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在任意不定常温度场和任意法向动载荷联合作用下中心开孔圆底扁球壳的动力问题

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摘　要

本文得出了在任意不定常温度场和任意法向动载荷联合作用下中心开孔圆底扁球壳的动力问题的解析解。我们假设温度沿壳体厚度直线分布。在第一部分，我们研究了常用边界条件下的中心开孔圆底扁球壳的自由振动。作为例子，我们计算了一边缘夹紧的扁球壳的自然基频（$m=0$），所得结果与E. Reissner[1] 的结果作了比较。频率方程的解法是钱伟长[2] 提出来的。这将在附录3中介绍。在第二部分，我们研究了在任意谐温度场和任意谐法向动载荷联合作用下的中心开孔圆底扁球壳的强迫振动。在第三部分，我们研究了在任意不定常温度场和任意法向动载荷联合作用下的具有初始条件的上述壳体的强迫振动。在附录1和2中，我们讨论了如何用应力函数来表示位移边界条件和 $m=1$ 情形的边界条件。

一、引　　言

工程结构除了对薄壳结构静力分析的要求外，还必须考虑其他更多的因素；例如温度作用，振动影响，稳定性等等。近代生产的迅速发展，对于动力学和热应力问题的研究的要求不断地增长。E. Reissner[1][5][6][8] 曾讨论了圆底扁球壳的自由振动。我们在文献[4]中讨论了任意分布横向载荷下的中心开孔圆底扁球壳静力平衡问题的通解。现在在这个基础上，进一步来讨论在任意不定常温度场和任意法向动载荷联合作用下中心开孔圆底扁球壳的动力问题。在第一部分，我们研究了常用边界条件下的中心开孔圆底扁球壳的自由振动。作为例子，我们计算了一边缘夹紧的扁球壳的自然基频($m=0$)。所得结果和E. Reissner[1] 的结果作了比较。频率方程的解法是钱伟长[2]提出来的。这将在附录3中介绍。在第二部分，我们研究了在任意谐温度场和任意谐法向动载荷联合作用下的中心开孔圆底扁球壳的强迫振动。在第三部分，我们研究了在任意不定常温度场和任意法向动载荷联合作用下具有初始条件的上述壳体的强迫振动。在附录1和2中，我们讨论了如何用应力函数来表示位移边界条件和 $m=1$ 情形的边界条件。当然，我们在这里只研究了非耦合的热弹性问题，关于耦合的热弹性问题将在另文讨论。

二、在任意不定常温度场和任意法向动载荷联合作用下的极座标扁壳动力问题的基本方程

考虑几何尺寸如图1所示的薄、弹性、等厚度的中心开孔圆底扁球壳，它的位移，内力，内力矩，载荷的正向如图2所示，图中R为壳体中面的半径，h为壳体厚度，b_1为内半径，a_1为外半径，r_1,θ为垂直于中心轴平面上的极座标，$q_1(r_1,\theta,t)$为法向表面动载荷强度，t为时间，$N_{r_1},N_\theta,N_{r_1\theta}$为单位长度薄膜内力，$M_{r_1}$为单位长度径向弯矩，$M_\theta$为单位长度切向弯矩，$M_{r_1\theta}$为单位长度扭矩，$Q_{r_1},Q_\theta$为单位长度径向、切向剪力，$w_1$为挠度函数，$u_1,v_1$为径向，切向位移。

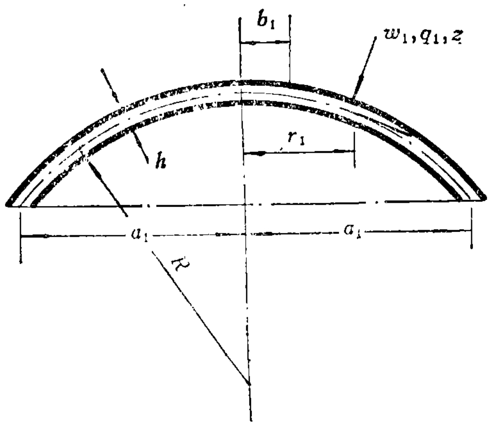

图1

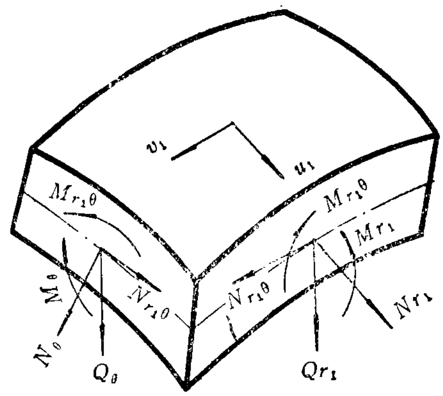

图2

由于壳体很薄，我们有足够的精确度认为温度是沿壳厚直线分布，即可设壳体中任一点$P(r_1,\theta,z)$处的温度t^*为

$$t^*(r_1,\theta,z,t)=t_1(r_1,\theta,t)+t_2(r_1,\theta,t)z \tag{2.1}$$

其中t_1和t_2为已知的壳体中面的位置和时间函数。

在任意不定常温度场和任意法向动载荷联合作用下的极座标扁球壳动力问题的基本方程为：

$$\left.\begin{aligned}&\frac{1}{Eh}\nabla_1^2\nabla_1^2\varphi_1+\frac{1}{R}\nabla_1^2 w_1+\alpha\nabla_1^2 t_1=0\\&D\nabla_1^2\nabla_1^2 w_1-\frac{1}{R}\nabla_1^2\varphi_1+D\alpha(1+\nu)\nabla_1^2 t_2=-\rho\frac{h}{g}\frac{\partial^2 w_1}{\partial t^2}+q_1(r_1,\theta,t)\end{aligned}\right\} \tag{2.2a,b}$$

其中E为杨氏弹性模量，$D=\dfrac{Eh^3}{12(1-\nu^2)}$为抗弯刚度，$\nu$为泊松比，$\varphi_1$为应力函数，$\rho$为材料密度，$g$为重力加速度，$\alpha$为线膨胀系数，

$$\nabla_1^2=\frac{\partial^2}{\partial r_1^2}+\frac{1}{r}\frac{\partial}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2}{\partial\theta^2} \tag{2.3}$$

为拉普拉斯算子。

弯矩、扭矩、剪力、反力与挠度函数间有下列关系：

$$\left.\begin{aligned} M_{r_1}&=-D\left[\frac{\partial^2 w_1}{\partial r_1^2}+\frac{\nu}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{\nu}{r_1^2}\frac{\partial^2 w_1}{\partial\theta^2}+\alpha(1+\nu)t_2\right]\\ M_\theta&=-D\left[\frac{1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2 w_1}{\partial\theta^2}+\nu\frac{\partial^2 w_1}{\partial r_1^2}+\alpha(1+\nu)t_2\right]\\ M_{r_1\theta}&=-D(1-\nu)\left(\frac{1}{r_1}\frac{\partial^2 w_1}{\partial r_1\partial\theta}-\frac{1}{r_1^2}\frac{\partial w_1}{\partial\theta}\right)\end{aligned}\right\}\quad(2.4\text{a,b,c})$$

$$\left.\begin{aligned} Q_{r_1}&=\frac{\partial M_{r_1}}{\partial r_1}+\frac{1}{r_1}\frac{\partial M_{r_1\theta}}{\partial\theta}+\frac{1}{r_1}(M_{r_1}-M_\theta)\\ &=-D\frac{\partial}{\partial r_1}[\nabla_1^2 w_1+(1+\nu)\alpha t_2]\\ Q_\theta&=\frac{\partial M_{r_1\theta}}{\partial r_1}+\frac{1}{r_1}\frac{\partial M_\theta}{\partial\theta}+\frac{2M_{r_1\theta}}{r_1}\\ &=-D\frac{1}{r_1}\frac{\partial}{\partial\theta}[\nabla_1^2 w_1+(1+\nu)\alpha t_2]\end{aligned}\right\}\quad(2.5\text{a,b})$$

$$V_{r_1}=Q_{r_1}+\frac{1}{r_1}\frac{\partial}{\partial\theta}(M_{r_1\theta})\quad(2.6)$$

其中V_{r_1}为单位长度径向反力。

薄膜内力与应力函数间有下列关系：

$$\left.\begin{aligned} N_{r_1}&=\frac{1}{r_1}\frac{\partial\varphi_1}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2\varphi_1}{\partial\theta^2}\\ N_\theta&=\frac{\partial^2\varphi}{\partial r_1^2}\\ N_{r_1\theta}&=\frac{1}{r_1^2}\frac{\partial\varphi_1}{\partial\theta}-\frac{1}{r_1}\frac{\partial^2\varphi_1}{\partial r_1\partial\theta}\end{aligned}\right\}\quad(2.7\text{a,b,c})$$

应力应变关系为

$$\left.\begin{aligned} \varepsilon_{r_1}&=\frac{\partial u_1}{\partial r_1}-\frac{w_1}{R}=\frac{1}{Eh}(N_{r_1}-\nu N_\theta)+\alpha t_1\\ \varepsilon_\theta&=\frac{1}{r_1}\frac{\partial v_1}{\partial\theta}+\frac{u_1}{r_1}-\frac{w_1}{R}=\frac{1}{Eh}(N_\theta-\nu N_{r_1})+\alpha t_1\\ \gamma_{r_1\theta}&=\frac{1}{r_1}\frac{\partial u_1}{\partial\theta}+r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)=\frac{2(1+\nu)}{Eh}N_{r_1\theta}\end{aligned}\right\}\quad(2.8\text{a,b,c})$$

其中 $\varepsilon_{r_1},\varepsilon_\theta,\gamma_{r_1\theta}$为应变分量。

我们考虑的边界条件有下列三种类型：

当 $r_1=a_1$(或$r_1=b_1$)时

a) 夹紧边

$$\left.\begin{aligned} w_1&=\frac{\partial w_1}{\partial r_1}=0\\ u_1&=v_1=0\end{aligned}\right\}\quad(2.9\text{a,b,c,d})$$

b) 简支边

$$\left.\begin{array}{l} w_1 = M_{r_1} = 0 \\ N_{r_1} = v_1 = 0 \end{array}\right\} \qquad (2.10a,b,c,d)$$

c) 悬空边

$$\left.\begin{array}{l} M_{r_1} = v_{r_1} = 0 \\ N_{r_1} = N_{r_1\theta} = 0 \end{array}\right\} \qquad (2.11a,b,c,d)$$

