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中心开口圆底扁薄球壳在任意分布横向载荷下的弯曲问题的一般解

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摘　要

本文在文献[1]的某些启示下，求出了任意分布载荷下中心开孔圆底扁薄球壳弯曲问题的一般解．就作者所知，目前各国已发表有关这类问题的文献为数并不多．

一、引　　言

在许多实用问题中，往往要在扁壳中心开孔，这样就形成了多连通扁壳．Власов В. З.[2]曾研究过在横向均匀载荷下中心开孔的圆底扁薄球壳的弯曲问题，应该指出，他的某些假设并不全面．本文在文献[1]的某些启示下求出了任意分布载荷下中心开孔圆底扁薄球壳弯曲问题的通解．就作者所知，目前各国已发表有关这类问题的文献为数并不多．

二、基本方程和边界条件

众所周知，圆底扁薄球壳的基本方程为

$$\left.\begin{aligned}&\frac{R}{Eh}\nabla_1^2\nabla_1^2\varphi_1+\nabla_1^2 w_1=0\\&\nabla_1^2\varphi_1-DR\nabla_1^2\nabla_1^2 w_1=-Rq_1(r_1,\theta)\end{aligned}\right\}\qquad(2.1\text{a,b})$$

弯矩、剪力、扭矩、反力间有下列关系

$$\left.\begin{aligned}&M_{r_1}=-D\left(\frac{\partial^2 w_1}{\partial r_1^2}+\frac{\nu}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{\nu}{r_1^2}\frac{\partial^2 w_1}{\partial\theta^2}\right)\\&M_\theta=-D\left(\frac{1}{r_1}\frac{\partial w_1}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2 w_1}{\partial\theta^2}+\nu\frac{\partial^2 w_1}{\partial r_1^2}\right)\\&M_{r_1\theta}=-D(1-\nu)\left(\frac{1}{r_1}\frac{\partial^2 w_1}{\partial r_1\partial\theta}-\frac{1}{r_1^2}\frac{\partial w_1}{\partial\theta}\right)\end{aligned}\right\}\qquad(2.2\text{a,b,c})$$

$$\left.\begin{aligned}&Q_{r_1}=-D\frac{\partial}{\partial r_1}(\nabla_1^2 w_1)\\&Q_\theta=-D\frac{1}{r_1}\frac{\partial}{\partial\theta}(\nabla_1^2 w_1)\end{aligned}\right\}\qquad(2.3\text{a,b})$$

$$V_{r_1}=Q_{r_1}+\frac{1}{r_1}\frac{\partial}{\partial\theta}(M_{r_1\theta}) \tag{2.4}$$

薄膜内力为

$$\left.\begin{aligned}N_{r_1}&=\frac{1}{r_1}\frac{\partial\varphi_1}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2\varphi_1}{\partial\theta^2}\\ N_\theta&=\frac{\partial^2\varphi_1}{\partial r_1^2}\\ N_{r_1\theta}&=\frac{1}{r_1^2}\frac{\partial\varphi_1}{\partial\theta}-\frac{1}{r_1}\frac{\partial^2\varphi_1}{\partial r_1\partial\theta}\end{aligned}\right\} \tag{2.5a,b,c}$$

应变分量与位移分量及薄膜内力的关系为:

$$\left.\begin{aligned}\varepsilon_{r_1}&=\frac{\partial u_1}{\partial r_1}-\frac{w_1}{R}=\frac{1}{Eh}(N_{r_1}-\nu N_\theta)\\ \varepsilon_\theta&=\frac{1}{r_1}\frac{\partial v_1}{\partial\theta}+\frac{u_1}{r_1}-\frac{w_1}{R}=\frac{1}{Eh}(N_\theta-\nu N_{r_1})\\ \gamma_{r_1\theta}&=\frac{1}{r_1}\frac{\partial u_1}{\partial\theta}+r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)=\frac{2(1+\nu)}{Eh}N_{r_1\theta}\end{aligned}\right\} \tag{2.6a,b,c}$$

其中

$$\nabla_1^2=\frac{\partial^2}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}+\frac{1}{r_1^2}\frac{\partial^2}{\partial\theta^2} \tag{2.7}$$

壳体基本尺寸如图1所示，内力正方向如图2所示，$D=\dfrac{Eh^3}{12(1-\nu^2)}$是抗弯刚度，$E$是杨氏

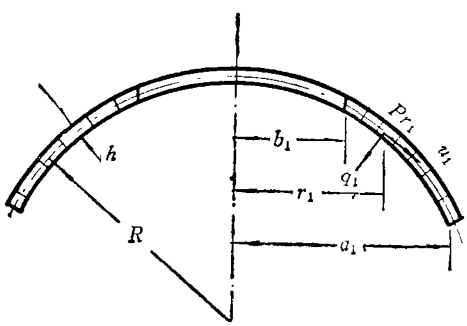

图1

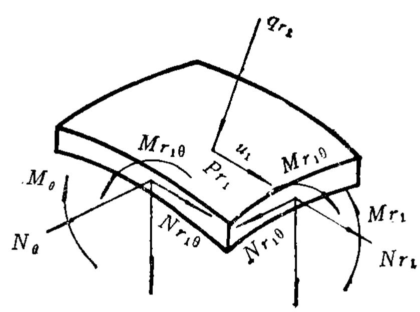

图2

模量，ν是泊松比，R是壳体中性面曲率半径，u_1, v_1各为径向和切向位移，φ_1，w_1为应力函数和挠度函数.

寻常边界条件有下列三种类型:

当$r_1=a_1$(或$r_1=b_1$)时，

a) 夹紧边

$$w_1=\frac{\partial w_1}{\partial r_1}=0\,,\quad u_1=v_1=0 \tag{2.8}$$

b) 简支边

$$w_1=M_{r_1}=0,\quad N_{r_1}=v_1=0 \tag{2.9}$$

c) 悬空边

$$M_{r_1}=V_{r_1}=0,\quad N_{r_1}=N_{r_1\theta}=0 \tag{2.10}$$

现在的问题是：要在边界条件(2.8)、(2.9)、(2.10)下求解方程(2.1a,b).

三、一 般 解 法

令 $r_1=r_0 r$，此处r_0为待定常数，则方程(2.1a,b)化为下列的形式：

$$\left.\begin{aligned}&\frac{R}{Ehr_0^2}\bar{\nabla}^2\bar{\nabla}^2\varphi+\bar{\nabla}^2 w=0\\&\bar{\nabla}^2\varphi-DR/r_0^2\bar{\nabla}^2\bar{\nabla}^2 w=-r_0^2Rq(r,\theta)\end{aligned}\right\} \tag{3.1a,b}$$

