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厚板在集中荷载作用下的弯曲*

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摘　要

本文根据[1]中提出的简化理论，利用两变元的δ-函数的性质[2]和级数解法，处理了在集中荷载作用下两对边简支，另两对边为任意的矩形厚板的弯曲问题．考虑了横向剪力对于弯曲变形的影响．当板的厚度 h 很小时，忽略公式中所有 h^2 以上的项，则所得的结果与薄板弯曲问题的相应解一致[3]．在本文的最后，我们还得到了在任意线分布荷载作用下相应问题的解．

一、基本方程和边界条件

根据[1]的简化理论，板弯曲时的基本方程为：

$$D\nabla^4 w=q-\frac{h^2(2-\nu)}{10(1-\nu)}\nabla^2 q \tag{1.1}$$

$$\Psi=-D\nabla^2 w-\frac{h^2(2-\nu)}{10(1-\nu)}q \tag{1.2}$$

式中 $\nabla^4(\quad)=\nabla^2\nabla^2(\quad)$，而 $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ 为调和算子．w 为板中面的挠度，q 为作用于板的横向荷载，Ψ 称为应力函数[1]；而

$$D=\frac{Eh^3}{12(1-\nu^2)} \tag{1.3}$$

为板的抗弯刚度，E 为杨氏模量，ν 为泊松比．

求得 w 及 Ψ 以后，便可得到：

$$Q_x=\frac{\partial\Psi}{\partial x},\quad Q_y=\frac{\partial\Psi}{\partial y}, \tag{1.4}$$

$$\left.\begin{aligned}M_x&=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{h^2}{5}\frac{\partial Q_x}{\partial x}-\frac{qh^2\nu}{10(1-\nu)}\\M_y&=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)+\frac{h^2}{5}\frac{\partial Q_y}{\partial y}-\frac{qh^2\nu}{10(1-\nu)}\\M_{xy}&=(1-\nu)D\frac{\partial^2 w}{\partial x\partial y}-\frac{h^2}{10}\left(\frac{\partial Q_x}{\partial y}+\frac{\partial Q_y}{\partial x}\right)\end{aligned}\right\} \tag{1.5}$$

* 叶开沅推荐．

$$\left.\begin{aligned}\varphi_x&=-\frac{\partial w}{\partial x}+\frac{12(1+\nu)}{5Eh}Q_x\\ \varphi_y&=-\frac{\partial w}{\partial y}+\frac{12(1+\nu)}{5Eh}Q_y\end{aligned}\right\}\tag{1.6}$$

其中符号与Reissner理论[8]中的符号相一致，即Q_x,Q_y为横向剪力，M_x,M_y和M_{xy}分别为弯矩和扭矩，φ_x,φ_y为板中面的转角．

根据[1]的简化理论，在板的每一边有两个边界条件，例如：

1．当$x=a$边为简支时，则有条件：

$$w|_{x=a}=0,\quad M_x|_{x=a}=0\tag{1.7}$$

2．当$x=a$边为固支时，则有条件：

$$w|_{x=a}=0,\quad \varphi_x|_{x=a}=0\tag{1.8}$$

3．当$x=a$边为自由时，则有条件：

$$M_x|_{x=a}=0,\ \left.\left(Q_x-\frac{\partial M_{xy}}{\partial y}\right)\right|_{x=a}=0\tag{1.9}$$

对于$y=\text{const}$的边，有相类似的边界条件．

二、两对边简支的矩形板的一般解

设矩形板的$x=0$及$x=a$两边为简支；$y=0$及$y=b$两边为任意．在板的任意一点(ξ,η)处受有横向集中荷载p的作用．

为了满足$x=0$及$x=a$两边的边界条件，令挠度w为：

$$w=w(x,y;\xi,\eta)=\sum_{m=1}^{\infty}Y_m(y)\sin\frac{m\pi x}{a}\tag{2.1}$$

横向荷载q可表示为：

$$\begin{aligned}q&=q(x,y;\ \xi,\eta)=p\delta(x-\xi,y-\eta)\\ &=\delta(y-\eta)\frac{2p}{a}\sum_{m=1}^{\infty}\sin\frac{m\pi\xi}{a}\sin\frac{m\pi x}{a}\end{aligned}\tag{2.2}$$