初始条件

$$\left.\begin{array}{l} w_1 \Big|_{t=0} = f_1(r_1,\theta) \\ \dfrac{\partial w_1}{\partial t}\Big|_{t=0} = f_2(r_1,\theta) \end{array}\right\} \qquad (2.12a,b)$$

其中f_1, f_2为已知函数。

为了将方程(2.2)和边界条件(2.9)—(2.11)，初始条件(2.12)无量纲化，我们引进无量纲量：

$$\left.\begin{array}{l} \Phi = \dfrac{4b_1r_0^2}{DR}\varphi, W = \dfrac{16w}{b_1}, r = \dfrac{2r_1}{b_1r_0}, a = \dfrac{2a_1}{b_1r_0}, \\ b = \dfrac{2}{r_0}, \tau = \sqrt{\dfrac{Dg}{\rho h}}\left(\dfrac{2}{b_1r_0}\right)^2 t, \ U = \dfrac{32Ru}{b_1^2r_0}, V = \dfrac{32Rv}{b_1^2r_0}, \\ \bar{\varepsilon}_r = \dfrac{16R}{b_1}\varepsilon_r, \ \bar{\varepsilon}_\theta = \dfrac{16R}{b_1}\varepsilon_\theta^*, \ \bar{\gamma}_{r\theta} = \dfrac{16R}{b_1}\gamma_{r\theta}, \\ \overline{N}_r = \dfrac{b_1^3r_0^4}{DR}N_r, \ \overline{N}_\theta = \dfrac{b_1^3r_0^4}{DR}N_\theta^*, \overline{N}_{r\theta} = \dfrac{b_1^3r_0^4}{DR}N_{r\theta}, \\ \overline{M}_r = \dfrac{4b_1r_0^2}{D}M_r, \overline{M}_\theta = \dfrac{4b_1r_0^2}{D}M_\theta^*, \overline{M}_{r\theta} = \dfrac{4b_1r_0^2}{D}M_{r\theta}, \\ \overline{Q}_r = \dfrac{2r_0^3b_1^2}{D}Q_r, \ \overline{Q}_\theta = \dfrac{2r_0^3b_1^2}{D}Q_\theta^*, \overline{V}_r = \dfrac{2r_0^3b_1^2}{D}V_r, \\ Q = \dfrac{b_1^3r_0^4}{D}q(r,\theta;\tau), T = \dfrac{16R\alpha}{b_1}\bar{t}_1, T' = 4b_1r_0^2\alpha(1+\nu)\bar{t}_2, \\ F_1 = \dfrac{16}{b_1}f_1^*, \ F_2 = 4b_1r_0^2\sqrt{\dfrac{\rho h}{Dg}}f_2^* \end{array}\right\} \qquad (2.13)$$

其中

$$\left.\begin{array}{l} \varphi = \varphi_1(\beta r,\theta;\gamma\tau), w = w_1(\beta r,\theta;\gamma\tau), \varepsilon_r = \varepsilon_{r_1}(\beta r,\theta;\gamma\tau) \\ \varepsilon_\theta^* = \varepsilon_\theta(\beta r,\theta;\gamma\tau), \gamma_{r\theta} = \gamma_{r_1\theta}(\beta r,\theta;\gamma\tau), N_{r\theta} = N_{r_1\theta}(\beta r,\theta;\gamma\tau), \\ N_\theta^* = N_\theta(\beta r,\theta;\gamma\tau), N_r = N_{r_1}(\beta r,\theta;\gamma\tau), \\ M_r = M_{r_1}(\beta r,\theta;\gamma\tau), M_\theta^* = M_\theta(\beta r,\theta;\gamma\tau), M_{r\theta} = M_{r_1\theta}(\beta r,\theta;\gamma\tau), \\ Q_r = Q_{r_1}(\beta r,\theta,\gamma\tau), Q_\theta^* = Q_\theta(\beta r,\theta;\gamma\tau), V_r = V_{r_1}(\beta r,\theta;\gamma\tau), \\ q = q_1(\beta r,\theta;\gamma\tau), u = u_1(\beta r,\theta;\gamma\tau), v = v_1(\beta r,\theta;\gamma\tau), \\ \bar{t}_1 = t_1(\beta r,\theta;\gamma\tau), \bar{t}_2 = t_2(\beta r,\theta;\gamma\tau), \\ f_1^* = f_1(\beta r,\theta), f_2^* = f_2(\beta r,\theta), \beta = \dfrac{b_1r_0}{2}, \\ \gamma = \sqrt{\dfrac{\rho h}{Dg}}\left(\dfrac{b_1r_0}{2}\right)^2, \ r_0 = \sqrt{\dfrac{4Rh}{b_1\sqrt{12(1-\nu^2)}}} \end{array}\right\} \qquad (12.14)$$

则方程(2.2)、(2.4)—(2.8)式变成：

$$\left.\begin{aligned}&\nabla^2\nabla^2\Phi+\nabla^2W+\nabla^2T=0\\&\nabla^2\nabla^2W-\nabla^2\Phi+\nabla^2T'=-\frac{\partial^2W}{\partial\tau^2}+Q\end{aligned}\right\}\tag{2.15a,b}$$

$$\left.\begin{aligned}&\overline{M}_r=-\left(\frac{\partial^2W}{\partial r^2}+\frac{\nu}{r}\frac{\partial W}{\partial r}+\frac{\nu}{r^2}\frac{\partial^2W}{\partial\theta^2}+T'\right)\\&\overline{M}_\theta=-\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2W}{\partial\theta^2}+\nu\frac{\partial^2W}{\partial r^2}+T'\right)\\&\overline{M}_{r\theta}=-(1-\nu)\left(\frac{1}{r}\frac{\partial^2W}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial W}{\partial\theta}\right)\end{aligned}\right\}\tag{2.16a,b,c}$$

$$\left.\begin{aligned}&\overline{Q}_r=-\frac{\partial}{\partial r}(\nabla^2W+T')\\&\overline{Q}_\theta=-\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2W+T')\end{aligned}\right\}\tag{2.17a,b}$$

$$\overline{V}_r=\overline{Q}_r+\frac{1}{r}\frac{\partial}{\partial\theta}(\overline{M}_{r\theta})\tag{2.18}$$

$$\left.\begin{aligned}&\overline{N}_r=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\\&\overline{N}_\theta=\frac{\partial^2\Phi}{\partial r^2}\\&\overline{N}_{r\theta}=\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}\end{aligned}\right\}\tag{2.19}$$

$$\left.\begin{aligned}&\bar{\varepsilon}_r=\frac{\partial U}{\partial r}-W=\nabla^2\Phi-(1+\nu)\frac{\partial^2\Phi}{\partial r^2}+T\\&\bar{\varepsilon}_\theta=\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{U}{r}-W=(1+\nu)\frac{\partial^2\Phi}{\partial r^2}-\nu\nabla^2\Phi+T\\&\bar{\gamma}_{r\theta}=\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\\&\qquad=2(1+\nu)\left(\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}\right)\end{aligned}\right\}\tag{2.20a,b,c}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\tag{2.21}$$

首先将$Q(r,\theta;\tau)$，$T(r,\theta;\tau)$，$T'(r,\theta;\tau)$展开成富里哀级数：

$$Q=Q_0+\sum_{m=1}^{\infty}Q_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{Q}_m\sin m\theta\tag{2.22}$$

$$T=T_0+\sum_{m=1}^{\infty}T_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{T}_m\sin m\theta\tag{2.23}$$

$$T'=T'_0+\sum_{m=1}^{\infty}T'_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{T}'_m\sin m\theta \tag{2.24}$$

此处

$$\left.\begin{aligned}(Q_0,T_0,T'_0)&=\frac{1}{2\pi}\int_0^{\pi}(Q,T,T')d\theta\\(Q_m,T_m,T'_m)&=\frac{1}{\pi}\int_0^{\pi}(Q,T,T')\cos m\theta\,d\theta\\(\widetilde{Q}_m,\widetilde{T}_m,\widetilde{T}'_m)&=\frac{1}{\pi}\int_0^{\pi}(Q,T,T')\sin m\theta\,d\theta\end{aligned}\right\} \tag{2.25}$$

今将$W,\Phi,\overline{M}_r,\overline{M}_\theta,\overline{M}_{r\theta},\overline{N}_r,\overline{N}_\theta,\overline{N}_{r\theta},\overline{Q}_r,\overline{Q}_\theta,\overline{V}_r,U,V,F_1,F_2$也写成如下的形式：

$$W=W_0+\sum_{m=1}^{\infty}W_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{W}_m\sin m\theta \tag{2.26}$$

$$\Phi=\Phi_0+\sum_{m=1}^{\infty}\Phi_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{\Phi}_m\sin m\theta \tag{2.27}$$

$$\overline{M}_r=M_{r0}+\sum_{m=1}^{\infty}M_{rm}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{M}_{rm}\sin m\theta \tag{2.28}$$

$$\overline{M}_\theta=M_{\theta0}+\sum_{m=1}^{\infty}M_{\theta m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{M}_{\theta m}\sin m\theta \tag{2.29}$$

$$\overline{M}_{r\theta}=M_{r\theta0}+\sum_{m=1}^{\infty}M_{r\theta m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{M}_{r\theta m}\sin m\theta \tag{2.30}$$

$$\overline{N}_r=N_{r0}+\sum_{m=1}^{\infty}N_{rm}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{N}_{rm}\sin m\theta \tag{2.31}$$

$$\overline{N}_\theta=N_{\theta0}+\sum_{m=1}^{\infty}N_{\theta m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{N}_{\theta m}\sin m\theta \tag{2.32}$$

$$\overline{N}_{r\theta}=N_{r\theta0}+\sum_{m=1}^{\infty}N_{r\theta m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{N}_{r\theta m}\sin m\theta \tag{2.33}$$

$$\overline{Q}_r=Q_{r0}+\sum_{m=1}^{\infty}Q_{rm}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{Q}_{rm}\sin m\theta \tag{2.34}$$

$$\overline{Q}_\theta=Q_{\theta0}+\sum_{m=1}^{\infty}Q_{\theta m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{Q}_{\theta m}\sin m\theta \tag{2.35}$$

$$\widetilde{V}_r=V_{r0}+\sum_{m=1}^{\infty}V_{rm}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{V}_{rm}\sin m\theta \tag{2.36}$$

$$U=U_0+\sum_{m=1}^{\infty}U_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{U}_m\sin m\theta \tag{2.37}$$

$$V=V_0+\sum_{m=1}^{\infty}V_m\cos m\theta+\sum_{m=1}^{\infty}\widetilde{V}_m\sin m\theta \tag{2.38}$$

$$F_1=F_{10}+\sum_{m=1}^{\infty}F_{1m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{F}_{1m}\sin m\theta \tag{2.39}$$

$$F_2=F_{20}+\sum_{m=1}^{\infty}F_{2m}\cos m\theta+\sum_{m=1}^{\infty}\widetilde{F}_{2m}\sin m\theta \tag{2.40}$$