其中

$$\bar{\nabla}^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{3.2}$$

引进下列无量纲量：

$$\left.\begin{aligned}&\varphi=\frac{\bar{q}(\alpha,\theta)b_1^4}{16h}\sqrt{12(1-\nu^2)}\,\psi(\alpha,\theta)\\&w=\frac{q(\alpha,\theta)b_1^4}{16D}\bar{\omega}(\alpha,\theta)\\&u=\bar{U}b_1,\quad v=\bar{V}b_1,\quad r=\frac{b_1}{2}\alpha\\&\bar{N}_r=\frac{b_1^2N_r}{D},\quad \bar{N}_\theta=\frac{b_1^2N_\theta}{D},\quad \bar{N}_{r\theta}=\frac{b_1^2N_{r\theta}}{D}\\&\bar{V}_r=\frac{b_1^2V_r}{D},\quad \bar{V}_\theta=\frac{b_1^2V_\theta}{D},\quad \bar{M}_r=\frac{b_1M_r}{D}\\&\bar{M}_\theta=\frac{b_1M_\theta}{D},\quad \bar{M}_{r\theta}=\frac{b_1M_{r\theta}}{D}\\&\bar{Q}_r=\frac{b_1^2Q_r}{D},\quad \bar{Q}_\theta=\frac{b_1^2Q_\theta}{D}\\&Q=\frac{q(\alpha,\theta)b_1^3}{D}\end{aligned}\right\} \tag{3.3}$$

则方程(3.1a)变成：

$$\nabla^2\nabla^2(\psi Q)+\frac{r_0^2b_1^2}{4Rh}\sqrt{12(1-\nu^2)}\nabla^2(\bar{\omega}Q)=0 \tag{3.4}$$

其中

$$\nabla^2=\frac{\partial^2}{\partial\alpha^2}+\frac{1}{\alpha}\frac{\partial}{\partial\alpha}+\frac{1}{\alpha^2}\frac{\partial^2}{\partial\theta^2} \tag{3.5}$$

取

$$r_0=\sqrt{\frac{4Rh}{b_1^2\sqrt{12(1-\nu^2)}}} \tag{3.6}$$

则方程(3 4)化成下面的简单形式

$$\nabla^2\nabla^2 p+\nabla^2\bar{m}=0 \tag{3.7}$$

其中$p=\psi Q, \bar{m}=\bar{\omega} Q$. 方程(3.1b)化成如下的形式

$$\nabla^2\nabla^2\bar{m}-\nabla^2 p=r_0^4 Q \tag{3.8}$$

方程(3.7)、(3.8)，便是圆底扁薄球壳平衡问题的基本方程. 对方程(3.7)两侧进行拉普拉斯运算，我们得出

$$\nabla^2\nabla^2\nabla^2 p+\nabla^2\nabla^2\bar{m}=0 \tag{3.9}$$

利用方程(3.8)，则方程(3.9)变成

$$\nabla^2\nabla^2\nabla^2 p+\nabla^2 p=K \tag{3.10}$$

其中$K=-r_0^4 Q$. 现在我们把 p, $\bar{m}$, K 展开成傅里叶级数形式：

$$(p,\bar{m},K)=(p_0,m_0,k_0)+\sum_{m=1}^{\infty}(p_m,m_m,k_m)\cos m\theta$$

$$+\sum_{m=1}^{\infty}(\bar{p}_m,\bar{m}_m,\bar{k}_m)\sin m\theta \tag{3.11}$$

其中

$$\left.\begin{aligned}(p_0,\ m_0,\ k_0)&=\frac{1}{2\pi}\int_0^{2\pi}(p,\bar{m},K)d\theta\\(p_m,\ m_m,\ k_m)&=\frac{1}{\pi}\int_0^{2\pi}(p,\bar{m},K)\cos m\theta d\theta\\(\bar{p}_m,\ \bar{m}_m,\ \bar{k}_m)&=\frac{1}{\pi}\int_0^{2\pi}(p,\bar{m},K)\sin m\theta d\theta\end{aligned}\right\} \tag{3.12}$$

将(3.11)代入方程(3.9)，我们即得

$$\nabla_m^2(\nabla_m^2\nabla_m^2+1)p_m=k_m \tag{3.13}$$

其中

$$\nabla_m^2=\frac{d^2}{d\alpha^2}+\frac{1}{\alpha}\frac{d}{d\alpha}-\frac{m^2}{\alpha^2} \tag{3.14}$$

现在我们按$m=0$, $m=1$, $m>1$三种不同情况分别来讨论.

1) $m=0$情况：

由方程(3.14)，我们有

$$\nabla_0^2(\nabla_0^2\nabla_0^2+1)p_0=k_0 \tag{3.15}$$

令

$$(\nabla_0^2\nabla_0^2+1)p_0=y_0$$

则方程(3.15)可以写成如下形式

$$\left.\begin{aligned}&\nabla_0^2 y_0=k_0\\&\nabla_0^2\nabla_0^2 p_0+p_0=y_0\end{aligned}\right\} \tag{3.16a,b}$$

由(3.16a)，我们有

$$\frac{1}{\alpha}\frac{d}{d\alpha}\left(\alpha\frac{dy_0}{d\alpha}\right)=k_0$$

解此方程即得

$$y_0=A_0'\ln\alpha+B_0'+\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}k_0\alpha d\alpha \tag{3.17}$$

其中

$$\alpha_1=\frac{b_1^4\sqrt{12(1-\nu^2)}}{\sqrt{Rh}} \tag{3.18}$$

令方程(3.16b)的特解为 y_0^*，齐次解为 p_{01}，p_{02}，则方程(3.15)的解在此情形可写成

$$p_0=y_0^*+p_{01}+p_{02} \tag{3.19}$$

这里的y_0^*，p_{01}，p_{02}将在 3)中给出.

2） $m=1$ 情况：

在此情况，方程(3.13)可书成

$$\left.\begin{array}{l}\nabla_1^2 y_1=k_1\\ \nabla_1^2\nabla_1^2 p_1+p_1=y_1\end{array}\right\} \tag{3.20a,b}$$

方程(3.20a)可写成

$$\frac{d}{d\alpha}\left[\frac{1}{\alpha}\frac{d}{d\alpha}(\alpha y_1)\right]=k_1$$

解此方程即得

$$y_1=A_1'\alpha+B_1'\alpha^{-1}+\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}k_1 d\alpha \tag{3.21}$$

和 1）一样，令(3.20b)的特解为y_1^*，齐次解为p_{11}，p_{12}，因此方程(3.20b)的解可以写成

$$p_1=y_1^*+p_{11}+p_{12} \tag{3.22}$$

这里的y_1^*，p_{11}，p_{12}也由 3 ）给出.

3） $m>1$ 情况：

$$\left.\begin{array}{l}\nabla_m^2 y_m=k_m\\ \nabla_m^2\nabla_m^2 p_m+p_m=y_m\end{array}\right\} \tag{3.23a,b}$$

令

$$y_m=\alpha^m z_m$$

即可得出

$$\frac{d}{d\alpha}\left(\alpha^{2m+1}\frac{dz_m}{d\alpha}\right)=\alpha^{m+1}k_m$$

因此方程(3.23a)可写成如下的可积形式

$$\frac{d}{d\alpha}\left[\alpha^{2m+1}\frac{d}{d\alpha}\left(\frac{y_m}{\alpha_m}\right)\right]=\alpha^{m+1}k_m \tag{3.24}$$

将上面方程积分，即得

$$y_m=A_m'\alpha^{-m}+B_m'\alpha^m+\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}k_m\alpha^{m+1}d\alpha \tag{3.25}$$