将w和q代入(1.1)，则对任何的m都有：

$$\begin{aligned}&Y_m^{(4)}(y)-2\left(\frac{m\pi}{a}\right)^2Y_m''(y)+\left(\frac{m\pi}{a}\right)^4Y_m(y)\\ &=\bar{h}(m\xi)\delta(y-\eta)+\overline{H}(m\xi)\delta''(y-\eta)\end{aligned}\tag{2.3}$$

式中

$$\left.\begin{aligned}\bar{h}(m\xi)&=\frac{2p}{2D}\left[1+\left(\frac{m\pi}{a}\right)^2\frac{h^2(2-\nu)}{10(1-\nu)}\right]\sin\frac{m\pi\xi}{a}\\ \overline{H}(m\xi)&=-\frac{p}{aD}\frac{h^2(2-\nu)}{5(1-\nu)}\sin\frac{m\pi\xi}{a}\end{aligned}\right\}\tag{2.4}$$

(2.3)的齐次方程的基本解系为：

$$\left.\begin{aligned}&Y_m(y)=\operatorname{ch}\frac{m\pi y}{a},\quad Y_{2m}(y)=\operatorname{sh}\frac{m\pi y}{a}\\&Y_{3m}(y)=\left(\frac{m\pi y}{a}\right)\operatorname{ch}\frac{m\pi y}{a},\quad Y_{4m}(y)=\left(\frac{m\pi y}{a}\right)\operatorname{sh}\frac{m\pi y}{a}\end{aligned}\right\}\tag{2.5}$$

因而通解可写成：

$$Y_m(y)=\sum_{i=1}^{4}C_{im}Y_{im}(y)\tag{2.6}$$

其中 $C_{im}(i=1,2,3,4)$为积分常数．解 $Y_m(y)$在 $y\neq\eta$ 处同时满足(2.3)和它的齐次方程．由于荷载不连续，所以 $y=\eta$ 将板划分为两部份，一般．在不同的部份上，C_{im} 是不同的．因此可设：

$$C_{im}=A_{im}+g_{im}I(y-\eta)\qquad(i=1,2,3,4)\tag{2.7}$$

其中

$$I(y-\eta)=\begin{cases}0 & y<\eta\\1 & y\geqslant\eta\end{cases}\tag{2.8}$$

为 Heaveside 阶梯函数．A_{im} 和 g_{im} 为新常数．由于

$$\frac{dI}{dy}=\delta(y-\eta)\tag{2.9}$$

所以

$$\frac{dC_{im}}{dy}=g_{im}\delta(y-\eta)\qquad(i=1,2,3,4)\tag{2.10}$$

现在，不失一般性，可设 (2.3) 的通解为[4]：

$$Y_m(y)=\sum_{i=1}^{4}C_{im}Y_{im}(y)+\sum_{i=4}^{N}x_i\delta^{(i-4)}(y-\eta)\tag{2.11}$$

即：

$$Y_m(y)=\sum_{i=1}^{4}A_{im}Y_{im}(y)+I(y-\eta)\sum_{i=1}^{4}g_{im}Y_{im}(y)+\sum_{i=4}^{N}x_i\delta^{(i-4)}(y-\eta)\tag{2.12}$$

式中N为大于 4 的一个整数．显然，(2.12)的第一部份为 (2.3) 的齐次方程的通解，而后两部份之和为(2.3)的特解．常数g_{im}，x_i应使通解(2.11)或(2.12)满足(2.3)，而A_{im}由边界条件决定．$\delta^{(k)}(y-\eta)$表示$\delta(y-\eta)$对y的k阶导数．

现在计算$Y_m(y)$的各阶导数．注意到：

$$\sum_{i=1}^{4}\frac{dC_{im}}{dy}Y_{im}^{(k)}(y)=x_{4-k-1}\delta(y-\eta),\quad(k=0,1,2,3)\tag{2.13}$$

