_2-A_2)+(1-\nu)(D_2+E_2-B_2)+2(1+\nu)(E_1-C_1)=-8[8\lambda^2+2(1-\nu)] \tag{4.12d}$$

$$6\lambda^2(D_2+E_2-B_2)+6(1-\nu)(E_2-C_2)+3(1+\nu)(D_1+E_1-B_1)=-8[4\lambda^2+2(3-\nu)] \tag{4.12e}$$

$$6\lambda^2(F_2-D_2)+(1-\nu)(F_2-E_2)+(1+\nu)(F_1-E_1)=8[2\lambda^2+1-\nu] \tag{4.12f}$$

(4.11)，(4.12)各式可以分成两个独立部分：(4.11a,b,c)，(4.12a,b,c) 6个方程式，它们可以用来求解6个待定常量A_1，B_1，C_1，A_2，B_2，C_2．把求得的结果代入(4.11d,e,f)和(4.12d,e,f)，就得求解D_1，E_1，F_1，D_2，E_2，F_2的6个方程式．这样就求得了所有12个待定常量．从而求得了(4.8a，b)所代表的$U(\xi,\eta)$，$V(\xi,\eta)$的一级近似解$U_2(\xi,\eta)$，

$V_2(\xi, \eta)$。

从(4.11a，b)消去C_2，得用A_1，B_1，C_1表示的B_2的表达式，再把它代入(4.11c)，求得用A_1，B_1，C_1表示的A_2的表达式。把A_2，B_2的表达式和(4.11a)的C_2的表达式放在一起，构成下列矩阵方程。

$$15(1+\nu)\mathbf{\Phi}_2+\boldsymbol{\alpha}(1/\lambda^2)\mathbf{\Phi}_1=\mathbf{Q}_1(1/\lambda^2) \tag{4.13}$$

其中$\mathbf{\Phi}_1$，$\mathbf{\Phi}_2$，$\boldsymbol{\alpha}(1/\lambda^2)$，$\mathbf{Q}(1/\lambda^2)$分别为下列矩阵，而且$\boldsymbol{\alpha}(1/\lambda^2)$，$\mathbf{Q}_1(1/\lambda^2)$都是$1/\lambda^2$的函数

$$\mathbf{\Phi}_1=\begin{Bmatrix} A_1 \\ B_1 \\ C_1 \end{Bmatrix},\quad \mathbf{\Phi}_2=\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix},\quad \mathbf{Q}_1\left(\frac{1}{\lambda^2}\right)=\begin{Bmatrix} 128/\lambda^2+8(5-3\nu) \\ -80/\lambda^2+80 \\ 240/\lambda^2+40(1-\nu) \end{Bmatrix} \tag{4.14a,b,c}$$

$$\boldsymbol{\alpha}\left(\frac{1}{\lambda^2}\right)=\begin{bmatrix} 160/\lambda^2+5(1-\nu) & -32/\lambda^2-10(1-\nu) & 18/\lambda^2+30(1-\nu) \\ -160/\lambda^2-5(1-\nu) & 50/\lambda^2+10(1-\nu) & 15(1-\nu) \\ 210/\lambda^2+5(1-\nu) & 5(1-\nu) & 0 \end{bmatrix} \tag{4.14d}$$

同样，从(4.12a，b，c)中通过相似的运算，可以求得下列矩阵方程

$$15(1+\nu)\mathbf{\Phi}_1+\boldsymbol{\alpha}(\lambda^2)\mathbf{\Phi}_2=\mathbf{Q}_1(\lambda^2) \tag{4.15}$$

其中，$\mathbf{\Phi}_1$，$\mathbf{\Phi}_2$见(4.14a，b)，而$\boldsymbol{\alpha}(\lambda^2)$，$\mathbf{Q}_1(\lambda^2)$则和(4.14c，d)中的$\boldsymbol{\alpha}(1/\lambda^2)$，$\mathbf{Q}_1(1/\lambda^2)$相似，只要把(4.14c，d)中的$1/\lambda^2$用$\lambda^2$代替就得到了。

从(4.13)，(4.15)，我们解得$\mathbf{\Phi}_1$，$\mathbf{\Phi}_2$

$$\mathbf{\Phi}_1=[225(1-\nu)^2\mathbf{I}-\boldsymbol{\alpha}(\lambda^2)\boldsymbol{\alpha}(1/\lambda^2)]^{-1}\{-\boldsymbol{\alpha}(\lambda^2)\mathbf{Q}_1(1/\lambda^2)+15(1+\nu)\mathbf{Q}_1(\lambda^2)\} \tag{4.16a}$$

$$\mathbf{\Phi}_2=[225(1+\nu)^2\mathbf{I}-\boldsymbol{\alpha}(1/\lambda^2)\boldsymbol{\alpha}(\lambda^2)]^{-1}\{-\boldsymbol{\alpha}(1/\lambda^2)\mathbf{Q}_1(\lambda^2)+15(1+\nu)\mathbf{Q}_1(1/\lambda^2)\} \tag{4.16b}$$

其中$\mathbf{I}$为对角线单元矩阵。

用(4.11a，b)式消去(4.11d，e，f)中的B_2，C_2后，可以导出下列矩阵方程：

$$6(1+\nu)\mathbf{\Psi}_2+\boldsymbol{\beta}(1/\lambda^2)\mathbf{\Phi}_1+\boldsymbol{\gamma}(1/\lambda^2)\mathbf{\Psi}_1=-\mathbf{Q}_2(1/\lambda^2) \tag{4.17}$$

其中$\mathbf{\Phi}_1$见(4.14a)，$\mathbf{\Psi}_1$，$\mathbf{\Psi}_2$，$\mathbf{Q}_2(1/\lambda^2)$分别为

$$\mathbf{\Psi}_1=\begin{Bmatrix} D_1 \\ E_1 \\ F_1 \end{Bmatrix},\quad \mathbf{\Psi}_2=\begin{Bmatrix} D_2 \\ E_2 \\ F_2 \end{Bmatrix},\quad \mathbf{Q}_2\left(\frac{1}{\lambda^2}\right)=\begin{Bmatrix} 32 \\ 96/\lambda^2+32(1-\nu) \\ -16(1-\nu) \end{Bmatrix} \tag{4.18a,b,c}$$