我们令方程(3.23b)的特解为y_m^*，齐次解为p_{m1}, p_{m2}，则方程(3.23b)的解可写成如下的形式

$$p_m=y_m^*+p_{m1}+p_{m2} \tag{3.26}$$

现在我们来求解 $p_{m1}, p_{m2}(m=0,1,2,\cdots)$，$y_m^*(m=0,1,2,\cdots)$. 方程 (3.23b) 的两个齐次解应满足以下两个独立的二阶齐次方程：

$$\left.\begin{array}{l}\nabla_m^2 p_{m1}-i p_{m1}=0\\ \nabla_m^2 p_{m2}+i p_{m2}=0\end{array}\right\} \tag{3.27a,b}$$

其中 $i=\sqrt{-1}$. 方程(3.27)的解为

$$
\left.\begin{aligned}
p_{m1}&=F'_mJ_m(ai^{1/2})+G'_mT_m(ai^{1/2})\\
p_{m2}&=E'_mJ_m(ai^{-1/2})+H'_mT_m(ai^{-1/2})
\end{aligned}\right\}\tag{3.28}
$$

其中 J_m 为 m 阶贝塞尔函数，T_m 为 m 阶柱函数。F'_m，G'_m，E'_m，H'_m 为待定积分常数. 由于贝塞尔函数中含有复数变量，为了用实数变量表示，我们记

$$
\left.\begin{aligned}
i^mJ_m(ai^{1/2})&=\mathrm{ber}_m\,a+i\,\mathrm{bei}_m\,a\\
i^{-m}T_m(ai^{1/2})&=\mathrm{ker}_m\,a+i\,\mathrm{kei}_m\,a\\
i^{-m}J_m(ai^{-1/2})&=\mathrm{ber}_m\,a-i\,\mathrm{bei}_m\,a\\
i^mT_m(ai^{-1/2})&=\mathrm{ker}_m\,a-i\,\mathrm{kei}_m\,a
\end{aligned}\right\}\tag{3.29}
$$

其中ber，ker，bei．kei 为汤姆逊函数．利用(3.29)式，(3.26)式可写成下面的形式

$$
p_m=y_m^*+F_m\mathrm{ber}_m\,a+G_m\mathrm{bei}_m\,a+E_m\,\mathrm{ker}_m\,a+H_m\,\mathrm{kei}_m\,a\tag{3.30}
$$

此处F_m，G_m，E_m，H_m为待定积分常数．对于1)，2)两种情况，(3.30)式同样可以使用，仅需令 $m=0$，$m=1$ 就可以了．

至于特解y_m^*，我们可以这样来考虑：

由方程(3.1a)，我们得到

$$
\overline{\nabla}^2\,\overline{\nabla}^2 w=-\frac{R}{\mathrm{Ehr_0}^2}\,\overline{\nabla}^2\,\overline{\nabla}^2\,\overline{\nabla}^2\varphi\tag{3.31}
$$

将(3.31)代入方程(3.1b)式，我们得到

$$
\overline{\nabla}^2\varphi+D\frac{R^2}{\mathrm{Ehr_0}^4}\,\overline{\nabla}^2\,\overline{\nabla}^2\,\overline{\nabla}^2\varphi=-r_0^2Rq\tag{3.32}
$$

或写成

$$
\overline{\nabla}^2\left(\overline{\nabla}^2\,\overline{\nabla}^2\varphi+\frac{16}{b_1^4}\varphi\right)=-\frac{16}{b_1^4}r_0^2Rq\tag{3.33}
$$

令(3.33)中

$$
\overline{\nabla}^2\,\overline{\nabla}^2\varphi+\frac{16}{b_1^4}\varphi=y_1\tag{3.34}
$$

则方程(3.33)可写成下面的组合形式

$$
\left.\begin{aligned}
\overline{\nabla}^2 y_1&=-\frac{16}{b_1^4}r_0^2Rq\\
\overline{\nabla}^2\,\overline{\nabla}^2\varphi+\frac{16}{b_1^4}\varphi&=y_1
\end{aligned}\right\}\tag{3.35a,b}
$$

由(3.35a)，我们有

$$
\nabla^2\overline{y_1}=\frac{\qquad}{b_1^3}\frac{Eh^2}{\sqrt{12(1-\nu^2)}}K\tag{3.36}
$$

为使两边取得一致的无量纲量，我们可取

$$
y=\frac{b_1^3\sqrt{12(1-\nu^2)}}{Eh^2}\overline{y_1}
$$

则方程(3.36)可以化成如下的简单形式

$$
\nabla^2y=K\tag{3.37}
$$

将 y 展开成傅里叶级数形式

$$y=y_0+\sum_{m=1}^{\infty} y_m\cos m\theta+\sum_{m=1}^{\infty}\overline{y}_m\sin m\theta \tag{3.38}$$

将(3.38)式和(3.11)式代入方程(3.37)，我们得到和(3.23a)相同的方程

$$\nabla_m{}^2 y_m=k_m \quad (m=0,1,2,\cdots) \tag{3.39}$$

由(3.35b)，令

$$\lambda^4=\frac{16}{b_1{}^4} \tag{3.40}$$

并利用 $\bar{y}_i$ 与 y 的关系式，我们有

$$\nabla^2\nabla^2\varphi+\varphi=\frac{1}{\lambda^4}\frac{Eh^2}{b_1{}^3\sqrt{12(1-\nu^2)}}y \tag{3.41}$$

应用(3.38)式代入(3.41)式，并利用(3.17)，(3.21)，(3.25)等式，我们有

$$\begin{aligned}\nabla^2\nabla^2\varphi+\varphi=&\frac{1}{\lambda^4}\frac{Eh^2}{b_1{}^3\sqrt{12(1-\nu^2)}}\Big\{A_0'\ln\alpha+B_0'+\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}k_0\alpha\, d\alpha\\&+\Big[A_1'\alpha+B_1'\alpha^{-1}+\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}k_1 d\alpha\Big]\cos\theta+\sum_{m=2}^{\infty}\Big[A_m'\alpha^{-m}\\&+B_m'\alpha^m+\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}k_m\alpha^{m+1}d\alpha\Big]\cos m\theta\\&+\sum_{m=1}^{\infty}\bar{y}_m\sin m\theta\Big\}\end{aligned} \tag{3.42}$$

此处 $\bar{y}_m$ 和 y_m 形式完全一致，仅需把积分常数改为 $\bar{A}_1'$、$\bar{B}_1'$、$\bar{A}_m'$、$\bar{B}_m'$，$k_m(k=1,2,\cdots)$ 改为 $\bar{k}_m(m=1,2,\cdots)$，显然这个方程的特解为

$$\begin{aligned}\varphi^*=&\frac{1}{\lambda^4}\frac{Eh^2}{b_1{}^3\sqrt{12(1-\nu^2)}}\Big\{A_0'\ln\alpha+B_0+(A_1\alpha+B_1\alpha^{-1})\cos\theta\\&+\sum_{m=2}^{\infty}(A_m'\alpha^{-m}+B_m\alpha^m)\cos m\theta+(\bar{A}_1\alpha+\bar{B}_1'\alpha^{-1})\sin\theta\\&+\sum_{m=2}^{\infty}(\bar{A}_m'\alpha^{-m}+\bar{B}_m'\alpha^m)\sin m\theta\Big\}\\&-\frac{Eh^2}{2\pi\lambda^2 b_1{}^3\sqrt{12(1-\nu^2)}}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta(\delta,\vartheta)\,\mathrm{kei}\,\rho_1\,\delta d\delta d\vartheta\end{aligned}$$