其中$W_m(m=0,1,2,\cdots)$,$\widetilde{W}_m(m=1,2\cdots)$;$\Phi_m(m=0,1,2,\cdots)$,$\widetilde{\Phi}_m$ $(m=1,2,\cdots)$ 是所求的未知函数，而$M_{rm}(m=0,1,2,\cdots)$,$\widetilde{M}_{rm}(m=1,2,\cdots)\cdots$,$V_m(m=0,1,2,\cdots)$,$\widetilde{V}_m(m=1,2,\cdots)$均是它们的导来函数，将公式(2.26)—(2.36)代入(2.15)—(2.19)，比较三角级数的诸项系数，即得

$$\begin{matrix}M_{rm}\\ \widetilde{M}_{rm}\end{matrix}=-\left(\frac{\partial^2}{\partial r^2}+\frac{\nu}{r}\frac{\partial}{\partial r}-\nu\frac{m^2}{r^2}\right)\begin{matrix}W_m\\ \widetilde{W}_m\end{matrix}-\begin{matrix}T'_m(m=0,1,2,\cdots)\\ \widetilde{T}'_m\ (m=1,2,\cdots)\end{matrix} \tag{2.41}$$

$$\begin{matrix}M_{\theta m}\\ \widetilde{M}_{\theta m}\end{matrix}=-\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\nu\frac{\partial^2}{\partial r^2}\right)\begin{matrix}W_m\\ \widetilde{W}_m\end{matrix}-\begin{matrix}T'_m(m=0,1,2,\cdots)\\ \widetilde{T}'_m(m=1,2,\cdots)\end{matrix} \tag{2.42}$$

$$\begin{matrix}M_{r\theta m}\\ \widetilde{M}_{r\theta m}\end{matrix}=\mp m(1-\nu)\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\begin{matrix}\widetilde{W}_m\\ W_m\end{matrix}\ (m=1,2,\cdots) \tag{2.43}$$

$$\begin{matrix}N_{rm}\\ \widetilde{N}_{rm}\end{matrix}=\left(\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}\right)\begin{matrix}\Phi_m(m=0,1,2\cdots)\\ \widetilde{\Phi}_m(m=1,2,\cdots)\end{matrix} \tag{2.44}$$

$$\begin{matrix}N_{\theta m}\\ \widetilde{N}_{\theta m}\end{matrix}=\frac{\partial^2}{\partial r^2}\left(\begin{matrix}\Phi_m\\ \widetilde{\Phi}_m\end{matrix}\right)\begin{matrix}(m=0,1,2,\cdots)\\ (m=1,2,\cdots)\end{matrix} \tag{2.45}$$

$$\begin{matrix}N_{r\theta m}\\ \widetilde{N}_{r\theta m}\end{matrix}=\pm m\left(\frac{1}{r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\begin{matrix}\widetilde{\Phi}_m\\ \Phi_m\end{matrix}\ (m=1,2,\cdots) \tag{2.46}$$

$$\begin{matrix}Q_{rm}\\ \widetilde{Q}_{rm}\end{matrix}=-\frac{\partial}{\partial r}\left(\nabla_m^2\begin{matrix}W_m\\ \widetilde{W}_m\end{matrix}+\begin{matrix}T'_m\\ \widetilde{T}'_m\end{matrix}\right)\begin{matrix}(m=0,1,2\cdots)\\ (m=1,2,\cdots)\end{matrix} \tag{2.47}$$

$$\begin{matrix}Q_{\theta m}\\ \widetilde{Q}_{\theta m}\end{matrix}=\mp\frac{m}{r}\left(\nabla_m^2\begin{matrix}\widetilde{W}_m\\ W_m\end{matrix}+\begin{matrix}\widetilde{T}'_m\\ T'_m\end{matrix}\right)(m=1,2,\cdots) \tag{2.48}$$

$$\begin{matrix}V_{rm}\\ \widetilde{V}_{rm}\end{matrix}=\begin{matrix}Q_{rm}\\ \widetilde{Q}_{rm}\end{matrix}+(1-\nu)m^2\left(\frac{1}{r^2}\frac{\partial}{\partial r}-\frac{1}{r^3}\right)\begin{matrix}W_m(m=0,1,2\cdots)\\ \widetilde{W}_m(m=1,2,\cdots)\end{matrix} \tag{2.49}$$

$U_m(m=0,1,2,\cdots)$, $\widetilde{U}_m(m=1,2,\cdots)$, V_m $(m=0,1,2,\cdots)$, $\widetilde{V}_m$ $(m=1,2,\cdots)$ 和 W_m $(m=0,1,2,\cdots)$;$\widetilde{W}_m(m=1,2,\cdots)$; $\Phi_m(m=0,1,2,\cdots)$, $\widetilde{\Phi}_m(m=1,2,\cdots)$;$T_m(m=0,1,2,\cdots)$,

$\tilde{T}_m(m=1,2,\cdots)$的关系在附录1中讨论。将方程(2.26)—(2.40)，(2.22)—(2.24)代入方程(2.15)和边界条件(2.9)—(2.11)和初始条件(2.12)并利用关系(2.13)，(2.14)—(2.49)和附录1中的方程(11′)—(14′)，比较三角级数诸项的系数，即得下列二种类型的边值问题。第一类基本方程为

$$\left.\begin{aligned}&\nabla_m^2\nabla_m^2\Phi_m+\nabla_m^2W_m+\nabla_m^2T_m=0\\&\nabla_m^2\nabla_m^2W_m-\nabla_m^2\Phi_m+\nabla_m^2T'_m=-\frac{\partial^2W_m}{\partial\tau^2}+Q_m\\&\qquad(m=0,1,2,\cdots)\end{aligned}\right\}\tag{2.50a,b}$$

其中

$$\nabla_m^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}\tag{2.51}$$

边界条件为

当$r=a$（或$r=b$）时

a) 夹紧边

$$\left.\begin{aligned}&W_m=\frac{\partial W_m}{\partial r}=U_m=0\ (m=0,1,2,\cdots)\\&\tilde{V}_m=0\ ,(m=1,2,\cdots)\end{aligned}\right\}\tag{2.52a,b,c,d}$$

b) 简支边

$$\left.\begin{aligned}&W_m=M_{rm}=N_{rm}=0\ (m=0,1,2\cdots)\\&\tilde{V}_m=0\qquad(m=1,2,\cdots)\end{aligned}\right\}\tag{2.53a,b,c,d}$$

c) 悬空边

$$\left.\begin{aligned}&M_{rm}=V_{rm}=N_{rm}=0\quad(m=0,1,2,\cdots)\\&\tilde{N}_{r\theta m}=0\qquad(m=1,2,\cdots)\end{aligned}\right\}\tag{2.54a,b,c,d}$$

初始条件为

$$W_m\Big|_{\tau=0}=F_{1m},\ \frac{\partial W_m}{\partial\tau}\Big|_{\tau=0}=F_{2m}(m=0,1,2,\cdots)\tag{2.55a,b}$$

第二类基本方程为

$$\left.\begin{aligned}&\nabla_m^2\nabla_m^2\tilde{\Phi}_m+\nabla_m^2\tilde{W}_m+\nabla_m^2\tilde{T}_m=0\\&\nabla_m^2\nabla_m^2\tilde{W}_m-\nabla_m^2\tilde{\Phi}_m+\nabla_m^2\tilde{T}'_m=-\frac{\partial^2\tilde{W}_m}{\partial\tau^2}+\tilde{Q}_m\\&\qquad(m=1,2,\cdots)\end{aligned}\right\}\tag{2.56a,b}$$

边界条件为

当$r=a$（或$r=b$）时

a) 夹紧边

$$\tilde{W}_m=\frac{\partial\tilde{W}_m}{\partial r}=\tilde{U}_m=V_m=0\ ,(m=1,2,\cdots)\tag{2.57a,b,c,d}$$

b) 简支边

$$\tilde{W}_m=\tilde{M}_{rm}=\tilde{N}_{rm}=V_m=0\ ,(m=1,2,\cdots)\tag{2.58a,b,c,d}$$

c) 悬空边

$$\widetilde{M}_{rm}=\widetilde{V}_{rm}=\widetilde{N}_{rm}=N_{r\theta m}=0,(m=1,2,\cdots) \tag{2.59a,b,c,d}$$

初始条件为

$$\widetilde{W}_m\Big|_{\tau=0}=\widetilde{F}_{1m},\quad \frac{\partial\widetilde{W}_m}{\partial\tau}\Big|_{\tau=0}=\widetilde{F}_{2m}(m=1,2,\cdots) \tag{2.60a,b}$$

上述两组方程，边界条件和初始条件便是我们所求的基本边值问题。

三、中心开孔圆底扁球壳的自由振动

现在我们来计算中心开孔圆底扁球壳的自由振动。由基本方程(2.50a,b)，并令

$$T_m=T'_m=0,\quad Q_m=0,$$

在此情况，我们有

$$\left.\begin{aligned}&\nabla_m^2\nabla_m^2\Phi_m+\nabla_m^2 W_m=0\\&\nabla_m^2\nabla_m^2 W_m-\nabla_m^2\Phi_m=-\frac{\partial^2 W_m}{\partial\tau^2}\\&(m=0,1,2,\cdots)\end{aligned}\right\} \tag{3.1a,b}$$