其中x_{4-k-1}是新常数．于是有：

$$Y_m^{(k)}(y)=\sum_{i=1}^{4}C_{im}Y_{im}^{(k)}(y)+\sum_{i=4-k}^{N}x_i\delta^{(i-4+k)}(y-\eta)$$

$$(k=0,1,2,3,4) \tag{2.14}$$

将(2.14)代入(2.3)，并利用 δ- 函数及其各阶导数的过滤性质，得到：

$$\sum_{k=0}^{4}f_k x_{k+i}=K_i \qquad (i=0,1,\cdots N) \tag{2.15}$$

其中f_k为(2.3)左端各项的系数，而K_i为(2.3)右端$\delta^{(i)}(y-\eta)$的系数．即

$$\left.\begin{aligned}&f_0=1,\ f_1=0,\ f_2=-2\left(\frac{m\pi}{a}\right)^2,\ f_3=0,\ f_4=\left(\frac{m\pi}{a}\right)^4\\&K_0=\bar{h}(m\xi),\ K_1=0,\ K_2=\bar{H}(m\xi),\ K_i=0\ (i>2)\end{aligned}\right\} \tag{2.16}$$

(2.15)是含有$N+1$个未知量x_i的$N+1$个方程，因为系数行列式不为零，所以唯一解x_i:

$$\left.\begin{aligned}&x_0=\bar{h}(m\xi)+2\left(\frac{m\pi}{a}\right)^2\bar{H}(m\xi),\ x_1=0\\&x_2=\bar{H}(m\xi),\ x_i=0\ (i>2)\end{aligned}\right\} \tag{2.17}$$

现在回到(2.13)，利用δ-函数的过滤性质，并注意到(2.10)，得到求解g_{im}的方程组：

$$\sum_{i=1}^{4}g_{im}Y_{im}^{(k)}(\eta)=x_{4-k-1},\quad (k=0,1,2,3) \tag{2.18}$$

此方程组的系数行列式是$Y_{im}(y)$在$y=\eta$处的 Wronskian 行列式，所以不为零，于是得到唯一的g_{im}:

$$\left.\begin{aligned}g_{1m}(\xi,\eta)=&I_0(m\xi)\left\{\mathrm{sh}\frac{m\pi\eta}{a}-\frac{m\pi\eta}{a}\mathrm{ch}\frac{m\pi\eta}{a}\right\}\\&-\left(\mathrm{sh}\frac{m\pi\eta}{a}\right)\frac{a}{m\pi}\bar{H}(m\xi)\\g_{2m}(\xi,\eta)=&\frac{a}{m\pi}\mathrm{ch}\frac{m\pi\eta}{a}\bar{H}(m\xi)-I_0(m\xi)\left\{\mathrm{ch}\frac{m\pi\eta}{a}\right.\\&\left.-\frac{m\pi\eta}{a}\mathrm{sh}\frac{m\pi\eta}{a}\right\}\\g_{3m}(\xi,\eta)=&I_0(m\xi)\,\mathrm{ch}\frac{m\pi\eta}{a}\\g_{4m}(\xi,\eta)=&-I_0(m\xi)\,\mathrm{sh}\frac{m\pi\eta}{a}\end{aligned}\right\} \tag{2.19}$$

式中

$$\begin{aligned}I_0(m\xi)=&\frac{1}{2}\left(\frac{a}{m\pi}\right)^3\left\{\bar{h}(m\xi)+\left(\frac{m\pi}{a}\right)^2\bar{H}(m\xi)\right\}\\=&\frac{p}{aD}\left(\frac{a}{m\pi}\right)^3\sin\frac{m\pi\xi}{a}\end{aligned} \tag{2.20}$$

由于(2.17)，所以(2.3)的通解(2.12)应该为：

$$Y_m(y)=\sum_{i=1}^{4}A_{im}Y_{im}(y)+I(y-\eta)\sum_{i=1}^{4}g_{im}Y_{im}(y) \tag{2.21}$$

其中 g_{im} 由(2.19)给出.

因而板的挠度w为:

$$w=w(x,y;\xi,\eta)$$
$$=\sum_{m=1}^{\infty}\left\{\sum_{i=1}^{4}A_{im}Y_{im}(y)+I(y-\eta)\sum_{i=1}^{4}g_{im}Y_{im}(y)\right\}\sin\frac{m\pi x}{a} \tag{2.22}$$

应力函数Ψ为:

$$\Psi=-2D\sum_{m=1}^{\infty}\{A_{3m}Y_{2m}(y)+A_{4m}Y_{1m}(y)$$
$$+I(y-\eta)(g_{3m}Y_{2m}(y)+g_{4m}Y_{1m}(y))\}\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi x}{a} \tag{2.23}$$