其中的 $\mathrm{kei}\rho_1$ 见附录 2 (4′)及(5′)，

$$\begin{aligned}\eta(\delta,\vartheta)=&\int_{\alpha_1}^{\delta}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\delta}k_0\alpha\, d\alpha+\Big(\frac{1}{\delta}\int_{\alpha_1}^{\delta}\alpha\, d\alpha\int_{\alpha_1}^{\delta}k_1\, d\alpha\Big)\cos\vartheta\\&+\sum_{m=2}^{\infty}\Big(\delta^m\int_{\alpha_1}^{\delta}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\delta}k_m\alpha^{m+1}d\alpha\Big)\cos m\vartheta\end{aligned}$$

$$+\left(\frac{1}{\delta}\int_{\alpha_1}^{\delta} ad\,a\int_{\alpha_1}^{\delta} \bar{k}_1 da\right)\sin\vartheta+\sum_{m=2}^{\infty}\left(\delta^m\int_{\alpha_1}^{\delta}\frac{1}{a^{2m+1}}da\int_{\alpha_1}^{\delta}\bar{k}_m a^{m+1}da\right)\sin m\vartheta$$

利用(3.3)式及φ与ψ，ψ与p的关系式，我们即得

$$p^*=\Big\{A_0'\ln a+B_0+(A_1'a+B_1'a^{-1})\cos\theta$$

$$+\sum_{m=2}^{\infty}(A_m'a^{-m}+B_m'a^m)\cos m\theta+(\bar{A}_1'a+\bar{B}_1'a^{-1})\sin\theta$$

$$+\sum_{m=2}^{\infty}\left(\bar{A}_m a^{-m}+\bar{B}_m' a^m\right)\sin\theta\Big\}-\frac{2r_0^2}{\pi b_1^2}\int_{\alpha_1}^{\alpha_2}\eta(\delta,\vartheta)\mathrm{kei}\rho_1\delta d\delta d\vartheta \quad (3.43)$$

其中

$$\alpha_2=\frac{a_1\sqrt[4]{12(1-\nu^2)}}{\sqrt{Rh}} \quad (3.44)$$

利用(3.11)式和(3.16b)，(3.20b)，(3.23b) 我们得到下面的一系列特解

$$\left.\begin{aligned}
y_0^*&=A_0'\ln a+B_0'-\frac{2r_0^2}{\pi b_1^2}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta_0(\delta,\vartheta)\mathrm{kei}_0\rho_1\delta\, d\delta d\,\vartheta\\
y_1^*&=A_1'a+B_1'a^{-1}-\frac{2r_0^2}{\pi b_1^2}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta_1(\delta,\vartheta)\mathrm{kei}_1\rho_1\delta d\delta d\vartheta\\
y_m^*&=A_m'a^{-m}+B_m'a^m-\frac{2r_0^2}{\pi b_1^2}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta_m(\delta,\vartheta)\mathrm{kei}_m\rho_1\delta d\delta d\vartheta\ (m\geqslant 2)\\
\bar{y}_1^*&=\bar{A}_1'a+\bar{B}_1'a^{-1}-\frac{2r_0^2}{\pi b_1^2}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\bar{\eta}_1(\delta,\vartheta)\mathrm{kei}_1\rho_1\delta d\delta d\vartheta\\
\bar{y}_m^*&=\bar{A}_m'a^{-m}+\bar{B}_m'a^m-\frac{2r_0^2}{\pi b_1^2}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\bar{\eta}_m(\delta,\vartheta)\mathrm{kei}_m\rho_1\delta d\delta d\vartheta\ (m\geqslant 2)
\end{aligned}\right\} \quad (3.45)$$

其中

$$\left.\begin{aligned}
\eta_0(\delta,\vartheta)&=\int_{\alpha_1}^{\delta}\frac{1}{a}da\int_{\alpha_1}^{\delta}k_0 a da\\
\eta_1(\delta,\vartheta)&=\left(\frac{1}{\delta}\int_{\alpha_1}^{\delta}ada\int_{\alpha_1}^{\delta}k_1 da\right)\cos\vartheta\\
\eta_m(\delta,\vartheta)&=\left(\delta^m\int_{\alpha_1}^{\delta}\frac{1}{a^{2m+1}}da\int_{\alpha_1}^{\delta}k_m a^{m+1}da\right)\cos m\vartheta\ (m\geqslant 2)\\
\bar{\eta}_1(\delta,\vartheta)&=\left(\frac{1}{\delta}\int_{\alpha_1}^{\delta}ad\,a\int_{\alpha_1}^{\delta}\bar{k}_1 da\right)\sin\vartheta\\
\bar{\eta}_m(\delta,\vartheta)&=\left(\delta^m\int_{\alpha_1}^{\delta}\frac{1}{a^{2m+1}}da\int_{\alpha_1}^{\delta}\bar{k}_m a^{m+1}da\right)\sin m\vartheta\ (m\geqslant 2)
\end{aligned}\right\} \quad (3.46)$$

当 $a<\delta$ 时，

$\mathrm{kei}_0\rho_1=\mathrm{kei}_0\delta\,\mathrm{ber}_0 a+\mathrm{ker}_0\,\delta\,\mathrm{bei}_0 a$

$\mathrm{kei}_1\rho_1=2(\mathrm{kei}_1\,\delta\mathrm{ber}_1 a+\mathrm{ker}_1\delta\mathrm{bei}_1 a)\cos\vartheta$

$\mathrm{kei}_m\rho_1=2(\mathrm{kei}_m\,\delta\mathrm{ber}_m a+\mathrm{ker}_m\,\delta\mathrm{bei}_m a)\cos m\vartheta,(m\geqslant 2)$

$\mathrm{kei}_1\rho_1=2(\mathrm{kei}_1\,\delta\mathrm{ber}_1 a+\mathrm{ker}_1\,\delta\mathrm{bei}_1\,a)\sin\vartheta$

$\mathrm{kei}_m\rho_1=2(\mathrm{kei}_m\,\delta\,\mathrm{ber}_m\alpha+\mathrm{ker}_m\,\delta\mathrm{bei}_m\alpha)\sin m\vartheta\ \ (m\geqslant 2)$ (3.47)

当 $\alpha>\delta$ 时

$\mathrm{kei}_0\rho_1=\mathrm{bei}_0\,\delta\,\mathrm{ker}_0\alpha+\mathrm{ber}_0\delta\,\mathrm{kei}_0\alpha$

$\mathrm{kei}_1\rho_1=2(\mathrm{bei}_1\,\delta\mathrm{ker}_1\,\alpha+\mathrm{ber}_1\delta\,\mathrm{kei}_1\alpha)\cos\vartheta$

$\mathrm{kei}_m\rho_1=2(\mathrm{bei}_m\,\delta\mathrm{ker}_m\,\alpha+\mathrm{ber}_m\,\delta\,\mathrm{kei}_m\alpha)\cos m\,\vartheta\ \ (m\geqslant 2)$

$\mathrm{kei}_1\rho_1=2(\mathrm{bei}_1\delta\,\mathrm{ker}_1\,\alpha+\mathrm{ber}_1\delta\,\mathrm{kei}_1\,\alpha)\sin\vartheta$

$\mathrm{kei}_m\rho_1=2(\mathrm{bei}_m\delta\,\mathrm{ker}_m\,\alpha+\mathrm{ber}_m\delta\,\mathrm{kei}_m\,\alpha)\sin m\vartheta\ \ (m\geqslant 2)$ (3.48)

现在我们来求折合挠度函数$\overline{m}$.