令

$$\left.\begin{aligned}&\Phi_m(r,\tau)=e^{i\omega^*_{mn}t}\varphi_{mn}(r)=e^{i\omega_{mn}\tau}\varphi_{mn}(r)\\&W_m(r,\tau)=e^{i\omega^*_{mn}t}W_{mn}(r)=e^{i\omega_{mn}\tau}W_{mn}(r)\\&(m=0,1,2\cdots;\ n=1,2\cdots)\end{aligned}\right\} \tag{3.2a,b}$$

其中 $i=\sqrt{-1}$，ω^*_{mn} 为圆频率，ω_{mn}为折合圆频率，两者有下列的关系：

$$\frac{\omega^*_{mn}}{\omega_{mn}}=\frac{\tau}{t}=\sqrt{\frac{Dg}{\rho h}}\left(\frac{2}{b_1r_0}\right)^2=\frac{1}{R}\sqrt{\frac{Eg}{\rho}} \tag{3.3}$$

将(3.2)代入方程(3.1)，我们得到

$$\left.\begin{aligned}&\Delta_m\Delta_m\varphi_{mn}+\Delta_m W_{mn}=0\\&\Delta_m\Delta_m W_{mn}-\Delta_m\varphi_{mn}=\omega^2_{mn}W_{mn}\\&(m=0,1,2,\cdots;n=1,2,\cdots)\end{aligned}\right\} \tag{3.4a,b}$$

式中

$$\Delta_m=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2} \tag{3.5}$$

由文献[3]，对(3.4a)积分，则有

$$\left.\begin{aligned}&\Delta_m\varphi_{mn}+W_{mn}=\psi_{mn}(A_{mn},B_{mn})\\&(m=0,1,2,\cdots;n=1,2,\cdots)\end{aligned}\right\} \tag{3.6}$$

式中

当 $m=0$，$\psi_{on}(A_{on},B_{on})=A_{on}\ln r+B_{on}$ (3.7)

$m\geqslant 1$，$\psi_{mn}(A_{mn},B_{mn})=A_{mn}r^{-m}+B_{mn}r^m$ (3.8)

其中A_{mn}，B_{mn}($m=0,1,2,\cdots$；$n=1,2,\cdots$)为待定积分常数，注意ψ_{mn}适合下列方程

$$\Delta_m \psi_{mn} = 0\ (m=0,1,2,\cdots;\ n=1,2\cdots) \tag{3.9}$$

以(3.6)代入(3 4b)，我们得到

$$\Delta_m \Delta_m W_{mn} - k_{mn}^4 W_{mn} = \psi_{mn}(A_{mn}, B_{mn}) \tag{3.10}$$

其中

$$-k^4{}_n = 1 - \omega^2_{mn} (m=0,1,2,\cdots; n=1,2,\cdots) \tag{3.11}$$

方程(3.10)的解可写成如下的形式

$$W_{mn} = W_{mn1}^{(0)} + W_{mn2}^{(0)} + W_{mn3} \tag{3.12}$$

它们各自满足下列的方程

$$(\Delta_m \Delta_m - k_{mn}^4) W_{mn3} = \psi_{mn}(A_{mn}, B_{mn}) \tag{3.13}$$

$$(\Delta_m + k_{mn}^2) W_{mn1}^{(0)} = 0 \tag{3.14}$$

$$(\Delta_m - k_{mn}^2) W_{mn2}^{(0)} = 0 \tag{3.15}$$

由文献[4]，我们可得出

$$W_{mn3} = -\frac{1}{k_{mn}^4} \psi_{mn}(A_{mn}, B_{mn}) (m=0.1,2,\cdots; n=1,2,\cdots) \tag{3.16}$$

$$W_{mn1}^{(0)} = E_{mn} J_m(k_{mn} r) + F_{mn} Y_m(k_{mn} r)$$

$$(m=0,1,2,\cdots;\ n=1,2,\cdots) \tag{3.17}$$

$$W_{mn2}^{(0)} = G_{mn} J_m(ik_{mn} r) + H_{mn} Y_m(ik_{mn} r)$$

$$(m=0,1,2,\cdots;\ n=1,2,\cdots) \tag{3.18}$$

其中E_{mn}，F_{mn}，G_{mn}，$H_{mn}(m=0,1,2,\cdots;\ n=1,2,\cdots)$为待定积分常数，$J_m$，$Y_m$为第一类和第二类贝塞尔函数。因此方程(3.10)的解可写成如下的形式

$$W_{mn} = E_{mn} J_m(k_{mn} r) + F_{mn} Y_m(k_{mn} r) + G_{mn} J_m(ik_{mn} r)$$

$$+ H_{mn} Y_m(ik_{mn} r) - \frac{1}{k_{mn}^4} \psi_{mn}(A_{mn},\ B_{mn})$$

$$(m=0,1,2,\cdots;\ \cdots n=1,2,\cdots) \tag{3.19}$$

以(3.19)代入(3.6)

$$\Delta_m \varphi_{mn} = \left(1 + \frac{1}{k_{mn}^4}\right) \psi_{mn} - E_{mn} J_m(k_{mn} r)$$

$$- F_{mn} Y_m(k_{mn} r) - G_{mn} J_m(ik_{mn} r)$$

$$- H_{mn} Y_m(ik_{mn} r) \quad (m=0,1,2,\cdots; n=1,2,\cdots) \tag{3.20}$$

由文献[4]，我们立即得出

$$\text{当 } m=0, \varphi_{0n} = \varphi_{0n}(C_{0n},\ D_{0n}) + \left(1 + \frac{1}{k_{0n}^4}\right)\left[\frac{1}{4} A_{0n} r^2 \ln r\right.$$

$$\left. + \frac{1}{4}(B_{0n} - A_{0n}) r^2\right] + \frac{1}{k_{0n}^2} E_{0n} J_0(k_n r)$$

$$+ \frac{1}{k_{0n}^2} F_{0n} Y_0(k_n r) - \frac{1}{k_{0n}^2} G_{0n} J_0(ik_{0n} r)$$

$$-\frac{1}{k_{0n}^2}H_{0n}Y_0(ik_{0n}r)\quad(n=1,2,\cdots)\tag{3.21}$$

当$m=1$，$\varphi_{1n}=\psi_{1n}(C_{1n},D_{1n})+\left(1+\frac{1}{k_{1n}^4}\right)\left[\frac{1}{2}A_{1n}\left(r\ln r-\frac{r}{2}\right)\right.$

$$\left.+\frac{1}{8}B_{1n}r^3\right]+\frac{1}{k_{1n}^2}E_{1n}J_1(k_{1n}r)+\frac{1}{k_{1n}^2}F_{1n}Y_1(k_{1n}r)$$

$$-\frac{1}{k_{1n}^2}G_{1n}J_1(ik_{1n}r)-\frac{1}{k_{1n}^2}H_{1n}Y_1(ik_{1n}r)$$

$$(n=1,2,\cdots)\tag{3.22}$$

当$m>1$，$\varphi_{mn}=\psi_{mn}(C_{mn},D_{mn})+\left(1+\frac{1}{k_{mn}^4}\right)\left[-\frac{A_{mn}}{4(m-1)}r^{-m+2}\right.$

$$\left.+\frac{B_{mn}}{4(m+1)}r^{m+2}\right]+\frac{1}{k_{mn}^2}E_{mn}J_m(k_{mn}r)$$

$$+\frac{1}{k_{mn}^2}F_{mn}Y_m(k_{mn}r)-\frac{1}{k_{mn}^2}G_{mn}J_m(ik_{mn}r)$$

$$-\frac{1}{k_{mn}^2}H_{mn}Y_m(ik_{mn}r)\quad(n=1,2,\cdots)\tag{3.23}$$

其中

$$\left.\begin{aligned}&\psi_{0n}(C_{0n},\ D_{0n})=C_{0n}\ln r+D_{0n}\\&\psi_{mn}(C_{mn},\ D_{mn})=C_{mn}r^{-m}+D_{mn}r^m\\&(m=1,2,\cdots;\ n=1,2,\cdots)\end{aligned}\right\}\tag{3.24}$$

C_{mn}，$D_{mn}(m=0,1,2,\cdots;\ n=1,2,\cdots)$为待定积分常数，应该指出，在(3.22)中，E. Reissner[8]遗漏了$-\frac{1}{4}A_{1n}r$项，其余的结果与我们得到的相同。在$m=0$时，即轴对称情况，在附录2中我们证明了$A_{0n}=0$，在(3.21)中D_{0n}是一个可有可无的常数。因为用φ_{0n}来表示边界条件时，D_{0n}总是微分掉的，因此我们可令之为零。在此情况一共有六个待定积分常数C_{0n}，B_{0n}，E_{0n}，F_{0n}，G_{0n}，H_{0n}刚好由内外两边六个边界条件决定[见(2.52)－(2.54)，应令$T_m=T'_m=0$]。在$m=1$时，由位移单值性，我们在附录1中证明了$A_{1n}=0$，和$m=0$一样，$D_{1n}=0$，因此在此情况也一共有六个待定积分常数C_{1n}, B_{1n}, E_{1n}, F_{1n}, G_{1n}, H_{1n}，也刚好由内外两边六个边界条件决定，这将在附录2中详加讨论。

方程(3.4)的解，我们可写成如下的形式

$$\begin{aligned}&W_{mn}(r)=W_{mn}(A_{mn},B_{mn},E_{mn},F_{mn},G_{mn},H_{mn},k_{mn},r)\\&\varphi_{mn}(r)=\varphi_{mn}(A_{mn},B_{mn},C_{mn},D_{mn},E_{mn},F_{mn},G_{mn},H_{mn},k_{mn},r)\\&(m=0,1,2,\cdots;\ n=1,2,\cdots)\end{aligned}\tag{3.25}$$

第二类问题的方程(2.56)（令$\widetilde{T}_m=\widetilde{T}'_m=0$，$\widetilde{Q}_m=0$），边界条件(2.57)－(2.59)（令$\widetilde{T}_m=\widetilde{T}'_m=0$）与第一类问题的方程(2.50)（令$T_m=T'_m=0$，$Q_m=0$），边界条件(2.52)—(2.54)（令$T_m=T'_m=0$）在自由振动时形式相同，因此两者的自然频率的值也相同。