横向剪力为:

$$\left.\begin{aligned}Q_x=&-2D\sum_{m=1}^{\infty}\{A_{3m}Y_{2m}(y)+A_{4m}Y_{1m}(y)\\&+I(y-\eta)(g_{3m}Y_{2m}(y)+g_{4m}Y_{1m}(y))\}\left(\frac{m\pi}{a}\right)^3\cos\frac{m\pi x}{a}\\Q_y=&-2D\sum_{m=1}^{\infty}\{A_{3m}Y_{1m}(y)+A_{4m}Y_{2m}(y)\\&+I(y-\eta)(g_{3m}Y_{1m}(y)+g_{4m}Y_{2m}(y))\}\left(\frac{m\pi}{a}\right)^3\sin\frac{m\pi x}{a}\end{aligned}\right\} \tag{2.24}$$

弯矩和扭矩为:

$$\left.\begin{aligned}M_x=&D\sum_{m=1}^{\infty}\left\{\sum_{i=1}^{4}C_{im}\left[\left(\frac{m\pi}{a}\right)^2Y_{im}(y)-\nu Y''_{im}(y)\right]\right.\\&+\frac{2h^2}{5}[A_{3m}Y_{2m}(y)+A_{4m}Y_{1m}(y)+I(y\\&-\eta)(g_{3m}Y_{2m}(y)+g_{4m}Y_{1m}(y))]\left(\frac{m\pi}{a}\right)^4\\&\left.+\delta(y-\eta)\frac{ph^2\nu}{5aD}\sin\frac{m\pi\xi}{a}\right\}\sin\frac{m\pi x}{a}\\M_y=&D\sum_{m=1}^{\infty}\left\{\sum_{i=1}^{4}C_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(y)-Y''_{im}(y)\right]\right.\\&-\frac{2h^2}{5}[A_{3m}Y_{2m}(y)+A_{4m}Y_{1m}(y)+I(y\end{aligned}\right\} \tag{2.25}$$

$$-\eta)\,(g_{3m}Y_{2m}(y)+g_{4m}Y_{1m}(y))]\left(\frac{m\pi}{a}\right)^4\Big\}\sin\frac{m\pi x}{a}$$

$$M_{xy}=(1-\nu)\,D\sum_{m=1}^{\infty}\Big\{\Big[\sum_{i=1}^{4}C_{im}Y'_{im}(y)\Big]\left(\frac{m\pi}{a}\right)$$

$$+\frac{2h^2}{5(1-\nu)}[A_{3m}Y_{1m}(y)+A_{4m}Y_{2m}(y)+I(y$$

$$-\eta)\,(g_{3m}Y_{1m}(y)+g_{4m}Y_{2m}(y))]\left(\frac{m\pi}{a}\right)^4\Big\}\cos\frac{m\pi x}{a}$$

中面的转角为：

$$\varphi_x=-\sum_{m=1}^{\infty}\Big\{\Big[\sum_{i=1}^{4}C_{im}Y_{im}(y)\Big]\left(\frac{m\pi}{a}\right)$$

$$+\frac{2h^2}{5(1-\nu)}\Big[A_{3m}Y_{2m}(y)+A_{4m}Y_{1m}(y)$$

$$+I(y-\eta)(g_{3m}Y_{2m}(y)+g_{4m}Y_{1m}(y))\Big]\left(\frac{m\pi}{a}\right)^3\Big\}\cos\frac{m\pi x}{a}$$

$$\varphi_y=-\sum_{m=1}^{\infty}\Big\{\sum_{i=1}^{4}C_{im}Y'_{im}(y)$$

$$+\frac{2h^2}{5(1-\nu)}\Big[A_{3m}Y_{1m}(y)+A_{4m}Y_{2m}(y)$$

$$+I(y-\eta)(g_{3m}Y_{1m}(y)+g_{4m}Y_{2m}(y))\Big]\left(\frac{m\pi}{a}\right)^3\Big\}\sin\frac{m\pi x}{a} \qquad (2.26)$$