由方程(3.7)，我们有

$$\nabla^2\overline{m}=-\nabla^2\nabla^2 p \tag{3.49}$$

以傅里叶级数展开式(3.11)代入上面的方程(3.49)，我们得到

$$\nabla_m{}^2 m_m=-\nabla_m{}^2\nabla_m{}^2 p_m\ \ (m=0,1,2,\cdots) \tag{3.50}$$

现在我们分三种情形$m=0$，$m=1$，$m\geqslant 2$来讨论.

1） $m=0$ 情形

$$\nabla_0{}^2 m_0=-\nabla_0{}^2\nabla_0{}^2 p_0 \tag{3.51}$$

对上面方程进行积分，即得

$$m_0=A_0\ln\alpha+B_0-\nabla_0{}^2 p_0 \tag{3.52}$$

以(3.19)式代入上式，即得

$$m_0=A_0\ln\alpha+B_0-\nabla_0{}^2 y_0{}^*-\nabla_0{}^2(p_{01}+p_{02}) \tag{3.53}$$

此式还可以写成另一种等价的式子，将方程(3.16b)代到方程(3.51)，即得

$$\nabla_0{}^2 m_0=-y_0+p_0 \tag{3.54}$$

以(3.17)，(3.19)和(3.45)的第一个式子代入方程(3.54)，我们得到

$$\begin{aligned}\nabla_0{}^2 m_0=&-A_0'\ln\alpha-B_0'-\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}k_0\alpha d\alpha\\&+A_0'\ln\alpha+B_0'-\frac{2r_0{}^2}{\pi b_1{}^2}\int_{\alpha_1}^{\alpha}\int_0^{2\pi}\eta_0(\delta,\vartheta)\mathrm{kei}_0\rho_1\delta d\delta d\vartheta\\&+F_0\mathrm{ber}_0\alpha+G_0\mathrm{bei}_0\alpha+E_0\mathrm{ker}_0\alpha+H_0\mathrm{kei}_0\alpha\end{aligned}$$

利用部分积分，我们有

$$\begin{aligned}m_0=&A_0\ln\alpha+B_0-\frac{1}{4}(\alpha^2\ln\alpha-\alpha^2)\int_{\alpha_1}^{\alpha}k_0\alpha d\alpha\\&-\frac{1}{4}\int_{\alpha_1}^{\alpha}k_0\alpha^3 d\alpha-\frac{1}{4}\ln\alpha\int_{\alpha_1}^{\alpha}k_0\alpha^3 d\alpha+\frac{1}{4}\int_{\alpha_1}^{\alpha}k_0\alpha^3\ln\alpha d\alpha\\&+\frac{1}{4}\alpha^2\int_{\alpha_1}^{\alpha}k_0\alpha\ln\alpha\,d\alpha\\&-\frac{2r_0{}^2}{\pi b_1{}^2}\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta_0(\delta,\vartheta)\mathrm{kei}_0\rho_1\,\delta d\delta d\vartheta\alpha d\alpha\\&+F_0\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{ber}_0\alpha\alpha d\alpha+G_0\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{bei}_0\rho_1\alpha d\alpha\end{aligned}$$

$$+E_0\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{ker}_0\rho_1\alpha d\alpha+H_0\int_{\alpha_1}^{\alpha}\frac{1}{\alpha}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{kei}_0\rho_1\alpha d\alpha \quad (3.55)$$

2） $m=1$ 情形

由方程(3.50)，我们有

$$\nabla_1^2 m_1=-\nabla_1^2\nabla_1^2 p_1 \quad (3.56)$$

和情形1)一样，我们有

$$m_1=A_1\alpha+B_1\alpha^{-1}-\frac{1}{16}\alpha^3\int_{\alpha_1}^{\alpha}k_1 d\alpha+\frac{1}{16\alpha}\int_{\alpha_1}^{\alpha}k_1\alpha^4 d\alpha$$

$$+\frac{1}{4}\alpha\ln\alpha\int_{\alpha_1}^{\alpha}k_1\alpha^2 d\alpha-\frac{1}{4}\alpha\int_{\alpha_1}^{\alpha}k_1\alpha^2\ln\alpha d\alpha$$

$$-\frac{2r_0^2}{\pi b_1^2}\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta_1(\delta,\vartheta)\mathrm{kei}_1\rho_1\delta d\delta d\vartheta d\alpha$$

$$+F_1\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}\mathrm{ber}_1\alpha d\alpha+G_1\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}\mathrm{bei}_1\alpha d\alpha$$

$$+E_1\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}\mathrm{ker}_1\alpha d\alpha+H_1\frac{1}{\alpha}\int_{\alpha_1}^{\alpha}\alpha d\alpha\int_{\alpha_1}^{\alpha}\mathrm{kei}_1\alpha d\alpha \quad (3.57)$$

3） $m\geqslant 2$ 情形

由方程(3.50)，我们有

$$\nabla_m^2 m_m=-\nabla_m^2\nabla_m^2 p_m \quad (3.58)$$

和情形 1) 一样，我们有

$$m_m=A_m\alpha^{-m}+B_m\alpha^m$$

$$-\frac{1}{8m}\Big[\frac{1}{m-1}\frac{1}{\alpha^{m-2}}\int_{\alpha_1}^{\alpha}k_m\alpha^{m+1}-\frac{1}{m-1}\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{m-3}}k_m d\alpha$$

$$+\frac{1}{m+1}\alpha^{m+2}\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{m-1}}k_m d\alpha-\frac{1}{m+1}\alpha^m\int_{\alpha_1}^{\alpha}\alpha^{m+3}k_m d\alpha\Big]$$

$$-\frac{2r_0^2}{\pi b_1^2}\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}\int_{\alpha_1}^{\alpha_2}\int_0^{2\pi}\eta_m(\delta,\vartheta)\mathrm{kei}_m\rho_1\delta d\delta d\vartheta\alpha^{m+1}d\alpha$$

$$+F_m\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{ber}_m\alpha\alpha^{m+1}d\alpha$$

$$+G_m\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{bei}_m\alpha^{m+1}d\alpha$$

$$+E_m\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{ker}_m\alpha\alpha^{m+1}d\alpha$$

$$+H_m\alpha^m\int_{\alpha_1}^{\alpha}\frac{1}{\alpha^{2m+1}}d\alpha\int_{\alpha_1}^{\alpha}\mathrm{kei}_m\alpha\alpha^{m+1}d\alpha \quad (3.59)$$