把所得结果(3.25)代入壳体内外两边的边界条件(2.52)—(2.54)(令$T_m=T'_m=0$)，

利用了(3.2)式，并注意所有待定积分常数$A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, F_{mn}, G_{mn}, H_{mn}$集合不能同时等于零，因而一般情况，我们得到一个八阶行列式方程，亦即频率方程。

作为一个例子，我们利用附录3中钱伟长[2]所提出的方法求解一个中心不开孔，边界夹紧的扁球壳的轴对称振型($m=0$)的基频($n=1$)。

实　　例

设壳体的基本尺寸为：壳厚$h=7$厘米，壳体中面半径$R=3850$厘米，外半径$a_1=692.39$厘米，杨氏弹性模量$E=2\times10^5$公斤/厘米^2，泊松比$\nu=0.15$，材料密度$\rho=2.4$克/厘米^3。

在中心不开孔的情形，φ_{01}函数与W_{01}函数如下式

$$W_{01}=E_{01}J_0(k_{01}r)+G_{01}J_0(ik_{01}r)-\frac{1}{k_{01}^4}B_{01} \tag{3.26}$$

$$\varphi_{01}=\frac{1}{4}\left(1+\frac{1}{k_{01}^4}\right)B_{01}r^2+\frac{1}{k_{01}^2}E_{01}J_0(k_{01}r)-\frac{1}{k_{01}^2}G_{01}J_0(ik_{01}r) \tag{3.27}$$

边界条件为($r=a$)

$$\left.\begin{aligned}&\int\Delta_0\varphi_{01}dr-(1+\nu)\frac{d\varphi_{01}}{dr}+\int W_{01}dr=0\\&W_{01}=0,\qquad \frac{dW_{01}}{dr}=0\end{aligned}\right\} \tag{3.28}$$

将(3.26)，(3.27)代入(3.28)，我们得到下列的联立方程组：

$$\left.\begin{aligned}&\left[1-\frac{1}{2}(1+\nu)\left(1+\frac{1}{k_{01}^4}\right)\right]a_1B_{01}-(1+\nu)\frac{1}{k_{01}}J_0'(k_{01}a_1)E_{01}\\&\quad+(1+\nu)\frac{1}{k_{01}}I_0(k_{01}a_1)G_{01}=0\\&-\frac{1}{k_{01}^4}B_{01}+J_0(k_{01}a_1)E_{01}+I_0(k_{01}a_1)G_{01}=0\\&J_0'(k_{01}a_1)E_{01}+I_0'(k_{01}a_1)G_{01}=0\end{aligned}\right\} \tag{3.29}$$

因B_{01}、E_{01}，G_{01}不能同时为零，则有

$$\begin{vmatrix}\left[1-\frac{1}{2}(1+\nu)\left(1+\frac{1}{k_{01}^4}\right)\right]a_1 & -(1+\nu)\frac{1}{k_{01}}J_0'(k_{01}a_1) & (1+\nu)\frac{1}{k_{01}}I_0'(k_{01}a_1)\\ -\frac{1}{k_{01}^4} & J_0(k_{01}a_1) & I_0(k_{01}a_1)\\ 0 & J_0'(k_{01}a_1) & I_0'(k_{01}a_1)\end{vmatrix}=0 \tag{3.30}$$

此处我们用到等式$J_0(ik_{01}a_1)=I_0(k_{01}a_1)$，$I_0$为修正的第一类贝塞尔函数，“$'$”表示对$r$微分。由上式，我们得到下面的频率方程

$$\left[1-\frac{1}{2}(1+\nu)\left(1+\frac{1}{k_{01}^4}\right)\right]a_1[J_0(k_{01}a_1)I_0'(k_{01}a_1)-I_0(k_{01}a_1)J_0'(k_{01}a_1)]$$

$$+\frac{1}{k_{01}^5}[-2(1+\nu)J_0'(k_{01}a_1)I_0'(k_{01}a_1)]=0 \tag{3.31}$$

将贝塞尔函数展开成$k_{01}r$的幂级数，然后代入(3.31)，最后我们得到下列含未知量k_{01}的无穷幂级数方程

$$1+a_1^* k_{01}^4+a_2 k_{01}^8+\cdots=0 \tag{3.32}$$

式中

$$\left.\begin{aligned}a_1^*&=\frac{\left\{-3.32\times10^{-1}\left[1-\frac{1}{2}(1+\nu)\right]-1.44\times10^{-3}(1+\nu)\left(\frac{1}{2}a_1\right)^4\right\}}{\left\{2\left[1-\frac{1}{2}(1+\nu)\right]+8.20\times10^{-2}(1+\nu)\left(\frac{1}{2}a_1\right)^4\right\}}\left(\frac{1}{2}a_1\right)^4\\a_2&=\frac{\left\{4.28\times10^{-3}\left[1-\frac{1}{2}(1+\nu)\right]+3.86\times10^{-6}(1+\nu)\left(\frac{1}{2}a_1\right)^4\right\}}{\left\{2\left[1-\frac{1}{2}(1+\nu)\right]+8.20\times10^{-2}(1+\nu)\left(\frac{1}{2}a_1\right)^4\right\}}\left(\frac{1}{2}a_1\right)^8\end{aligned}\right\} \tag{3.33}$$

方程(3.32)的最小根的二次近似值为

$$k_{01}^4=\sqrt{\frac{1}{-2a_2+a_1^2}} \tag{3.34}$$

由(3.11)式和(3.3)式，我们有

$$\omega_{01}^*=\frac{1}{R}\left(\frac{Eg}{\rho}\right)^{1/2}[1+k_{01}^4]^{1/2} \tag{3.35}$$

其中 h_{01}^4 由(3.34)式给出，代入前述数据，即得

$$k_{01}^4=0.254$$

最后有

$$\omega_{01}^*=83.12\text{次/秒}$$

这个结果与E. Reissner[1]用下面的公式所得的结果十分接近

$$\omega=\left(\frac{Eg}{\rho}\right)^{1/2}\frac{h}{a_1^2}\left[\frac{\mu^4+K^4(1-\nu)/(1+\nu)}{12(1-\nu^2)}\right]^{1/2}=84.10\text{次/秒}$$

式中 $K^4=\frac{1+\nu}{1-\nu}\frac{hEa_1^4}{R^2D}$，$\mu$ 为下面方程的根

$$J_0(\mu)I_1(\mu)+J_1(\mu)I_0(\mu)+\frac{4K^4J_1(\mu)I_1(\mu)}{\mu(\mu^4-K^4)}=0 \tag{3.36}$$

四、在谐载荷和谐温度场下的中心开孔圆底扁球壳的强迫振动

把方程(2.50)写成如下的形式

$$\left.\begin{aligned}&\nabla_m^2\nabla_m^2\Phi_m+\nabla_m^2W_m+\nabla_m^2T_m=0\\&\nabla_m^2\nabla_m^2W_m-\nabla_m^2\Phi_m+\nabla_m^2T_m+\frac{\partial^2W_m}{\partial\tau^2}=Q_m\\&\qquad(m=0,1,2,\cdots)\end{aligned}\right\} \tag{4.1a,b}$$

载荷和温度场为谐函数，它们可写成

$$\left.\begin{aligned} Q_m &= \overline{Q}_m(r)\, e^{i\Omega_{m\text{I}}\tau} \\ T_m &= t_m(r)\, e^{i\Omega_{m\text{II}}\tau} \\ T'_m &= t'_m(r)\, e^{i\Omega_{m\text{II}}\tau} \end{aligned}\right\} \tag{4.2a,b,c}$$

令

$$\left.\begin{aligned} W_m &= W_{m\text{I}}(r)\, e^{i\Omega_{m\text{I}}\tau} + W_{m\text{II}}(r) e^{i\Omega_{m\text{II}}\tau} \\ \Phi_m &= \varphi_{m\text{I}}(r) e^{i\Omega_{m\text{I}}\tau} + \varphi_{m\text{II}}(r) e^{i\Omega_{m\text{II}}\tau} \end{aligned}\right\} \tag{4.3a,b}$$

$W_{m\text{I}},\varphi_{m\text{I}};W_{m\text{II}},\varphi_{m\text{II}}$各自适合下列的方程

$$\left.\begin{aligned} &\Delta_m\Delta_m\varphi_{m\text{I}} + \Delta_m W_{m\text{I}} = 0 \\ &\Delta_m\Delta_m W_{m\text{I}} - \Delta_m\varphi_{m\text{I}} - \Omega^2_{m\text{I}} W_{m\text{I}} = \overline{Q}_m \\ &\qquad (m=0,1,2,\cdots) \end{aligned}\right\} \tag{4.4a,b}$$

$$\left.\begin{aligned} &\Delta_m\Delta_m\varphi_{m\text{II}} + \Delta_m W_{m\text{II}} + \Delta_m t_m = 0 \\ &\Delta_m\Delta_m W_{m\text{II}} - \Delta_m\varphi_{m\text{II}} + \Delta_m t'_m - \Omega^2_{m\text{II}} W_{m\text{II}} = 0 \\ &\qquad (m=0,1,2,\cdots) \end{aligned}\right\} \tag{4.5a,b}$$

令$-K^4_{m\text{I}} = 1-\Omega^2_{m\text{I}}$，$-K^4_{m\text{II}} = 1-\Omega^2_{m\text{II}}$，由(3.25)和文献[4]，方程(4.4)的解可写成如下的形式

$$W_{m\text{I}} = W^{(0)}_{m\text{I}}(A_{m\text{I}},B_{m\text{I}},E_{m\text{I}},F_{m\text{I}},G_{m\text{I}},H_{m\text{I}},K_{m\text{I}},r) + W^{(*)}_{m\text{I}}(r,K_{m\text{I}},\overline{Q}_m),\ (m=0,1,2,\cdots) \tag{4.6}$$