$\boldsymbol{\beta}(1/\lambda^2)$，$\boldsymbol{\gamma}(1/\lambda^2)$都是$1/\lambda^2$的函数的矩阵

$$\boldsymbol{\beta}\left(\frac{1}{\lambda^2}\right)=\begin{bmatrix} -88/\lambda^2-4(1-\nu) & 8/\lambda^2+5(1-\nu) & -6(1-\nu) \\ 24/\lambda^2+2(1-\nu) & -(1-\nu) & 0 \\ 24/\lambda^2+2(1-\nu) & -(1-\nu) & 0 \end{bmatrix} \tag{4.18d}$$

$$\boldsymbol{\gamma}\left(\frac{1}{\lambda^2}\right)=\begin{bmatrix} -48/\lambda^2-3(1-\nu) & 12/\lambda^2+9(1-\nu) & 0 \\ 60/\lambda^2+3(1-\nu) & 3(1-\nu) & 0 \\ 24/\lambda^2+3(1-\nu) & -3(1-\nu) & 36/\lambda^2+6(1-\nu) \end{bmatrix} \tag{4.18e}$$

同样，从(4.12d,e,f)中可以导出

$$6(1+\nu)\mathbf{\Psi}_1+\boldsymbol{\beta}(\lambda^2)\mathbf{\Phi}_2+\boldsymbol{\gamma}(\lambda^2)\mathbf{\Psi}_2=-\mathbf{Q}_2(\lambda^2) \tag{4.19}$$

其中$\mathbf{\Phi}_2$见(4.14b)，$\mathbf{\Psi}_1$，$\mathbf{\Psi}_2$见(4.18a，b)，$\boldsymbol{\beta}(\lambda^2)$，$\boldsymbol{\gamma}(\lambda^2)$，$\mathbf{Q}_2(\lambda^2)$诸矩阵见(4.18c，d，e)，但将(4.18)中的$1/\lambda^2$改写为λ^2。

从(4.17)，(4.19)中，可解出$\mathbf{\Psi}_1$，$\mathbf{\Psi}_2$，得

$$\mathbf{\Psi}_1=[36(1+\nu)^2\mathbf{I}-\boldsymbol{\gamma}(\lambda^2)\boldsymbol{\gamma}(1/\lambda^2)]^{-1}\{-6(1+\nu)\mathbf{Q}_2(\lambda^2)+\boldsymbol{\gamma}(\lambda^2)\mathbf{Q}_2(1/\lambda^2)$$

$$+6(1+\nu)\beta(\lambda^2)\Phi_2+\Upsilon(\lambda^2)\beta(1/\lambda^2)\Phi_1\} \tag{4.20a}$$

$$\Psi_2=[36(1+\nu)^2\mathrm{I}-\Upsilon(1/\lambda^2)\Upsilon(\lambda^2)]^{-1}\{-6(1+\nu)Q_2(1/\lambda^2)+\Upsilon(1/\lambda^2)Q_2(\lambda^2)$$

$$+6(1+\nu)\beta(1/\lambda^2)\Phi_1+\Upsilon(1/\lambda^2)\beta(\lambda^2)\Phi_2\} \tag{4.20b}$$

其中Φ_1，Φ_2是(4.16a，b)求得的．于是(4.16)，(4.20)给出了A_1，B_1，C_1，A_2，B_2，C_2，D_1，E_1，F_1，D_2，E_2，F_2等12个待定系数．这就求得了U_2，V_2，即U，V的一级近似解．

这里应该指出，当$\lambda^2=1/\lambda^2=1$时，弹性椭圆板归化为圆薄板．我们从它的对称性，可以设

$$A_2=C_1=C_2=A_1, \qquad B_1=B_2=2A_1 \tag{4.21}$$

于是(4.11a，b，c)和(4.12a，b，c)都可以化为

$$A_1=(7-\nu)/6 \tag{4.22}$$

亦即

$$A_1=A_2=C_1=C_2=(7-\nu)/6, \qquad B_1=B_2=(7-\nu)/3 \tag{4.23}$$

如果把(4.23)的结果代入(4.11d，e，f)，(4.12d，e，f)，取$\lambda^2=1/\lambda^2=1$，而且

$$D_1=E_1=D_2=E_2, \qquad F_1=F_2 \tag{4.24}$$

则(4.11d，e)，(4.12d，e)都相等，给出

$$D_1=E_1=D_2=E_2=-(13-3\nu)/6 \tag{4.25}$$

而(4.11f)，(4.12f)给出

$$F_1-D_1=3-\nu \tag{4.26}$$

在利用了(4.25)后，上式给出

$$F_1=F_2=(5-3\nu)/6 \tag{4.27}$$

于是，对圆薄板($\lambda^2=1/\lambda^2=1$)而言，

$$\frac{U_2(\xi,\eta)}{\xi}=\frac{V_2(\xi,\eta)}{\eta}=\frac{1}{6}(1-\xi^2-\eta^2)\{(7-\nu)(1-\xi^2-\eta^2)^2-(1+\nu)(1-\xi^2-\eta^2)$$

$$-(1+\nu)\} \tag{4.28}$$

这和圆薄板(1948，钱伟长)[1]的结果相同．表1(A)，1(B)，1(C)分别为当ν取0.25,0.33,0.35时，(4.8a，b)中诸系数的值．

表1(A) $\nu=0.25$**时，(4.8a，b)**U_2，V_2**中诸系数的值**

λ	A_1	B_1	C_1	D_1	E_1	F_1
1	1.12500	2.25000	1.12500	−2.04167	−2.04167	0.70834
2	1.15057	2.79600	1.39835	−2.15911	−2.92955	0.88968
3	1.18189	3.17442	1.49157	−2.25220	−3.30192	1.00153
4	1.20011	3.37940	1.53371	−2.30863	−3.47115	1.06840
5	1.20975	3.49342	1.55575	−2.34384	−3.55892	1.11010
λ	A_2	B_2	C_2	D_2	E_2	F_2
1	1.12500	2.25000	1.12500	−2.04167	−2.04167	0.70834
2	1.13309	2.06325	0.78911	−2.02992	−1.42616	0.62377
3	1.13762	2.04430	0.68017	−2.04007	−1.29319	0.61164
4	1.13968	2.04486	0.64402	−2.04620	−1.25386	0.60930
5	1.14074	2.04744	0.62940	−2.04968	−1.23862	0.60884

表1(B) $\nu=0.30$时，(4.8a, b)U_2, V_2中诸系数的值

λ	A_1	B_1	C_1	D_1	E_1	F_1
1	1.11667	2.23333	1.11667	−2.01667	−2.01667	0.68333
2	1.12585	2.78453	1.39402	−2.08489	−2.92156	0.81479
3	1.13279	3.17261	1.48901	−2.10788	−3.30292	0.87458
4	1.12114	3.38259	1.53210	−2.08840	−3.47524	0.89767
5	1.09958	3.49866	1.55467	−2.05252	−3.56378	0.90489
λ	A_2	B_2	C_2	D_2	E_2	F_2
1	1.11667	2.23333	1.11667	−2.01667	−2.01667	0.68333
2	1.13057	2.05329	0.77981	−2.02120	−1.40966	0.61730
3	1.13643	2.03825	0.67519	−2.03573	−1.28684	0.60949
4	1.13899	2.04100	0.64216	−2.04365	−1.25235	0.60856
5	1.14030	2.04481	0.62922	−2.04803	−1.23919	0.60862