其中A_m，$B_m(m=0,1,2,\cdots)$为待定积分常数．对于$\bar{p}_m$，$\bar{m}_m(m=1,2,\cdots)$的解的形式与上面完全相同，只需把A_m，B_m，A_m'，B_m'，E_m，F_m，G_m，H_m换成$\bar{A}_m$，$\bar{B}_m$，$\bar{A}_m'$，$\bar{B}_m'$，$\bar{E}_m$，$\bar{F}_m$，$\bar{G}_m$，$\bar{H}_m$； k_m换成$\bar{k}_m$；$\mathrm{kei}_m\rho_1(m=1,2,\cdots)$换成$\mathrm{kei}_m\rho_1$，$\eta_m(\delta,\vartheta)(m=1,2,\cdots)$换成$\bar{\eta}_m(\delta,\vartheta)$．

将(2.2a,b,c)，(2.3a,b)，(2.4)，(2.5a,b,c)换成无量纲形式，我们有下列的关系式

$$\left.\begin{aligned}\overline{M}_r&=-\frac{1}{4r_0^2}\left(\frac{\partial^2\overline{m}}{\partial\alpha^2}+\frac{\nu}{\alpha}\frac{\partial\overline{m}}{\partial\alpha}+\frac{\nu}{\alpha^2}\frac{\partial^2\overline{m}}{\partial\theta^2}\right)\\ \overline{M}_\theta&=-\frac{1}{4r_0^2}\left(\frac{1}{\alpha}\frac{\partial\overline{m}}{\partial\alpha}+\frac{1}{\alpha^2}\frac{\partial^2\overline{m}}{\partial\theta^2}+\nu\frac{\partial^2\overline{m}}{\partial\alpha^2}\right)\\ \overline{M}_{r\theta}&=-(1-\nu)\frac{1}{4r_0^2}\left(\frac{1}{\alpha}\frac{\partial^2\overline{m}}{\partial\alpha\partial\theta}-\frac{1}{\alpha^2}\frac{\partial\overline{m}}{\partial\theta}\right)\end{aligned}\right\}\qquad(3.60\mathrm{a,b,c})$$

$$\left.\begin{aligned}\overline{Q}_r&=-\frac{1}{2r_0^3}\frac{\partial}{\partial\alpha}(\nabla^2\overline{m})\\ \overline{Q}_\theta&=-\frac{1}{2r_0^3}\frac{1}{\alpha}\frac{\partial}{\partial\theta}(\nabla^2\overline{m})\end{aligned}\right\}\qquad(3.61\mathrm{a,b})$$

$$\overline{V}_r=-\frac{1}{2r_0^3}\left[\frac{\partial}{\partial\alpha}(\nabla^2\overline{m})+\frac{1-\nu}{\alpha}\frac{\partial}{\partial\theta}\left(\frac{1}{\alpha}\frac{\partial^2\overline{m}}{\partial\alpha\partial\theta}-\frac{1}{\alpha^2}\frac{\partial\overline{m}}{\partial\theta}\right)\right]\qquad(3.62)$$

$$\left.\begin{aligned}\overline{N}_r&=\frac{b_1\sqrt{12(1-\nu^2)}}{4r_0^3h}\left(\frac{1}{\alpha}\frac{\partial p}{\partial\alpha}+\frac{1}{\alpha^2}\frac{\partial^2p}{\partial\theta^2}\right)\\ \overline{N}_\theta&=\frac{b_1\sqrt{12(1-\nu^2)}}{4r_0^3h}\frac{\partial^2p}{\partial\alpha^2}\\ \overline{N}_{r\theta}&=\frac{b_1\sqrt{12(1-\nu^2)}}{4r_0^3h}\left(\frac{1}{\alpha^2}\frac{\partial p}{\partial\theta}-\frac{1}{\alpha}\frac{\partial^2p}{\partial\alpha\partial\theta}\right)\end{aligned}\right\}\qquad(3.63a,b,c)$$

积分(2.6a,b,c)也换成无量纲量，我们得到u,v的无量纲表达式

$$\overline{U}=\frac{1}{b_1}\left\{\frac{\sqrt{12(1-\nu^2)}D}{8Eh^2}\int\nabla^2p\,d\alpha-\frac{D(1+\nu)\sqrt{12(1-\nu^2)}}{8Eh^2}\frac{\partial p}{\partial\alpha}+\frac{b_1^2}{32R}\int\overline{m}\,d\alpha+r_0f(\theta)\right\}\qquad(3.64)$$

$$\overline{V}=\frac{1}{b_1}\left\{\frac{-(1+\nu)D\sqrt{12(1-\nu^2)}}{8Eh^2\alpha}\frac{\partial p}{\partial\theta}-\frac{D\alpha\sqrt{12(1-\nu^2)}}{8Eh^2}\int\frac{1}{\alpha^2}\frac{\partial}{\partial\theta}\int\nabla^2p\,d\alpha d\alpha-\frac{\alpha b_1^2}{32R}\int\frac{1}{\alpha^2}\frac{\partial}{\partial\theta}\int\overline{m}\,d\alpha d\alpha+r_0f'(\theta)+r_0\frac{b_1}{2}\alpha g(\theta)\right\}\qquad(3.65)$$

其中$f(\theta)$，$g(\theta)$在附录3中讨论.

边界条件(2.8)，(2.9)，(2.10)化为无量纲量，即有

当$\alpha=\alpha_1$(或$\alpha=\alpha_2$)

1) 夹紧边

$$\left.\begin{aligned}&\overline{m}=0,\quad\frac{\partial\overline{m}}{\partial\alpha}=0\\ &\overline{U}=0,\quad\overline{V}=0\end{aligned}\right\}\qquad(3.66)$$

2) 简支边

$$\left.\begin{aligned}&\overline{m}=0,\quad\overline{M}_r=0\\ &\overline{N}_{r\theta}=0,\quad\overline{V}=0\end{aligned}\right\}\qquad(3.67)$$

3） 悬空边

$$\left.\begin{aligned} M_r=0,\ \bar{V}_r=0 \\ \bar{N}_r=0,\ \bar{N}_{r\theta}=0 \end{aligned}\right\} \tag{3.68}$$

利用内外两边的边界条件就可决定八个待定积分常数.

附 录 1

本文的汤姆逊函数的定义如下：

$$\mathrm{ber}_m\,\alpha=\sum_{K=0,1,\dots}^{\infty}\frac{(-1)^{m+K}\left(\frac{1}{2}\alpha\right)^{m+2K}}{K!(m+K)!}\cos\frac{(m+2K)\pi}{4}$$

$$\mathrm{bei}_m\,\alpha=\sum_{K=0,1,\dots}^{\infty}\frac{(-1)^{m+K+1}\left(\frac{1}{2}\alpha\right)^{m+2K}}{K!(m+K)!}\sin\frac{(m+2K)\pi}{4}$$

$$\begin{aligned}\mathrm{ker}_m\,\alpha=&\left(\ln\frac{2}{\alpha}-\gamma\right)\mathrm{ber}_m\,\alpha+\frac{\pi}{4}\mathrm{bei}_m\,\alpha\\ &+\frac{1}{2}\sum_{K=0,1,\dots}^{m-1}\frac{(-1)^{m+K}(m-K-1)!}{K!}\left(\frac{1}{2}\alpha\right)^{2K-m}\cos\frac{(m+2K)\pi}{4}\\ &+\frac{1}{2}\sum_{K=0,1,\dots}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{K}+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m+K}\right)\\ &\times\frac{(-1)^{m+K}\left(\frac{1}{2}\alpha\right)^{m+2K}}{K!(m+K)!}\cos\frac{(m+2K)\pi}{4}\end{aligned}$$

$$\begin{aligned}\mathrm{kei}_m\alpha=&\left(\ln\frac{2}{\alpha}-\gamma\right)\mathrm{bei}_m\,\alpha-\frac{\pi}{4}\mathrm{ber}_m\,\alpha\\ &+\frac{1}{2}\sum_{K=0,1,\dots}^{m-1}\frac{(-1)^{m+K}(m-K-1)!}{K!}\left(\frac{1}{2}\alpha\right)^{2K-m}\sin\frac{(m+2K)\pi}{4}\\ &-\frac{1}{2}\sum_{K=0,1,\dots}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{K}+1+\frac{1}{2}+\cdots+\frac{1}{m+K}\right)\\ &\times\frac{(-1)^{m+K}\left(\frac{1}{2}\alpha\right)^{m+2K}}{K!(m+K)!}\sin\frac{(m+2K)\pi}{4}\end{aligned}$$