其中

$$\begin{aligned} W^{(*)}_{m\text{I}} =& \frac{\pi^2}{4} Y_m(iK_{m\text{I}}r)\int_b^r rJ_m(iK_{m\text{I}}r)Y_m(K_{m\text{I}}r)\int_b^r rJ_m(K_{m\text{I}}r)\overline{Q}_m\,dr^2 \\ &-\frac{\pi^2}{4} Y_m(iK_{m\text{I}}r)\int_b^r rJ_m(iK_{m\text{I}}r)J_m(K_{m\text{I}}r)\int_b^r rY_m(K_{m\text{I}}r)\overline{Q}_m\,dr^2 \\ &-\frac{\pi^2}{4} J_m(iK_{m\text{I}}r)\int_b^r rY_m(iK_{m\text{I}}r)Y_m(K_{m\text{I}}r)\int_b^r rJ_m(K_{m\text{I}}r)\overline{Q}_m\,dr^2 \\ &+\frac{\pi^2}{4} J_m(iK_{m\text{I}}r)\int_b^r rY_m(iK_{m\text{I}}r)J_m(K_{m\text{I}}r)\int_b^r rY_m(K_{m\text{I}}r)\overline{Q}_m\,dr^2 \\ &\qquad (m=0,1,2,\cdots) \end{aligned} \tag{4.7}$$

$W^{(0)}_{m\text{I}}$与(3.25)中的W_{mn}构造完全一样，这里在右上角加“0”是为了不致与(4.6)左端表示的通解的符号$W_{m\text{I}}$混淆，对于$\varphi^{(0)}_{m\text{I}}$同此

$$\varphi_{m\text{I}} = \varphi^{(0)}_{m\text{I}}(A_{m\text{I}},B_{m\text{I}},C_{m\text{I}},D_{m\text{I}},E_{m\text{I}},F_{m\text{I}},G_{m\text{I}},H_{m\text{I}},K_{m\text{I}},r) + \varphi^{(*)}_{m\text{I}}(r,K_{m\text{I}},\overline{Q}_m),\ (m=0,1,2,\cdots) \tag{4.8}$$

其中

$$\left.\begin{aligned}
&\text{当}\quad m=0,\ \varphi_{0\mathrm{I}}^{(*)}=-\int_b^r\frac{dr}{r}\int_b^r rW_{0\mathrm{I}}^{(*)}\,dr\\
&\text{当}\quad m=1,\ \varphi_{1\mathrm{I}}^{(*)}=-\frac{1}{r}\int_b^r r\,dr\int_b^r W_{1\mathrm{I}}^{(*)}\,dr\\
&\text{当}\quad m>1,\ \varphi_{m\mathrm{I}}^{(*)}=-r^m\int_b^r\frac{dr}{r^{2m+1}}\int_b^r r^{m+1}W_{m\mathrm{I}}^{(*)}\,dr
\end{aligned}\right\}\qquad(4.9)$$

$A_{m\mathrm{I}}$、$B_{m\mathrm{I}}$，…，$H_{m\mathrm{I}}$ 为八个待定积分常数，利用关系，(4.2)、(4.3)，由边界条件(2.52)—(2.54)决定[1)]。

同理方程(4.5)的解可写成

$$W_{m\mathrm{II}}=W_{m\mathrm{II}}^{(0)}(A_{m\mathrm{II}},B_{m\mathrm{II}},E_{m\mathrm{II}},F_{m\mathrm{II}},G_{m\mathrm{II}},H_{m\mathrm{II}},K_{m\mathrm{II}},r)$$
$$+W_{m\mathrm{I}}^{(*)}(r,K_{m\mathrm{II}},Ⓗ_m),\ (m=0,1,2,\cdots)\qquad(4.10)$$

$$\varphi_{m\mathrm{II}}=\varphi_{m\mathrm{II}}^{(0)}(A_{m\mathrm{II}},B_{m\mathrm{II}},C_{m\mathrm{II}},D_{m\mathrm{II}},E_{m\mathrm{II}},F_{m\mathrm{II}},G_{m\mathrm{II}},K_{m\mathrm{II}},r)$$
$$+\varphi_{m\mathrm{I}}^{(*)}(r,K_{m\mathrm{II}},Ⓗ_m)+\varphi_{m\mathrm{I}}^{(*)}(r,t_m)\qquad(4.11)$$

其中

$$Ⓗ_m=-\Delta_m t_m'-t_m\qquad(4.12)$$

$$\left.\begin{aligned}
&\text{当}\quad m=0,\ \varphi_{0\mathrm{I}}^{(*)}(r,t_0)=-\int_b^r\frac{dr}{r}\int_b^r r\,t_0\,dr\\
&\text{当}\quad m=1,\ \varphi_{1\mathrm{I}}^{(*)}(r,t_1)=-\frac{1}{r}\int_b^r r\,dr\int_b^r t_1\,dr\\
&\text{当}\quad m>1,\ \varphi_{m\mathrm{I}}^{(*)}(r,t_m)=-r^m\int_b^r\frac{dr}{r^{2m+1}}\int_b^r r^{m+1}t_m\,dr
\end{aligned}\right\}\qquad(4.13)$$

$A_{m\mathrm{II}}$，$B_{m\mathrm{II}}$，…，$H_{m\mathrm{II}}$ 为另外八个待定积分常数，利用关系(4.2)、(4.3)，由边界条件(2.52)—(2.54)决定[2)]。

第二类问题的基本方程(2.56)—(2.59)的求解与上述完全类似，无需另行叙述。

[2)1)] 由边界条件(2.52)—(2.54)决定待定积分常数，注意把它化为两组代数方程组各相应于方程(4.4)和(4.5)，相应于(4.5)的边界条件表达式中才含有函数T_m和T_m'。

五、具有初始条件的中心开孔的圆底扁球壳的强迫振动

现在我们在边界条件(2.52)—(2.54)和初始条件(2.55)下求解基本方程(2.50)。我们把方程(2.50)写成下面的形式

$$\left.\begin{aligned}
&\nabla_m^2\nabla_m^2\Phi_m+\nabla_m^2W_m=-\nabla_m^2T_m\\
&\nabla_m^2\nabla_m^2W_m-\nabla_m^2\Phi_m+\frac{\partial^2W_m}{\partial\tau^2}=Q_m-\nabla_m^2T_m'
\end{aligned}\right\}\qquad(5.1a,b)$$

首先，令方程(5.1)的解为

$$\left.\begin{aligned} W_m &= Y_m(r,\tau)+\xi(r,\tau) \\ \Phi_m &= \chi_m(r,\tau)+\eta(r,\tau) \end{aligned}\right\} \tag{5.2a,b}$$

其中Y_m,χ_m为所求的未知函数，ξ_m,η_m为预先给定的已知函数，依具体情况而定，它们满足这样的非齐次边界条件，即在(2.52)—(2.54)中将W_m,Φ_m换成ξ_m和η_m。将(5.2)代入(5.1)，我们有

$$\left.\begin{aligned} &\nabla_m^2\nabla_m^2\chi_m+\nabla_m^2 Y_m=-\nabla_m^2\Psi_m \\ &\nabla_m^2\nabla_m^2 Y_m-\nabla_m^2\chi_m+\frac{\partial^2 Y_m}{\partial T^2}=\Psi_m' \end{aligned}\right\} \tag{5.3a,b}$$

其中

$$\left.\begin{aligned} &\Psi_m=T_m+\xi_m+\nabla_m^2\eta_m \\ &\Psi_m'=Q_m-\nabla_m^2 T_m'-\nabla_m^2\nabla_m^2\xi_m-\frac{\partial^2\xi_m}{\partial\tau^2}+\nabla_m^2\eta_m \end{aligned}\right\} \tag{5.4a,b}$$

Y_m,χ_m满足这样的边界条件，即在(2.52)—(2.54)中，将W_m,Φ_m换成Y_m,χ_m，同时令T_m和T_m'等于零。

我们先求方程(5.3)的齐次解$Y_m^{(0)}$，$\chi_m^{(0)}$，它们满足下面的方程

$$\left.\begin{aligned} &\nabla_m^2\nabla_m^2\chi_m^{(0)}+\nabla_m^2 Y_m^{(0)}=0 \\ &\nabla_m^2\nabla_m^2 Y_m^{(0)}-\nabla_m^2\chi_m^{(0)}+\frac{\partial^2 Y_m^{(0)}}{\partial\tau^2}=0 \end{aligned}\right\} \tag{5.5a,b}$$

令

$$\left.\begin{aligned} \chi_m^{(0)} &= \bar{\chi}_m^{(0)}(r)\,\bar{\bar{\chi}}_m^{(0)}(\tau) \\ Y_m^{(0)} &= \bar{Y}_m^{(0)}(r)\,\bar{\bar{Y}}_m^{(0)}(\tau) \end{aligned}\right\} \tag{5.6a,b}$$

将(5.6)代入(5.5)，得

$$\frac{\Delta_m\Delta_m\bar{\chi}_m^{(0)}(r)}{\Delta_m\bar{Y}_m^{(0)}(r)}=-\frac{\bar{\bar{Y}}_m^{(0)}(\tau)}{\bar{\bar{\chi}}_m^{(0)}(r)}=-1$$

$$\frac{\Delta_m\Delta_m\bar{Y}_m^{(0)}(r)-\Delta_m\bar{\chi}_m^{(0)}(r)}{\bar{Y}_m^{(0)}(r)}=-\frac{\dfrac{d^2\bar{\bar{Y}}_m^{(0)}(\tau)}{d\tau^2}}{\bar{\bar{Y}}_m^{(0)}(\tau)}=k=\omega_{mn}^2$$

由此即得

$$\bar{\bar{Y}}_m^{(0)}(\tau)=\bar{\bar{\chi}}_m^{(0)}(\tau) \tag{5.7}$$

$$\left.\begin{aligned} &\Delta_m\Delta_m\bar{\chi}_m^{(0)}(r)+\Delta_m\bar{Y}_m^{(0)}(r)=0 \\ &(\Delta_m\Delta_m-\omega_{mn}^2)\bar{Y}_m^{(0)}(r)-\Delta_m\bar{\chi}_m^{(0)}(r)=0 \end{aligned}\right\} \tag{5.8}$$

$$\frac{d^2\bar{\bar{Y}}_m^{(0)}(\tau)}{d\tau^2}+\omega_{mn}^2\bar{\bar{Y}}_m^{(0)}(\tau)=0 \tag{5.9}$$