表1(C) $\nu=0.35$时，(4.8a, b)U_2, V_2中诸系数的值

λ	A_1	B_1	C_1	D_1	E_1	F_1
1	1.10833	2.21667	1.10833	−1.99167	−1.99167	0.65833
2	1.10100	2.77331	1.38966	−2.00963	−2.91403	0.73711
3	1.08266	3.17142	1.48643	−1.95854	−3.30471	0.73948
4	1.03923	3.38664	1.53048	−1.85643	−3.48026	0.71275
5	0.98369	3.50486	1.55358	−1.74105	−3.56961	0.67981
λ	A_2	B_2	C_2	D_2	E_2	F_2
1	1.10833	2.21667	1.10833	−1.99167	−1.99167	0.65833
2	1.12804	2.04328	0.77066	−2.01244	−1.39371	0.61122
3	1.13524	2.03215	0.67057	−2.03137	−1.28142	0.60775
4	1.13830	2.03710	0.64080	−2.04110	−1.25190	0.60818
5	1.13985	2.04217	0.62959	−2.04636	−1.24083	0.60871

从上表可以看到，在圆薄板时，$\lambda^2=1/\lambda^2=1$，表中各系数和(4.23)，(4.25)，(4.27)完全相同。

以上是一级近似的解。

五、二级近似的位移解

把一级近似解(4.4)，(4.8a, b)代入二级近似方程(3.5a)，即得求解$W_3(\xi, \eta)$，α_3的微分方程：

$$\begin{aligned}
&\frac{\partial^4 W_3}{\partial\xi^4}+2\lambda^2\frac{\partial^4 W_3}{\partial\xi^2\partial\eta^2}+\lambda^4\frac{\partial^4 W_3}{\partial\eta^4}-8(3+2\lambda^2+3\lambda^4)\alpha_3\\
&=48(1-3\xi^2-\eta^2)\{(7A_1+\nu\lambda^2C_2)\xi^6+[5(A_1+B_1)+3\nu\lambda^2(B_2+C_2)]\xi^4\eta^2\\
&\quad+[3(B_1+C_1)+5\lambda^2\nu(A_2+B_2)]\eta^4\xi^2+(C_1+7\nu\lambda^2A_2)\eta^6+[-5(A_1-D_1)\\
&\quad-\nu\lambda^2(C_2-E_2)]\xi^4+[-3(B_1-D_1-E_1)-3\nu\lambda^2(B_2-D_2-E_2)]\xi^2\eta^2\\
&\quad-[(C_1-E_1)+5\nu\lambda^2(A_2-D_2)]\eta^4-[3(D_1-F_1)+\nu\lambda^2(E_2-F_2)]\xi^2\\
&\quad-[(E_1-F_1)+3\nu\lambda^2(D_2-F_2)]\eta^2-[F_1+\nu\lambda^2F_2]\}\\
&\quad+48\lambda^2(1-\xi^2-3\eta^2)\{(7\nu A_1+\lambda^2C_2)\xi^6+[5\nu(A_1+B_1)+3\lambda^2(B_2+C_2)]\xi^4\eta^2
\end{aligned}$$

$$+[3\nu(B_1+C_1)+5\lambda^2(A_2+B_2)]\eta^4\xi^2+[\nu C_1+7\lambda^2A_2]\eta^6+[-5\nu(A_1-D_1)$$
$$-\lambda^2(C_2-E_2)]\xi^4$$
$$+[-3\nu(B_1-D_1-E_1)-3\lambda^2(B_2-D_2-E_2)]\xi^2\eta^2+[-\nu(C_1-E_1)-5\lambda^2(A_2-D_2)]\eta^4$$
$$+[-3\nu(D_1-F_1)-\lambda^2(E_2-F_2)]\xi^2+[-\nu(E_1-F_1)-3\lambda^2(D_2-F_2)]\eta^2$$
$$-[\nu F_1+\lambda^2F_2]\}$$
$$-192(1-\nu)\lambda^2\xi\eta\{(A_1+B_1+3C_2)\eta\xi^5+2(B_1+C_1+B_2+C_2)\xi^3\eta^3$$
$$+(3C_1+A_2+B_2)\eta^5\xi+[-B_1+E_1+D_1-2(C_2-E_2)]\eta\xi^3+[-2(C_1-E_1)$$
$$-(B_2-E_2-D_2)]\xi\eta^3+(-E_1+F_1-E_2+F_2)\eta\xi\}$$
$$+384(1-\xi^2-\eta^2)^2\{(3+\nu\lambda^2)\xi^4+[1+2(2+\nu)\lambda^2+\lambda^4]\xi^2\eta^2$$
$$+(\nu+3\lambda^2)\lambda^2\eta^4-(1+\nu\lambda^2)\xi^2-(\nu+\lambda^2)\lambda^2\eta^2\} \qquad (5.1)$$

同时，设$W_3(\xi,\ \eta)$的解可以写成

$$W_3(\xi,\ \eta)=(1-\xi^2-\eta^2)^2\{G\xi^8+H\xi^6\eta^2+I\xi^4\eta^4+J\xi^2\eta^6+K\eta^8+M\xi^6$$
$$+N\xi^4\eta^2+L\xi^2\eta^4+P\eta^6+Q\xi^4+R\xi^2\eta^2+S\eta^4+T\xi^2+X\eta^2\} \qquad (5.2)$$

其中G, H, I, J, K, M, N, L, P, Q, R, S, T, X为14个待定常数。这里可以看到(5.2)式的$W_3(\xi,\ \eta)$满足边界条件(3.6a, b, e)。把(5.2)式代入(5.1)式的左边，得

$$\frac{\partial^4W_3}{\partial\xi^4}+2\lambda^2\frac{\partial^4W_3}{\partial\xi^2\partial\eta^2}+\lambda^4\frac{\partial^4W_3}{\partial\eta^4}$$