对于 α 大值.

$$\mathrm{ber}_m\,\alpha\approx\frac{e^{\alpha/\sqrt{2}}}{\sqrt{2\pi\alpha}}\left[L_m(\alpha)\cos\left(\frac{\alpha}{\sqrt{2}}-\frac{\pi}{8}+\frac{m\pi}{2}\right)-M_m(\alpha)\sin\left(\frac{\alpha}{\sqrt{2}}-\frac{\pi}{8}+\frac{m\pi}{2}\right)\right]$$

$$\mathrm{bei}_m\alpha\approx\frac{e^{\alpha/\sqrt{2}}}{\sqrt{2\pi\alpha}}\left[M_m(\alpha)\cos\left(\frac{\alpha}{\sqrt{2}}-\frac{\pi}{8}+\frac{m\pi}{2}\right)+L_m(\alpha)\sin\left(\frac{\alpha}{\sqrt{2}}-\frac{\pi}{8}+\frac{m\pi}{2}\right)\right]$$

$$\mathrm{ker}_m\,\alpha\approx\left(\frac{\pi}{2\alpha}\right)^{1/2}e^{-\alpha/\sqrt{2}}\left[L_m(-\alpha)\cos\left(\frac{\alpha}{\sqrt{2}}+\frac{\pi}{8}+\frac{m\pi}{2}\right)\right.$$

$$+M_m(-\alpha)\sin\left(\frac{\alpha}{\sqrt{2}}+\frac{\pi}{8}+\frac{m\pi}{2}\right)\Big]$$

$$\mathrm{kei}_m\,\alpha\approx\left(\frac{\pi}{2\alpha}\right)^{1/2}e^{-\alpha/\sqrt{2}}\Big[M_m(-\alpha)\cos\left(\frac{\alpha}{\sqrt{2}}+\frac{\pi}{8}+\frac{m\pi}{2}\right)$$

$$-L_m(-\alpha)\sin\left(\frac{\alpha}{\sqrt{2}}+\frac{\pi}{8}+\frac{m\pi}{2}\right)\Big]$$

其中γ是欧拉常数0.5772157.

$$L_m(\alpha)=1-\frac{4m^2-1^2}{1!8\alpha}\cos\frac{\pi}{4}+\frac{(4m^2-1^2)(4m^2-3^2)}{2!(8\alpha)^2}\cos\frac{2\pi}{4}-\cdots$$

$$M_m(\alpha)=\frac{4m^2-1^2}{1!8\alpha}\sin\frac{\pi}{4}-\frac{(4m^2-1^2)(4m^2-3^2)}{2!(8\alpha)^2}\sin\frac{2\pi}{4}+\cdots$$

附　录　2

在弹性基础上圆薄板的基本方程是

$$\nabla_\rho^2\nabla_\rho^2 w+w=\frac{q}{\lambda^4 D}\tag{1'}$$

其中　$\lambda^4=\frac{k}{D}$，它与方程(3.41)的形式是相同的.

我们考虑在(δ,ϑ)坐标上作用一集中载荷F，如图3所示. 则由于载荷F而引起P点的挠度可以得出如下的式子

$$w_P=-\frac{F}{2\pi\lambda^2 D}\mathrm{kei}\,\rho_1\tag{2'}$$

其中

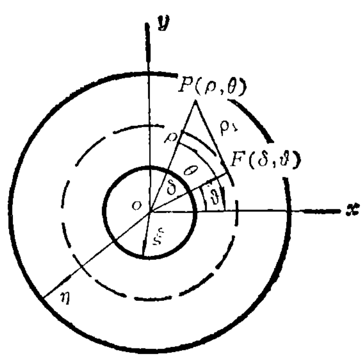

图　3

$$\rho_1^2=\rho^2+\delta^2-2\delta\rho\cos(\theta-\vartheta)$$

利用贝塞尔函数的性质，我们得到下面的一些结果：

$$\rho<\delta,\ T_0(\rho_1 i^{1/2})=T_0(\delta i^{1/2})J_0(\rho i^{1/2})+2\sum_{m=1}^{\infty}T_m(\delta i^{1/2})J_m(\rho i^{1/2})\cos m(\theta-\vartheta)$$

$$\rho>\delta,\ T_0(\rho_1 i^{1/2})=J_0(\delta i^{1/2})T_0(\rho i^{1/2})+2\sum_{m=1}^{\infty}J_m(\delta i^{1/2})T_m(\rho i^{1/2})\cos m(\theta-\vartheta)\tag{3'}$$

以汤姆逊函数表示，则有

$\rho<\delta$,

$\ker\rho_1+i\,\mathrm{kei}\,\rho_1=(\ker_0\delta\,\mathrm{ber}_0\rho-\mathrm{kei}_0\,\delta\,\mathrm{bei}_0\,\rho)+i\,(\mathrm{kei}_0\,\delta\,\mathrm{ber}_0\,\rho$

$$+\ker_0\delta\,\mathrm{bei}_0\rho)+2\sum_{m=1}^{\infty}[(\ker_m\delta\,\mathrm{ber}_m\rho-\mathrm{kei}_m\delta\,\mathrm{bei}_m\rho)$$

$$+i(\mathrm{kei}_m\delta\,\mathrm{ber}_m\rho+\ker_m\delta\,\mathrm{bei}_m\rho)]\cos m(\theta-\vartheta) \tag{4'}$$

亦即有

$$\mathrm{kei}\,\rho_1=\mathrm{kei}_0\delta\,\mathrm{ber}_0\rho+\ker_0\delta\,\mathrm{bei}_0\rho+2\sum_{m=1}^{\infty}(\mathrm{kei}_m\delta\,\mathrm{ber}_m\rho$$

$$+\ker_m\delta\,\mathrm{bei}_m\rho)\cos m(\theta-\vartheta) \tag{4*}$$

类似地可得

$$\rho>\delta,\ \mathrm{kei}\,\rho_1=\mathrm{bei}_0\delta\ker_0\rho+\mathrm{ber}_0\delta\,\mathrm{kei}\,\rho+2\sum_{m=1}^{\infty}(\mathrm{bei}_m\delta\ker_m\rho$$

$$+\mathrm{ber}_m\delta\,\mathrm{kei}_m\rho)\cos m(\theta-\vartheta) \tag{5'}$$

附 录 3

由方程(2.6a)，我们得到

$$u_1=\frac{1}{Eh}\int\nabla_1^2\varphi_1dr_1-\frac{1+\nu}{Eh}\frac{\partial\varphi_1}{\partial r_1}+\frac{1}{R}\int w_1dr_1+f(\theta) \tag{1''}$$