方程(5.8)为中心开孔圆底扁球壳的自由振动方程，只有当ω_{mn}值($n=1,2,\cdots$)等于本征值

的才有解，相应于此本征值的两组本征函数为$\{W_{mn}\}$，$\{\varphi_{mn}\}$，因此方程(5.8)的齐次解为

$$Y_m^{(0)}=\sum_{n=1}^{\infty}(\bar{A}_{mn}\cos\omega_{mn}\tau+\bar{B}_{mn}\sin\omega_{mn}\tau)W_{mn}(r) \tag{5.10}$$

$$\chi_m^{(0)}=\sum_{n=1}^{\infty}(\bar{A}_{mn}\cos\omega_{mn}\tau+\bar{B}_{mn}\sin\omega_{mn}\tau)\varphi_{mn}(r) \tag{5.11}$$

函数集只有夹紧边是正交的，对于其他边界条件的非正交$\{W_{mn}\}$由B. И. Смирнов著，«高等数学教程»，第四卷一分册，(1958)，pp.10-11所指出的方法可构成一正交集$\{\overline{W}_{mn}\}$，这样我们有

$$\Psi_m=\sum_{n=1}^{\infty}\overline{\Psi}_{mn}(\tau)\overline{W}_{mn}(r)=\sum_{n=1}^{\infty}\Psi_{mn}(\tau)W_{mn}(r) \tag{5.12}$$

$$\Omega_m=\sum_{n=1}^{\infty}\bar{f}_{mn}\overline{W}_{mn}(r)=\sum_{n=1}^{\infty}f_{mn}(\tau)W_{mn}(r) \tag{5.13}$$

$$(W_m-\xi_m)_{\tau=0}=\sum_{n=1}^{\infty}\bar{a}_{mn}\overline{W}_{mn}(r)=\sum_{n=1}^{\infty}a_{mn}W_{mn}(r) \tag{5.14}$$

$$\frac{\partial}{\partial\tau}(W_m-\xi_m)_{\tau=0}=\sum_{n=1}^{\infty}\bar{b}_{mn}\overline{W}_{mn}(r)=\sum_{n=1}^{\infty}b_{mn}W_{mn}(r) \tag{5.15}$$

其中a_{mn}，b_{mn}为常数，

$$\Omega_m=Q_m-\nabla_m^2T_m'-\nabla_m^2\nabla_m^2\xi_m-\frac{\partial^2\xi_m}{\partial\tau^2}+\nabla_m^2\eta_m-\sum_{n=1}^{\infty}\Psi_{mn}(\tau)W_{mn}(r) \tag{5.16}$$

方程(5.3)的特解可写成

$$\left.\begin{aligned}Y_m^{(*)}&=\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*W_{mn}(r)\\ \chi_m^{(*)}&=\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(t-\tau^*)f_{mn}(\tau^*)d\tau^*\varphi_{mn}(r)\end{aligned}\right\} \tag{5.17}$$

下面我们来证此解为真。

证：以(5.17)代入(5.3a)，我们有

$$\sum_{n=1}^{\infty}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*(\Delta_m\Delta_m\varphi_{mn}+\Delta_mW_{mn})$$

$$=-\nabla_m^2\Psi_m=-\Delta_m\sum_{n=1}^{\infty}\Psi_{mn}(\tau)W_{mn}(r)$$

将上式积分，我们得到

$$\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*[\Delta_m\varphi_{mn}+W_{mn}$$

$$-\psi_{mn}(A_{mn},B_{mn})]=-\sum_{n=1}^{\infty}\Psi_{mn}(\tau)W_{mn}$$

同时由(5.36)，

$$\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*\{[\Delta_m\Delta_m-\omega_{mn}^2]W_{mn}$$

$$-\Delta_m\varphi_{mn}\}+\sum_{n=1}^{\infty}f_{mn}(\tau)W_{mn}=Q_m-\nabla_m^2T_m'-\nabla_m^2\nabla_m^2\xi_m-\frac{\partial^2\xi_m}{\partial\tau^2}+\nabla_m^2\eta_m$$

将上面两式相加，即得

$$\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*\{[\Delta_m\Delta_m+1-\omega_{mn}^2]W_{mn}$$

$$-\psi_{mn}(A_{mn},B_{mn})\}+\sum_{n=1}^{\infty}f_{mn}(\tau)W_{mn}=\sum_{n=1}^{\infty}f_{mn}(\tau)W_{mn}$$

因此(5.17)是方程(5.3)的特解，(5.3)的通解可写成

$$\left.\begin{aligned}Y_m=&\sum_{n=1}^{\infty}[\bar{A}_{mn}\cos\omega_{mn}\tau+\bar{B}_{mn}\sin\omega_{mn}\tau\\&+\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*]W_{mn}\\\chi_m=&\sum_{n=1}^{\infty}[\bar{A}_{mn}\cos\omega_{mn}\tau+\bar{B}_{mn}\sin\omega_{mn}\tau\\&+\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*]\varphi_{mn}\end{aligned}\right\}\qquad(5.18)$$

现在再来决定积分常数$\bar{A}_{mn}$，$\bar{B}_{mn}$，当$\tau=0$，

$$\bar{A}_{mn}=a_{mn},\quad\bar{B}_{mn}=b_{mn}/\omega_{mn}$$

因此

$$W_m=\xi_m+\sum_{n=1}^{\infty}[a_{mn}\cos\omega_{mn}\tau+\frac{b_{mn}}{\omega_{mn}}\sin\omega_{mn}\tau$$

$$+\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*]W_{mn}$$

$$\Phi_{mn}=\eta_m+\sum_{n=1}^{\infty}[a_{mn}\cos\omega_{mn}\tau+\frac{b_{mn}}{\omega_{mn}}\sin\omega_{mn}\tau$$

$$+\frac{1}{\omega_{mn}}\int_0^{\tau}\sin\omega_{mn}(\tau-\tau^*)f_{mn}(\tau^*)d\tau^*]\varphi_{mn}$$

第二类基本方程(2.56)可完全类似求出，不再另行叙述。

六、讨 论

本文自由振动的解法和E. Reissner的解有本质地区别，我们的解法有一定的步骤，而E. Reissner是从直接假定出发的，因此在 $m=1$ 的情形，他的解是不完全的。

本文显然包含中心不开孔，任意不定常温度场和任意法向动载荷单独作用的情况。

在第五节中，最后问题归结为选择两个非齐次边界条件的函数 ξ_m，η_m，它们的求取并不困难。

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附 录 1

将(2.20a)积分，我们有

$$U=\int\nabla^2\Phi dr-(1+\nu)\frac{\partial\Phi}{\partial r}+\int W\,dr+\int T\,dr+f(\theta,\tau) \tag{1'}$$

其中 $f(\theta,\tau)$ 为一待定积分函数，我们在下面再讨论。将(1′)代入(2.20c)式，这样便有

$$V=-r\int\frac{dr}{r^2}\frac{\partial}{\partial\theta}\int\nabla^2\Phi\,dr-r\int\frac{dr}{r^2}\frac{\partial}{\partial\theta}\int W\,dr-r\int\frac{dr}{r^2}\frac{\partial}{\partial\theta}\int T dr$$

$$+f'(\theta,\tau)-(1+\nu)\frac{1}{r}\frac{\partial\Phi}{\partial\theta}+rg(\theta,\tau) \tag{2'}$$

其中 $g(\theta,\tau)$ 是另一待定积分函数，“′”是对 θ 微分，把(1′)，(2′)代入(2.20b)我们有

$$-\frac{\partial}{\partial\theta}\int\frac{dr}{r^2}\frac{\partial}{\partial\theta}\int(\nabla^2\Phi+W+T)\,dr+\frac{1}{r}\int(\nabla^2\Phi+W+T)\,dr$$

$$-(\nabla^2\Phi+W+T)=-\frac{1}{r}f''(\theta,\tau)-g'(\theta,\tau)-\frac{1}{r}f(\theta,\tau) \tag{3'}$$

根据第一类和第二类基本方程的求解，我们得到

$$-\frac{1}{r}f''(\theta,\tau)-\frac{1}{r}f(\theta,\tau)-g'(\theta,\tau)=-\frac{\partial}{\partial\theta}\int\frac{dr}{r^2}\frac{\partial}{\partial\theta}\int\psi dr$$

$$+\frac{1}{r}\int\psi dr-\psi \tag{4'}$$

式中

$$\psi=\psi_0(\bar{A}_0,\bar{B}_0)+\sum_{m=1}^{\infty}\psi_m(\bar{A}_m,\bar{B}_m)\cos m\theta+\sum_{m=1}^{\infty}\psi_m(\bar{A}'_m,\bar{B}'_m)\sin m\theta \tag{5'}$$

其中 $\bar{A}_m$, $\bar{B}_m(m=0,1,2,\cdots)$, $\bar{A}'_m$, $\bar{B}'_m(m=1,2,\cdots)$ 是含 τ 的积分函数，函数 $\psi_m(m=0,1,2,\cdots)$ 的定义如下：

$$\psi_0(\bar{A}_0,\bar{B}_0)=\bar{A}_0\ln r+\bar{B}_0$$
$$\psi_m(\bar{A}_m,\bar{B}_m)=\bar{A}_m r^{-m}+\bar{B}_m r^m,\quad(m\geqslant 1)$$

对于函数 $\psi_m(\bar{A}'_m,\bar{B}'_m)$ 的形式与上式相同，只不过积分函数不同而已，令 (4′) 的解的形式如下

$$\left.\begin{aligned}f(\theta,\tau)&=f_h(\theta,\tau)+f_p(\theta,\tau)\\ g(\theta,\tau)&=g_h(\theta,\tau)+g_p(\theta,\tau)\end{aligned}\right\} \tag{6'a,b}$$

其中 f_h, g_h 为(4′)的齐次解，f_p, g_p 为 (4′) 的特解。因此有

$$f_h(\theta,\tau)=\bar{C}_1(\tau)\cos\theta+\bar{C}_2(\tau)\sin\theta \tag{7'}$$

$$g_h(\theta,\tau)=\bar{C}_3(\tau) \tag{8'}$$

式中 $\bar{C}_1, \bar{C}_2, \bar{C}_3$ 是含 τ 的积分函数，由此可见 f_h 表示刚体转动，g_h 表示刚体平移。特解 f_p，g_p 满足下列方程

$$f_p''(\theta,\tau)+f_p(\theta,\tau)+r\,g_p'(\theta,\tau)=r\frac{\partial}{\partial\theta}\int\frac{dr}{r^2}\frac{\partial}{\partial\theta}\int\psi dr-\int\psi dr+r\psi \tag{9'}$$