$$=24\xi^8\{(495+30\lambda^2+\lambda^4)G+(15+2\lambda^2)\lambda^2H+\lambda^4I\}+24\xi^6\eta^2\{(420+56\lambda^2)G$$
$$+(210+112\lambda^2+15\lambda^4)H+(56+30\lambda^2)\lambda^2I+15\lambda^4J\}+120\xi^4\eta^4\{14G+(28+15\lambda^2)H$$
$$+(14+30\lambda^2+14\lambda^4)I+(15+28\lambda^2)\lambda^2J+14\lambda^4K\}+24\xi^2\eta^6\{15H+(30+56\lambda^2)I$$
$$+(15+112\lambda^2+210\lambda^4)J+(56+420\lambda^2)\lambda^2K\}+24\eta^8\{I+(2+15\lambda^2)J$$
$$+(1+30\lambda^2+495\lambda^4)K\}+8\xi^6\{-(1260+56\lambda^2)G-6\lambda^4I-(56+6\lambda^2)\lambda^2H$$
$$+(630+56\lambda^2+3\lambda^4)M+(28+6\lambda^2)\lambda^2N+3\lambda^4L\}+120\xi^4\eta^2\{-28G-(28+12\lambda^2)H$$
$$-(12+6\lambda^2)\lambda^2I-6\lambda^4J+6(1+\lambda^2)\lambda^2L+(28+6\lambda^2)M+(14+12\lambda^2+3\lambda^4)N$$
$$+3\lambda^4P\}+120\xi^2\eta^4\{-6H-(6+12\lambda^2)I-(12+28\lambda^2)\lambda^2J-28\lambda^4K+3M$$
$$+6(1+\lambda^2)N+(3+12\lambda^2+14\lambda^4)L+(6\lambda^2+28\lambda^4)P\}+8\eta^6\{-6I-(6+56\lambda^2)J$$
$$+3N+(6+28\lambda^2)L-(56+1260\lambda^2)\lambda^2K+(3+56\lambda^2+630\lambda^4)P\}$$
$$+24\xi^4\{70G+5\lambda^2H+\lambda^4I-(140+10\lambda^2)M-(10+2\lambda^2)\lambda^2N-2\lambda^4L$$
$$+(70+10\lambda^2+\lambda^4)Q+(5+2\lambda^2)\lambda^2R+\lambda^4S\}$$
$$+72\xi^2\eta^2\{5H+4\lambda^2I+5\lambda^4J-(8+10\lambda^2)\lambda^2L-10M$$
$$-(10+8\lambda^2)N+(10+4\lambda^2)Q-10\lambda^4P+(5+8\lambda^2+5\lambda^4)R+(4+10\lambda^2)\lambda^2S\}$$
$$+24\eta^4\{I+5\lambda^2J+70\lambda^4K-(2+10\lambda^2)L-2N-(10+140\lambda^2)\lambda^2P+Q+(2+5\lambda^2)R$$
$$+(1+10\lambda^2+70\lambda^4)S\}+24\xi^2\{\lambda^4L+15M+2\lambda^2N-(30+4\lambda^2)Q-2(2+\lambda^2)\lambda^2R$$
$$-2\lambda^4S+(15+4\lambda^2+\lambda^4)T+2(1+\lambda^2)\lambda^2X\}+24\eta^2\{2\lambda^2L+N+15\lambda^4P-2Q$$
$$-2(1+2\lambda^2)R-(4+30\lambda^2)\lambda^2S+2(1+\lambda^2)T+(1+4\lambda^2+15\lambda^4)X\}$$
$$+8\{3Q+\lambda^2R+3\lambda^4S-2(3+\lambda^2)T-2(1+3\lambda^2)\lambda^2X\} \qquad (5.3)$$

把(5.1)和(5.3)的各项系数相比较，ξ^8, $\xi^6\eta^2$, $\xi^4\eta^4$, $\xi^2\eta^6$, η^8的系数给出含有待定量G, H, I, J, K的五个方程式

$$\left.\begin{aligned}&(495+30\lambda^2+\lambda^4)G+(15+2\lambda^2)\lambda^2H+\lambda^4I=f_1\\&(420+56\lambda^2)G+(210+112\lambda^2+15\lambda^4)H+(56+30\lambda^2)\lambda^2I+15\lambda^4J=f_2\\&70G+(140+75\lambda^2)H+(70+150\lambda^2+70\lambda^4)I+(75+140\lambda^2)\lambda^2J+70\lambda^4K=f_3\\&15H+(30+56\lambda^2)I+(15+112\lambda^2+210\lambda^4)J+(56+420\lambda^2)\lambda^2K=f_4\\&I+(2+15\lambda^2)J+(1+30\lambda^2+495\lambda^4)K=f_5\end{aligned}\right\}\tag{5.4}$$

其中$f_i(i=1,2,3,4,5)$是λ^2和ν的函数，它们可以从(5.1)式的右端求得：

$$f_1=-6(7A_1+\nu\lambda^2C_2)-2\lambda^2(7\nu A_1+\lambda^2C_2)+16(3+\nu\lambda^2)\tag{5.5a}$$

$$\begin{aligned}f_2=&-2(7A_1+\nu\lambda^2C_2)-6[5(A_1+B_1)+3\nu\lambda^2(C_2+B_2)]-2\lambda^2[5\nu(A_1+B_1)+3\lambda^2(B_2+C_2)]\\&-6\lambda^2(7\nu A_1+\lambda^2C_2)-8(1-\nu)\lambda^2(A_1+B_1+3C_1)+32(3+\nu\lambda^2)\\&+16[1+2(2+\nu)\lambda^2+\lambda^4]\end{aligned}\tag{5.5b}$$

$$\begin{aligned}f_3=&-6[3(B_1+C_1)+5\nu\lambda^2(A_2+B_2)]-2[5(A_1+B_1)+3\nu\lambda^2(B_2+C_2)]-6\lambda^2[5\nu(A_1+B_1)\\&+3\lambda^2(C_2+B_2)]+2\lambda^2[3\nu(B_1+C_1)+5\lambda^2(A_2+B_2)]-16(1-\nu)\lambda^2[(B_1+C_1)\\&+(B_2+C_2)]+16(3+\nu\lambda^2)+32[1+2(2+\nu)\lambda^2+\lambda^4]+16(\nu+3\lambda^2)\lambda^2\end{aligned}\tag{5.5c}$$

$$\begin{aligned}f_4=&-6(C_1+7\nu\lambda^2A_2)-2[3(B_1+C_1)+5\nu\lambda^2(A_2+B_2)]-6\lambda^2[3\nu(B_1+C_1)\\&+5\lambda^2(A_2+B_2)]-2\lambda^2(\nu C_1+7\lambda^2A_2)-8(1-\nu)\lambda^2[3C_1+A_2+B_2]+32(\nu+3\lambda^2)\lambda^2\\&+16[1+2(2+\nu)\lambda^2+\lambda^4]\end{aligned}\tag{5.5d}$$

$$f_5=-2(C_1+7\nu\lambda^2A_2)-6\lambda^2(\nu C_1+7\lambda^2A_2)+16(\nu+3\lambda^2)\lambda^2\tag{5.5e}$$