由方程(2.6c)积分，有

$$\frac{v_1}{r_1}=-\frac{1+\nu}{Eh}\frac{1}{r_1^2}\frac{\partial\varphi_1}{\partial\theta}-\frac{1}{Eh}\int\frac{1}{r_1^2}\frac{\partial}{\partial\theta}\int\nabla_1^2\varphi_1dr_1dr_1-\frac{1}{R}\int\frac{1}{r_1^2}\frac{\partial}{\partial\theta}\int w_1dr_1dr_1$$

$$+\frac{f'(\theta)}{r_1}+g(\theta) \tag{2''}$$

将(1″)，(2″)代入(3.56)式，我们得到

$$\frac{1}{Eh}\frac{\partial}{\partial\theta}\int\frac{1}{r_1^2}\frac{\partial}{\partial\theta}\int\nabla_1^2\varphi_1dr_1dr_1+\frac{1}{R}\frac{\partial}{\partial\theta}\int\frac{1}{r_1^2}\frac{\partial}{\partial\theta}\int w_1dr_1dr_1$$

$$-\frac{1}{Eh}\frac{1}{r_1}\int\nabla_1^2\varphi_1dr_1-\frac{1}{Rr_1}\int w_1dr_1+\frac{w_1}{R}+\frac{1}{Eh}\nabla_1^2\varphi_1$$

$$=\frac{f''(\theta)}{r_1}+\frac{f(\theta)}{r_1}+g'(\theta) \tag{3''}$$

将此式换成无量纲形式

$$f''(\theta)+f(\theta)+r_0\frac{b_1}{2}\alpha g'(\theta)=\frac{D\alpha\sqrt{12(1-\nu^2)}}{8Eh^2r_0}\frac{\partial}{\partial\theta}\int\frac{1}{\alpha^2}\frac{\partial}{\partial\theta}\int\nabla^2p\,d\alpha d\alpha$$

$$+\frac{r_0b_1^2\alpha}{32R}\frac{\partial}{\partial\theta}\int\frac{1}{\alpha^2}\frac{\partial}{\partial\theta}\int\bar{m}\,d\alpha d\alpha-\frac{D\sqrt{12(1-\nu^2)}}{8Eh^2r_0}\int\nabla^2p\,d\alpha$$

$$-\frac{r_0b_1^2}{32R}\int\bar{m}\,d\alpha+\frac{r_0b_1^2\alpha}{32R}\bar{m}+\frac{D\sqrt{12(1-\nu^2)}}{8Eh^2r_0}\alpha\nabla^2p \tag{4''}$$

令(4″)的解写成如下的形式

$$\left.\begin{aligned}f(\theta)&=f_h(\theta)+f_p(\theta)\\g(\theta)&=g_h(\theta)+g_p(\theta)\end{aligned}\right\} \tag{5''}$$

其中 $f_h(\theta)$，$g_h(\theta)$ 为方程(4″)的齐次解，亦即下面方程的解

$$f_h''(\theta)+f_h(\theta)+r_0\frac{b_1}{2}\alpha g_h(\theta)=0 \tag{6''}$$

由此我们即得

$$f_h(\theta)=k_1'\cos\theta+k_2'\sin\theta,\quad g_h(\theta)=k_3' \tag{7''}$$

显然f_h, g_h仅分别表示转动和刚性位移．我们将p, $\bar{m}$的傅里叶级数展开式代入(4")，我们即得

$$f_p''(\theta)+f_p(\theta)+r_0\frac{b_1}{2}\alpha g_p'(\theta)=-\sum_{m=1}^{\infty}\frac{m^2D\alpha\sqrt{12(1-\nu^2)}\cos m\theta}{8Eh^2r_0}\int\frac{1}{\alpha^2}\int\nabla_m^2p_m d\alpha d\alpha$$

$$-\sum_{m=1}^{\infty}\frac{r_0b_1^2\alpha m^2\cos m\theta}{32R}\int\frac{1}{\alpha^2}\int m_m d\alpha d\alpha-\frac{D\sqrt{12(1-\nu^2)}}{8Eh^2r_0}\int\nabla_0^2p_0d\alpha$$

$$-\sum_{m=1}^{\infty}\frac{D\sqrt{12(1-\nu^2)}\cos m\theta}{8Eh^2r_0}\int\nabla_m^2p_m d\alpha-\frac{r_0b_1^2}{32R}\int m_0 d\alpha$$

$$-\sum_{m=1}^{\infty}\frac{r_0b_1^2\cos m\theta}{32R}\int m_m d\alpha+\frac{r_0b_1^2}{32R}m_0\alpha+\sum_{m=1}^{\infty}\frac{r_0b_1^2\cos m\theta}{32R}m_m\alpha$$

$$+\frac{D\sqrt{12(1-\nu^2)}\alpha}{8Eh^2r_0}\nabla_0^2p_0+\sum_{m=1}^{\infty}\frac{D\sqrt{12(1-\nu^2)}\alpha\cos m\theta}{8Eh^2r_0}\nabla_m^2p_m$$

$$-\sum_{m=1}^{\infty}\frac{m^2D\alpha\sqrt{12(1-\nu^2)}\sin m\theta}{8Eh^2r_0}\int\frac{1}{\alpha^2}\int\nabla_m^2\bar{p}_m d\alpha d\alpha$$

$$-\sum_{m=1}^{\infty}\frac{r_0b_1^2\alpha m^2\sin m\theta}{32R}\int\frac{1}{\alpha^2}\int\bar{m}_m d\alpha d\alpha$$

$$-\sum_{m=1}^{\infty}\frac{D\sqrt{12(1-\nu^2)}\sin m\theta}{8Eh^2r_0}\int\nabla_m^2\bar{p}_m d\alpha$$

$$-\sum_{m=1}^{\infty}\frac{r_0b_1^2\sin m\theta}{32R}\int\bar{m}_m d\alpha+\sum_{m=1}^{\infty}\frac{r_0b_1^2\sin m\theta}{32R}\bar{m}_m\alpha$$

$$+\sum_{m=1}^{\infty}\frac{D\sqrt{12(1-\nu^2)}\alpha\sin m\theta}{8Eh^2r_0}\nabla_m^2\bar{p}_m$$

从此式子，我们看出$f_p(\theta)=0$．而g_p仅包含待定积分常数A_m, B_m, E_m, F_m, G_m, H_m, A_m', B_m'和$\bar{A}_m$, $\bar{B}_m$, $\bar{E}_m$, $\bar{F}_m$, $\bar{G}_m$, $\bar{H}_m$, $\bar{A}_m'$, $\bar{B}_m'$的某些量而已．

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The General Solution of Bending of a Spherical Thin Shallow Shell with a Circular Hole at the Center under Arbitrary Transverse Loads

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Abstract

Basing several suggestions appeared in [1], we find out the general solution of the bending of a spherical thin shallow shell with a circular hole at the center. As we know, very few papers had touched upon these problems.