由此即得

$$\left.\begin{aligned}f_p(\theta,\tau)&=\bar{A}_1\theta\sin\theta-\bar{A}'_1\theta\cos\theta\\ g_p(\theta,\tau)&=\bar{A}_0\theta\end{aligned}\right\} \tag{10'a,b}$$

(10′)中的 f_p, g_p 是多值函数，为了使位移单值，因此必须有 $\bar{A}_0=\bar{A}_1=\bar{A}_1'=0$. 以(2.23)、(2.26)、(2.27)、(2.37)、(2.38)代入(1′) 和 (2′)，并略去刚体位移(7′)、(8′)式，则有

$$U_m=\int\nabla_m^2\Phi_m dr-(1+\nu)\frac{\partial\Phi_m}{\partial r}+\int W_m dr+\int T_m dr \tag{11'}$$

$$\tilde{U}_m=\int\nabla_m^2\tilde{\Phi}_m dr-(1+\nu)\frac{\partial\tilde{\Phi}_m}{\partial r}+\int\tilde{W}_m dr+\int\tilde{T}_m dr \tag{12'}$$

$$\tilde{V}_m=mr\int\frac{dr}{r^2}\int(\nabla_m^2\Phi_m+W_m+T_m)\,dr+(1+\nu)\frac{m}{r}\Phi_m \tag{13'}$$

$$V_m=-mr\int\frac{dr}{r^2}\int(\nabla_m^2\tilde{\Phi}_m+\tilde{W}_m+\tilde{T}_m)dr-(1+\nu)\frac{m}{r}\tilde{\Phi}_m \tag{14'}$$

利用(11′)—(14′)便可将位移边界条件用应力函数、挠度和温度来表示。

附 录 2

在 $m=0$ 的情况. (2,20)中的 U, V, W, Φ, T 均与 θ 无关，因此有

$$\left.\begin{aligned}\bar{\varepsilon}_r&=\frac{\partial U}{\partial r}-W=\nabla_0^2\Phi-(1+\nu)\frac{\partial^2\Phi}{\partial r^2}+T\\ \bar{\varepsilon}_\theta&=\frac{U}{r}-W=(1+\nu)\frac{\partial^2\Phi}{\partial r^2}-\nu\nabla_0^2\Phi+T\end{aligned}\right\}\quad(1'a,b)$$

作

$$\frac{\partial}{\partial r}(r\bar{\varepsilon}_\theta)=\frac{\partial U}{\partial r}-r\frac{\partial W}{\partial r}-W$$

$$=(1+\nu)\frac{\partial^2\Phi}{\partial r^2}+(1+\nu)r\frac{\partial^3\Phi}{\partial r^3}-\nu\nabla_0^2\Phi-\nu r\frac{\partial}{\partial r}\nabla_0^2\Phi+T+r\frac{\partial T}{\partial r}\quad(2')$$

从(2')减去(1'a)，我们有

$$\frac{\partial}{\partial r}(\nabla_0^2\Phi)+\frac{\partial W}{\partial r}+\frac{\partial T}{\partial r}=0\quad(3')$$

将(3')积分，即得

$$\nabla_0^2\Phi+W+T=S(\tau)\quad(4')$$

其中 $S(\tau)$为仅含 τ 的积分函数。当 $m=0$，由基本方程（2.50a），经积分后有

$$\nabla_0^2\Phi+W+T=\bar{A}_0\ln r+\bar{B}_0\quad(5')$$

比较(4')和(5')式，我们有

$$\bar{A}_0=0\quad(6')$$

在第三节中心开孔圆底扁球壳的自由振动中，上式便是

$$A_{0n}=0$$

在 $m=1$ 的情况，实际上也只有六个待定积分常数和六个边界条件。在附录 1 中我们已证明 $\bar{A}_1=0$，在自由振动的情况，便是 $A_{1n}=0$。用 φ_{1n} 表示边界条件时，可看出D_{1n} 可令之为零，对于悬空边界条件，在文献[8]中已指出边界条件 $N_{r1}=0$ 和$N_{r\theta 1}=0$ 是全同的，因此每边只有三个边界条件，即

当 $r=a$（或 $r=b$）时，

$$M_{r1}=V_{r1}=N_{r1}=0\quad(7')$$

对于夹紧边，边界条件 $U_1=0$ $\widetilde{V}_1=0$ 由附录 1(11')和(13')在$m=1$ 时可写成

当 $r=a$（或 $r=b$）时

$$\left.\begin{aligned}U_1&=\int(\nabla_1^2\Phi_1+W_1+T_1)dr-(1+\nu)\frac{\partial\Phi_1}{\partial r}=0\\ \widetilde{V}_1&=r\int\frac{dr}{r^2}\int(\nabla_1^2\Phi_1+W_1+T_1)dr+(1+\nu)\frac{1}{r}\Phi_1=0\end{aligned}\right\}\quad(8'a,b)$$

亦即

当 $r=a$（或 $r=b$）时

$$\left.\begin{aligned}&\int\psi_1(\bar{A}_1,\bar{B}_1)dr-(1+\nu)\frac{\partial\Phi}{\partial r}=0\\ &r\int\frac{dr}{r^2}\int\psi_1(\bar{A}_1,\bar{B}_1)dr+(1+\nu)\frac{1}{r}\Phi_1=0\end{aligned}\right\}\quad(9'a,b)$$

我们知道

$$\psi_1(\bar{A}_1,\bar{B}_1)=\bar{B}_1 r$$

代入上式，即得

当 $r=a$（或 $r=b$）时，

$$\left.\begin{aligned}&\frac{1}{2}\bar{B}_1r^2-(1+\nu)\frac{\partial\Phi_1}{\partial r}=0\\ &\frac{1}{2}\bar{B}_1r^2+(1+\nu)\frac{1}{r}\Phi_1=0\end{aligned}\right\}\quad(10'a,b)$$

若要上面二式全同，除非满足下列条件

$$-\frac{\partial \Phi_1}{\partial r}+\frac{1}{r}\Phi_1=0 \text{ [当 } r=a \text{ (或 } r=b \text{)]}$$

亦即

$$\int \nabla_1^2\Phi_1\,dr=0 \text{ [当 } r=a \text{ (或 } r=b \text{)]} \qquad (11')$$

因此夹紧边的边界条件为

当 $r=a$ (或 $r=b$) 时，

$$W_1=\frac{\partial W_1}{\partial r}=U_1=0 \qquad (12')$$

对于满足条件(11′)的情况，$\tilde{V}_1=0$ 亦能同时满足，不然这个条件便不能满足。

对于简支边的边界条件和夹紧边一样，边界条件 $\tilde{V}_1=0$ 只能条件地满足，此种情况，相应于(11′)的条件为

$$(1+\nu)\frac{\partial \Phi_1}{\partial r}+r\int\frac{dr}{r^2}\int\psi_1(\bar{A}_1,\bar{B}_1)dr=0$$

$$\text{[当 } r=a \text{ (或 } r=b \text{)]} \qquad (13')$$

简支边的边界条件为

当 $r=a$ (或 $r=b$) 时，

$$W_1=M_{r1}=N_{r1}=0 \qquad (14')$$

附 录 3

设幂级数方程

$$1-a_1X-a_2X^2-a_3X^3-\cdots\cdots\cdots\cdots=0 \qquad (1')$$

a_1,a_2,a_3 皆为实数。设方程(1′)的根依绝对值大小排列为 $X_1,X_2,X_3,\cdots$，令

$$1-a_1X-a_2X^2-\cdots=\left(1-\frac{X}{X_1}\right)\left(1-\frac{X}{X_2}\right)\cdots \qquad (2')$$

由公式

$$\ln(1-y)=-y-y^2/_2-y^3/_3-\cdots$$

因此将(2′)两端取对数，即可得到下面的关系式

$$\left.\begin{aligned}&\sum_{i=1}^{\infty}\frac{1}{X_i}=a_1\\&\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{X_i^2}=a_2+\frac{1}{2}a_1^2\\&\frac{1}{3}\sum_{i=1}^{\infty}\frac{1}{X_i^3}=a_3+a_1a_2+\frac{1}{3}a_1^3\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \qquad (3')$$

由于我们假设

$$|X_1|<|X_2|<|X_3|<\cdots$$

所以有

$$\left|\frac{1}{X_1}\right|>\left|\frac{1}{X_2}\right|>\left|\frac{1}{X_3}\right|>\cdots$$

若在（3′）中略去$\frac{1}{X_2}$，$\frac{1}{X_3}$，… 各值，则我们得到最小根的第一次近似

$$X_1^{(1)}=\frac{1}{a_1}$$

若略去$\frac{1}{X_2^2}$，$\frac{1}{X_3^2}$，…，则得到的最小根二次近似值为

$$X_1^{(2)}=\sqrt{\frac{1}{2a_2+a_1^2}}$$

依此类推，可以得到我们所需要的精确度。

假若要想求方程(1′)的第二根 X_2，可以 $\left(1-\frac{X}{X_1}\right)$ 去除原来级数而得到新的级数，然后用同样的方法求解。

用这个方法求频率方程的根是非常有效的。

Circular Shallow Spherical Shells with Circular Holes at the Center under Simultaneous Actions of Arbitrary Unsteady Temperature Field and Arbitrary Dynamic Normal Load

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Abstract

In this paper, under assumption that temperature is linearly distributed along the thickness of the shell, we deal with problems as indicated in the title and obtain general solutions of them which are expressed in analytic form.

In the first part, we investigate free vibration of circular shallow spherical shells with circular holes at the center under usual arbitrary boundary conditions. As an example, we calculate fundamental natural frequency of a circular shallow spherical shell whose edge is fixed (m=0). Results we get are expressed in analytic form and check well with E. Reissner's [1]. Method for calculating frequency equation is recently suggested by Chien Wei-zang and is to be introduced in appendix 3.

In the second part, we investigate forced vibration of shells as indicated in the title under arbitrary harmonic temperature field and arbitrary harmonic dynamic normal load.

In the third part, we investigate forced vibration of the above mentioned shells with initial conditions under arbitrary unsteady temperature field and arbitrary normal load.

In appendix 1 and 2, we discuss how to express displacement boundary conditions with stress function and boundary conditions in the case m=1.