我们必须指出，A_1，B_1，C_1，A_2，B_2，C_2已见(4.16a,b)，所以(5.5)式中的$f_i\ (i=1,2,3,4,5)$都是已知的．因而(5.4)式为求G，H，I，J，K五个待定系数的线性代数联立方程式，它可以用矩阵形式写成：

$$\boldsymbol{\mu}_1\mathbf{X}_1=\boldsymbol{\Omega}_1\tag{5.6}$$

其中

$$\mathbf{X}_1=(G\ \ H\ \ I\ \ J\ \ K)^T,\qquad \boldsymbol{\Omega}_1=(f_1\ \ f_2\ \ f_3\ \ f_4\ \ f_5)^T\tag{5.7a, b}$$

$$\boldsymbol{\mu}_1=\begin{bmatrix}495+30\lambda^2+\lambda^4 & (15+2\lambda^2)\lambda^2 & \lambda^4 & 0 & 0\\420+56\lambda^2 & 210+112\lambda^2+15\lambda^4 & (56+30\lambda^2)\lambda^2 & 15\lambda^4 & 0\\70 & 140+75\lambda^2 & 70+150\lambda^2+70\lambda^4 & (75+140\lambda^2)\lambda^2 & 70\lambda^4\\0 & 15 & 30+56\lambda^2 & 15+112\lambda^2+210\lambda^4 & (56+420\lambda^2)\lambda^2\\0 & 0 & 1 & 2+15\lambda^2 & 1+30\lambda^2+495\lambda^4\end{bmatrix}\tag{5.7c}$$

(5.6)式的解可以写为

$$\mathbf{X}_1=\boldsymbol{\mu}_1^{-1}\boldsymbol{\Omega}_1\tag{5.8}$$

(5.1)和(5.3)中，ξ^6，$\xi^4\eta^2$，$\xi^2\eta^4$，η^6各系数给出

$$\begin{aligned}&(1260+56\lambda^2)G+(56+6\lambda^2)\lambda^2H+6\lambda^4I-(630+56\lambda^2+3\lambda^4)M-(28+6\lambda^2)\lambda^2N\\&\quad-3\lambda^4L=f_6\end{aligned}\tag{5.9a}$$

$$\begin{aligned}&28G+(28+12\lambda^2)H+(12+6\lambda^2)\lambda^2I+6\lambda^4J-(28+6\lambda^2)M-(14+12\lambda^2+3\lambda^4)N\\&\quad-6\lambda^2(1+\lambda^2)L-3\lambda^4P=f_7\end{aligned}\tag{5.9b}$$

$$\begin{aligned}&6H+(6+12\lambda^2)I+(12+28\lambda^2)\lambda^2J+28\lambda^4K-3M-6(1+\lambda^2)N\\&\quad-(3+12\lambda^2+14\lambda^4)L-\lambda^2(6+28\lambda^2)P=f_8\end{aligned}\tag{5.9c}$$

$$6I+(6+56\lambda^2)J-3N-(6+28\lambda^2)L+(56+1260\lambda^2)\lambda^2K-(3+56\lambda^2+630\lambda^4)P=f_9\tag{5.9d}$$

其中G，H，I，J，K是已知的（见(5.8)），(5.9a,b,c,d)是决定L，M，N，P的四个线性代数联立方程式，用矩阵可以写成

$$\boldsymbol{\mu}_2\mathbf{X}_1-\boldsymbol{\theta}_2\mathbf{X}_2=-\frac{2}{5}\boldsymbol{\Omega}_2 \tag{5.10}$$

其中$\mathbf{X}_1$见(5.7a)，$\mathbf{X}_2$，$\boldsymbol{\mu}_2$，$\boldsymbol{\theta}_2$，$\boldsymbol{\Omega}_2$为

$$\mathbf{X}_2=\{M\ \ N\ \ L\ \ P\}^T,\qquad \boldsymbol{\Omega}_2=\{15f_6\ \ f_7\ \ f_8\ \ 15f_9\}^T \tag{5.11a, b}$$

$$\boldsymbol{\mu}_2=\begin{bmatrix}1260+56\lambda^2 & (56+6\lambda^2)\lambda^2 & 6\lambda^4 & 0 & 0\\ 28 & 28+12\lambda^2 & (12+6\lambda^2)\lambda^2 & 6\lambda^4 & 0\\ 0 & 6 & 6+12\lambda^2 & (12+28\lambda^2)\lambda^2 & 28\lambda^4\\ 0 & 0 & 6 & 6+56\lambda^2 & (56+1260\lambda^2)\lambda^2\end{bmatrix} \tag{5.11c}$$

$$\boldsymbol{\theta}_2=\begin{bmatrix}630+56\lambda^2+3\lambda^4 & (28+6\lambda^2)\lambda^2 & 3\lambda^4 & 0\\ 28+6\lambda^2 & 14+12\lambda^2+3\lambda^4 & 6\lambda^2(1+\lambda^2) & 3\lambda^4\\ 3 & 6(1+\lambda^2) & 3+12\lambda^2+14\lambda^4 & (6+28\lambda^2)\lambda^2\\ 0 & 3 & 6+28\lambda^2 & 3+56\lambda^2+630\lambda^4\end{bmatrix} \tag{5.11d}$$

解(5.10)，得

$$\mathbf{X}_2=\boldsymbol{\theta}_2^{-1}\{\boldsymbol{\mu}_2\mathbf{X}_1+2\boldsymbol{\Omega}_2/5\} \tag{5.12}$$

(5.11b)式中的f_6，f_7，f_8，f_9分别是

$$\begin{aligned}f_6=&(7A_1+\nu\lambda^2C_2)+3[5(A_1-D_1)+\nu\lambda^2(C_2-E_2)]+(7\nu A_1+\lambda^2C_2)\lambda^2\\&+[5\nu(A_1-D_1)\lambda^2+(C_2-E_2)\lambda^4]-8[2(3+\nu\lambda^2)+(1+\nu\lambda^2)]\end{aligned} \tag{5.13a}$$

$$\begin{aligned}f_7=&[5(A_1+B_1)+3\nu\lambda^2(B_2+C_2)]+9[B_1-D_1-E_1+\nu\lambda^2(B_2-D_2-E_2)]\\&+[5(A_1-D_1)+\nu\lambda^2(C_2-E_2)]+\lambda^2[5\nu(A_1+B_1)+3\lambda^2(B_2+C_2)]\\&+3\lambda^2[\nu(B_1-D_1-E_1)+\lambda^2(B_2-D_2-E_2)]+3\lambda^2[5\nu(A_1-D_1)+\lambda^2(C_2-E_2)]\\&+4(1-\nu)\lambda^2(B_1-E_1-D_1+2C_2-2E_2)-16[4+(4+3\nu)\lambda^2+\lambda^4]-8(2+3\nu\lambda^2+\lambda^4)\end{aligned} \tag{5.13b}$$

$$\begin{aligned}f_8=&[3(B_1+C_1)+5\nu\lambda^2(A_2+B_2)]+3[B_1-D_1-E_1+\nu\lambda^2(B_2-D_2-E_2)]\\&+3[C_1-E_1+5\nu\lambda^2(A_2-D_2)]+\lambda^2[3\nu(B_1+C_1)+5\lambda^2(A_2+B_2)]+\lambda^2[\nu(C_1-E_1)\\&+5\lambda^2(A_2-D_2)]+3\lambda^2[3\nu(B_1-D_1-E_1)+3\lambda^2(B_2-D_2-E_2)]+4(1-\nu)\lambda^2\\&\cdot[2(C_1-E_1)+B_2-E_2-D_2]-16[1+2(2+\nu)\lambda^2+\lambda^4]-16\lambda^2(\nu+3\lambda^2)\\&-16\lambda^2(\nu+\lambda^2)-8(1+\nu\lambda^2)\end{aligned} \tag{5.13c}$$

$$\begin{aligned}f_9=&(C_1+7\nu\lambda^2A_2)+[C_1-E_1+5\nu\lambda^2(A_2-D_2)]+(\nu C_1+7\lambda^2A_2)\lambda^2+3[\nu(C_1-E_1)\\&+5\lambda^2(A_2-D_2)]\lambda^2-16(\nu+3\lambda^2)\lambda^2-8(\nu+\lambda^2)\lambda^2\end{aligned} \tag{5.13d}$$

(5.12)式求得了$\mathbf{X}_2$，或M，N，L，P.

把(5.1)和(5.3)各项相比较，ξ^4，$\xi^2\eta^2$，η^4诸项的系数给出：

$$\begin{aligned}&70G+5\lambda^2H+\lambda^4I-(140+10\lambda^2)M-(10+2\lambda^2)\lambda^2N-2\lambda^4L+(70+10\lambda^2+\lambda^4)Q\\&\quad+(5+2\lambda^2)\lambda^2R+\lambda^4S=f_{10}\end{aligned} \tag{5.14a}$$

$$\begin{aligned}&15H+12\lambda^2I+15\lambda^4J-3(8+10\lambda^2)\lambda^2L-30M-3(10+8\lambda^2)N+3(10+4\lambda^2)Q\\&\quad-30\lambda^4P+3(5+8\lambda^2+5\lambda^4)R+3(4+10\lambda^2)\lambda^2S=f_{11}\end{aligned} \tag{5.14b}$$

$$\begin{aligned}&I+5\lambda^2J+70\lambda^4K-(2+10\lambda^2)L-2N-10(1+14\lambda^2)\lambda^2P+Q\\&\quad+(2+5\lambda^2)R+(1+10\lambda^2+70\lambda^4)S=f_{12}\end{aligned} \tag{5.14c}$$

其中f_{10}，f_{11}，f_{12}分别为

$$f_{10}=-10(1+\nu)\lambda^2(A_1-D_1)-2(\nu+\lambda^2)\lambda^2(C_2-E_2)+6(3+\nu\lambda^2)(D_1-F_1)+2(3\nu+\lambda^2)\lambda^2(E_2-F_2)+16(5+3\nu\lambda^2) \tag{5.15a}$$

$$f_{11}=2[3+4\lambda^2-3\nu\lambda^2](E_1-F_1)+2[4-3\nu+3\lambda^2]\lambda^2(E_2-F_2)+6(3\nu+\lambda^2)\lambda^2(D_2-F_2)+6(1+3\nu\lambda^2)(D_1-F_1)-6(1+\nu\lambda^2)(B_1-D_1-E_1)-6(\nu+\lambda^2)\lambda^2(B_2-D_2-E_2)+16[3+2(2+3\nu)\lambda^2+3\lambda^4] \tag{5.15b}$$

$$f_{12}=-2(1+\nu\lambda^2)(C_1-E_1)-10(\nu+\lambda^2)\lambda^2(A_2-D_2)+2(1+3\nu\lambda^2)(E_1-F_1)+6(\nu+3\lambda^2)\lambda^2(D_2-F_2)+16(3\nu+5\lambda^2)\lambda^2 \tag{5.15c}$$

(5.14)可以用矩阵方程表示如下：

$$\boldsymbol{\mu}_3\mathbf{X}_1+\boldsymbol{\theta}_3\mathbf{X}_2+\boldsymbol{\omega}_3\mathbf{X}_3=\boldsymbol{\Omega}_3 \tag{5.16}$$

其中$\mathbf{X}_1$，$\mathbf{X}_2$见(5.7a)，(5.11a)，而$\boldsymbol{\mu}_3$，$\boldsymbol{\theta}_3$，$\boldsymbol{\omega}_3$，$\boldsymbol{\Omega}_3$，$\mathbf{X}_3$分别为

$$\mathbf{X}_3=\begin{Bmatrix}Q\\R\\S\end{Bmatrix},\quad \boldsymbol{\Omega}_3=\begin{Bmatrix}f_{10}\\f_{11}\\f_{12}\end{Bmatrix},\quad \boldsymbol{\mu}_3=\begin{bmatrix}70 & 5\lambda^2 & \lambda^4 & 0 & 0\\0 & 15 & 12\lambda^2 & 15\lambda^4 & 0\\0 & 0 & 1 & 5\lambda^2 & 70\lambda^4\end{bmatrix} \tag{5.17a,b,c}$$

$$\boldsymbol{\theta}_3=\begin{bmatrix}-(140+10\lambda^2) & -(10+2\lambda^2)\lambda^2 & -2\lambda^4 & 0\\-30 & -3(10+8\lambda^2) & -3(8+10\lambda^2)\lambda^2 & -30\lambda^4\\0 & -2 & -(2+10\lambda^2) & (10+140\lambda^2)\lambda^2\end{bmatrix} \tag{5.17d}$$

$$\boldsymbol{\omega}_3=\begin{bmatrix}70+10\lambda^2+\lambda^4 & (5+2\lambda^2)\lambda^2 & \lambda^4\\3(10+4\lambda^2) & 3(5+8\lambda^2+5\lambda^4) & 3(4+10\lambda^2)\lambda^2\\1 & 2+5\lambda^2 & 1+10\lambda^2+70\lambda^4\end{bmatrix} \tag{5.17e}$$

从(5.16)式可以解出Q，R，S，即

$$\mathbf{X}_3=\boldsymbol{\omega}_3^{-1}\{\boldsymbol{\Omega}_3/3-\boldsymbol{\mu}_3\mathbf{X}_1-\boldsymbol{\theta}_3\mathbf{X}_2\} \tag{5.18}$$

把(5.1)和(5.3)各项相比较，ξ^2，η^2诸项的系数给出

$$\lambda^4L+15M+2\lambda^2N-(30+4\lambda^2)Q-(4+2\lambda^2)\lambda^2R-2\lambda^4S+(15+4\lambda^2+\lambda^4)T+2(1+\lambda^2)\lambda^2X=f_{13} \tag{5.19a}$$

$$2\lambda^2L+N+15\lambda^4P-2Q-2(1-2\lambda^2)R-(4+30\lambda^2)\lambda^2S+2(1+\lambda^2)T+(1+4\lambda^2+15\lambda^4)X=f_{14} \tag{5.19b}$$

其中f_{13}，f_{14}分别为

$$f_{13}=-6(1+\nu\lambda^2)(D_1-F_1)-2\lambda^2(\nu+\lambda^2)(E_2-F_2)+2(3+\nu\lambda^2)F_1+2(3\nu+\lambda^2)\lambda^2F_2-16(1+\nu\lambda^2) \tag{5.20a}$$

$$f_{14}=-2(1+\nu\lambda^2)(E_1-F_1)-6\lambda^2(\nu+\lambda^2)(D_2-F_2)+2(1+3\nu\lambda^2)F_1+2(\nu+3\lambda^2)\lambda^2F_2-16(\nu+\lambda^2)\lambda^2 \tag{5.20b}$$

(5.19a，b)可以用矩阵方程表示如下：

$$\boldsymbol{\theta}_4\mathbf{X}_2+\boldsymbol{\omega}_4\mathbf{X}_3+\boldsymbol{\pi}_4\mathbf{X}_4=\boldsymbol{\Omega}_4 \tag{5.21}$$

其中$\mathbf{X}_2$，$\mathbf{X}_3$见(5.11a)，(5.17a)，而$\mathbf{X}_4$，$\boldsymbol{\Omega}_4$，$\boldsymbol{\theta}_4$，$\boldsymbol{\omega}_4$，$\boldsymbol{\pi}_4$分别为

$$\mathbf{X}_4=\begin{Bmatrix}T\\X\end{Bmatrix},\quad \boldsymbol{\Omega}_4=\begin{Bmatrix}f_{13}\\f_{14}\end{Bmatrix},\quad \boldsymbol{\theta}_4=\begin{bmatrix}15 & 2\lambda^2 & \lambda^4 & 0\\0 & 1 & 2\lambda^2 & 15\lambda^4\end{bmatrix} \tag{5.22a,b,c}$$

$$\boldsymbol{\omega}_4=\begin{bmatrix}-(30+4\lambda^2) & -(4+2\lambda^2)\lambda^2 & -2\lambda^4\\-2 & -2(1+2\lambda^2) & -(4+30\lambda^2)\lambda^2\end{bmatrix} \tag{5.22d}$$

$$\pi_4=\begin{bmatrix}15+4\lambda^2+\lambda^4 & 2(1+\lambda^2)\lambda^2\\ 2(1+\lambda^2) & 1+4\lambda^2+15\lambda^4\end{bmatrix} \tag{5.22e}$$

(5.21)的解给出T，X。它们是

$$X_4=\pi_4^{-1}(\Omega_4-\theta_4X_2-\omega_4X_3) \tag{5.23}$$

表2(A)　　$\nu=0.25$时，(5.2)式中诸系数及a_3的值

λ	G	H	I	J	K
1	-0.52083×10^{-2}	-0.20833×10^{-1}	-0.31250×10^{-1}	-0.20833×10^{-1}	-0.52083×10^{-2}
2	-0.39984×10^{-1}	-0.52988×10^{-1}	-0.19562×10^{-1}	-0.51014×10^{-2}	-0.51610×10^{-3}
3	-0.11305	-0.87392×10^{-1}	-0.79593×10^{-2}	-0.16386×10^{-2}	-0.12148×10^{-3}
4	-0.21263	-0.11740	-0.30407×10^{-2}	-0.64242×10^{-3}	-0.41687×10^{-4}
5	-0.31431	-0.14190	-0.10610×10^{-2}	-0.29418×10^{-3}	-0.17816×10^{-4}

λ	M	N	L	P	Q
1	0.36458×10^{-1}	0.10938	0.10938	0.36458×10^{-1}	-0.11719
2	0.24550	0.21857	0.36759×10^{-1}	0.41229×10^{-2}	-0.64741
3	0.64191	0.33111	0.11728×10^{-1}	0.10113×10^{-2}	-1.45782
4	1.11020	0.42083	0.39038×10^{-2}	0.35335×10^{-3}	-2.22038
5	1.52727	0.48999	0.11689×10^{-2}	0.15263×10^{-3}	-2.79620

λ	R	S	T	X	a_3
1	-0.23438	-0.11719	0.33681	0.33681	0.53733
2	-0.33633	-0.15188×10^{-1}	0.93368	0.21976	0.58232
3	-0.44380	-0.36244×10^{-2}	1.55911	0.21914	0.63610
4	-0.52117	-0.10883×10^{-2}	1.99581	0.22522	0.66999
5	-0.57538	-0.33718×10^{-3}	2.27442	0.23011	0.68934

表2(B)　　$\nu=0.30$时，(5.2)式中诸系数及a_3的值

λ	G	H	I	J	K
1	-0.50556×10^{-2}	-0.20222×10^{-1}	-0.30333×10^{-1}	-0.20222×10^{-1}	-0.50556×10^{-2}
2	-0.38868×10^{-1}	-0.51510×10^{-1}	-0.18997×10^{-1}	-0.49270×10^{-2}	-0.50150×10^{-3}
3	-0.11061	-0.85186×10^{-1}	-0.78521×10^{-2}	-0.15566×10^{-2}	-0.11820×10^{-3}
4	-0.20942	-0.11476	-0.31870×10^{-2}	-0.59740×10^{-3}	-0.40580×10^{-4}
5	-0.31099	-0.13913	-0.13148×10^{-2}	-0.26760×10^{-3}	-0.17343×10^{-4}

λ	M	N	L	P	Q
1	0.35389×10^{-1}	0.10617	0.10617	0.35389×10^{-1}	-0.11375
2	0.23927	0.21258	0.35785×10^{-1}	0.39870×10^{-2}	-0.63502
3	0.63083	0.32328	0.11751×10^{-1}	0.96847×10^{-3}	-1.44202
4	1.09773	0.41254	0.42840×10^{-2}	0.33463×10^{-3}	-2.20583
5	1.51572	0.48211	0.16476×10^{-2}	0.14302×10^{-3}	-2.78420

λ	R	S	T	X	a_3
1	-0.22750	-0.11375	0.33656	0.33656	0.54564
2	-0.32806	-0.14814×10^{-1}	0.93108	0.21977	0.58713
3	-0.43563	-0.36637×10^{-2}	1.55569	0.21906	0.63872
4	-0.51406	-0.12047×10^{-2}	1.99280	0.22512	0.67127
5	-0.56946	-0.45853×10^{-3}	2.27203	0.23001	0.69043

最后，比较(5.1)，(5.3)的常数项，给出

$$(3+2\lambda^2+3\lambda^4)\alpha_3=3Q+\lambda^2R+3\lambda^4S-2(3+\lambda^2)T-2(1+3\lambda^2)\lambda^2X+6(1+\nu\lambda^2)F_1+6\lambda^2(\nu+\lambda^2)F_2 \tag{5.24}$$

其中Q，R，S，T，X分别从(5.18)，(5.23)的χ_3，χ_4给出，F_1，F_2分别从(4.20)的Ψ_1，Ψ_2给出。

根据上述二阶摄动计算结果，$W_3(\xi,\eta)$表达式(5.2)式中的待定系数，都已确定。α_3亦即求得。当$\nu=0.25$，0.30，0.35时，各该系数在不同λ时的值见表2(A，B，C)。

表2(C)　　$\nu=0.35$时，(5.2)式中诸系数及α_3的值

λ	G	H	I	J	K
1	-0.48750×10^{-2}	-0.19500×10^{-1}	-0.29250×10^{-1}	-0.19500×10^{-1}	-0.48750×10^{-2}
2	-0.37537×10^{-1}	-0.49757×10^{-1}	-0.18321×10^{-1}	-0.47271×10^{-2}	-0.48411×10^{-3}
3	-0.10754	-0.82559×10^{-1}	-0.76774×10^{-2}	-0.14679×10^{-2}	-0.11426×10^{-3}
4	-0.20503	-0.11159	-0.32840×10^{-2}	-0.55036×10^{-3}	-0.39247×10^{-4}
5	-0.30603	-0.13578	-0.15304×10^{-2}	-0.24024×10^{-3}	-0.16773×10^{-4}
λ	M	N	L	P	Q
1	0.34125×10^{-1}	0.10238	0.10238	0.34125×10^{-1}	-0.10969
2	0.23168	0.20550	0.34586×10^{-1}	0.38301×10^{-2}	-0.61898
3	0.61615	0.31393	0.11656×10^{-1}	0.92098×10^{-3}	-1.41849
4	1.07935	0.40246	0.45874×10^{-2}	0.31437×10^{-3}	-2.18091
5	1.49685	0.47233	0.20724×10^{-2}	0.13275×10^{-3}	-2.76095
λ	R	S	T	X	α_3
1	-0.21938	-0.10969	0.33525	0.33525	0.55294
2	-0.31820	-0.14344×10^{-1}	0.92455	0.21931	0.59077
3	-0.42563	-0.36685×10^{-2}	1.54615	0.21860	0.64018
4	-0.50509	-0.13036×10^{-2}	1.98308	0.22469	0.67188
5	-0.56178	-0.56969×10^{-3}	2.26325	0.22964	0.69068

六、中心位移和均布载荷的关系

我们可以从(3.2d)写出二级近似的中心位移和均布载荷的关系式：

$$3Q/2(3+2\lambda^2+3\lambda^4)=\alpha_1W_m+\alpha_3W_m^3 \tag{6.1}$$

其中，Q为无量纲均布载荷，W_m为无量纲中心位移，见(2.4)。α_1和α_3分别见(4.4)和(5.24)，

$$\alpha_1=1 \tag{6.2}$$

α_3是λ^2和ν的函数。对于不同的λ^2，ν值，α_3的数值见表2(A，B，C)。

当$\lambda^2=1/\lambda^2=1$时

$$3/2(3+2\lambda^2+3\lambda^4)=3/16 \tag{6.3}$$

(6.1)式即为圆薄板的形式，见[1](1948)。而在$\lambda^2=1$，$\nu=0.30$时，

$$\alpha_3=0.54564 \tag{6.4}$$

这与[1](1948)的结果也是一致的。

根据(6.1)给出的关系式，可画出中心位移对均布载荷的曲线。

图2为当$\nu=0.3$，λ取不同的值时，中心位移W_m对均布载荷Q的曲线。

图3为当$\lambda=2$，ν取不同值时，中心位移W_m对均布载荷Q的曲线。

图2中"·"为文献[3]中Nash和Cooley (1959)给出的实验点($\lambda^2=4$)。

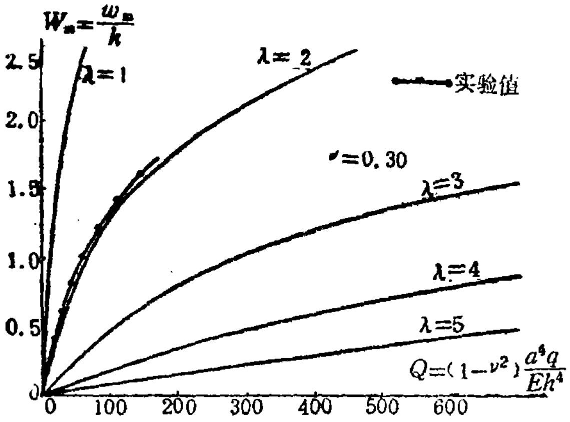

图2 当$\nu=0.30$时，λ^2取不同的值，W_m对Q的曲线

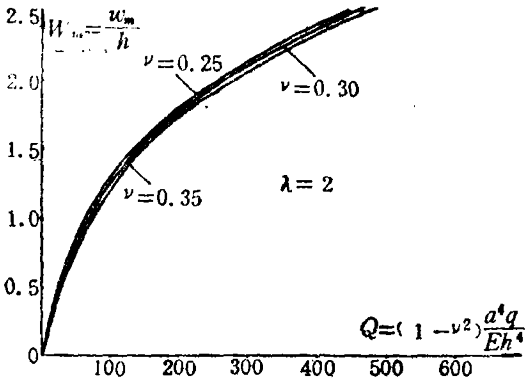

图3 当$\lambda=2$时，ν取不同值，W_m对Q的曲线

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Large Deflection Problem of a Clamped Elliptical Plate Subjected to Uniform Pressure

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Abstract

In this paper, the perturbation solution of large deflection problem of clamped elliptical plate subjected to uniform pressure is given on the basis of the perturbation solution of large deflection problem of similar clamped circular plate (1948)[1], (1954)[2]. The analytical solution of this problem was obtained in 1957. However, due to social difficulties, these results have never been published. Nash and Cooley (1959)[3] published a brief note of similar nature, in which only the case $\lambda=a/b=2$ is given. In this paper, the analytical solution is given in detail up to the 2nd approximation. The numerical solutions are given for various Poisson ratios $\nu=0.25$, 0.30, 0.35 and for various eccentricities $\lambda=1$, 2, 3, 4, 5, which can be used in the calculation of engineering designs.

Key words elliptical plate, large deflection, perturbation method