三、由边界条件决定常数A_{im}

下面，我们考虑三种情况：

1． 若 $y=0$ 及 $y=b$ 两边简支，则有条件：

$$w|_{y=0,b}=0,\quad M_y|_{y=0,b}=0 \qquad (3.1)$$

因而由(2.22)及(2.25)，得到关于 A_{im} 的线性方程组：

$$\sum_{i=1}^{4}A_{im}Y_{im}(0)=0$$

$$\sum_{i=1}^{4}A_{im}Y_{im}(b)=-\sum_{i=1}^{4}g_{im}Y_{im}(b)$$

$$\sum_{i=1}^{4}A_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(0)-Y''_{im}(0)\right]$$

$$\left.\begin{aligned}&-\frac{2h^2}{5}[A_{3m}Y_{2m}(0)+A_{4m}Y_{1m}(0)]\left(\frac{m\pi}{a}\right)^4=0\\&\sum_{i=1}^{4}A_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(b)-Y''_{im}(b)\right]\\&\quad-\frac{2h^2}{5}[A_{3m}Y_{2m}(b)+A_{4m}Y_{1m}(b)]\left(\frac{m\pi}{a}\right)^4\\&=-\sum_{i=1}^{4}g_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(b)-Y''_{im}(b)\right]\\&\quad+\frac{2h^2}{5}[g_{3m}Y_{2m}(b)+g_{4m}Y_{1m}(b)]\left(\frac{m\pi}{a}\right)^4\end{aligned}\right\}\tag{3.2}$$

由此得到：

$$\left.\begin{aligned}&A_{1m}=0\\&A_{2m}=-\frac{1}{\operatorname{sh}\alpha_m}\{g_{1m}\operatorname{ch}\alpha_m+g_{2m}\operatorname{sh}\alpha_m-g_{4m}\alpha_m\}\\&A_{3m}=-g_{3m}-g_{4m}\operatorname{cth}\alpha_m\\&A_{4m}=0\qquad(m=1,2,\cdots)\end{aligned}\right\}\tag{3.3}$$

式中

$$\alpha_m=\frac{m\pi b}{a}\tag{3.4}$$

2. 若 $y=0$ 及 $y=b$ 两边固支，则有条件：

$$w|_{y=0,b}=0,\quad \varphi_y|_{y=0,b}=0\tag{3.5}$$

而由(2.22)及(2.26)，得到关于 A_{im} 的线性方程组：

$$\left.\begin{aligned}&\sum_{i=1}^{4}A_{im}Y_{im}(0)=0\\&\sum_{i=1}^{4}A_{im}Y_{im}(b)=-\sum_{i=1}^{4}g_{im}Y_{im}(b)\\&\sum_{i=1}^{4}A_{im}Y'_{im}(0)+\frac{2h^2}{5(1-\nu)}[A_{3m}Y_{1m}(0)+A_{4m}Y_{2m}(0)]\left(\frac{m\pi}{a}\right)^3=0\\&\sum_{i=1}^{4}A_{im}Y'_{im}(b)+\frac{2h^2}{5(1-\nu)}[A_{3m}Y_{1m}(b)+A_{4m}Y_{2m}(b)]\left(\frac{m\pi}{a}\right)^3\\&=-\sum_{i=1}^{4}g_{im}Y'_{im}(b)-\frac{2h^2}{5(1-\nu)}[g_{3m}Y_{1m}(b)+g_{4m}Y_{2m}(b)]\left(\frac{m\pi}{a}\right)^3\end{aligned}\right\}\tag{3.6}$$

由此得到：

$$
\left.\begin{aligned}
&A_{1m}=0\\
&A_{2m}=-BA_{3m}\\
&A_{3m}=\frac{1}{\Delta_1}\{g_{1m}(\alpha_m+B\,\mathrm{ch}\,\alpha_m\cdot\mathrm{sh}\,\alpha_m)+g_{2m}B\,\mathrm{sh}^2\alpha_m+g_{3m}\alpha_m^2\}\\
&A_{4m}=\frac{1}{\Delta_1}\{-g_{1m}B\,\mathrm{sh}^2\alpha_m+g_{2m}(\alpha_m-B\,\mathrm{ch}\,\alpha_m\,\mathrm{sh}\,\alpha_m)\\
&\qquad+g_{3m}B(\alpha_m-B\,\mathrm{ch}\,\alpha_m\,\mathrm{sh}\,\alpha_m)+g_{4m}(\alpha_m^2-B^2\,\mathrm{sh}^2\alpha_m)\}\\
&\qquad\qquad\qquad(m=1,2\cdots)
\end{aligned}\right\}\tag{3.7}
$$

式中

$$B=1+\frac{2h^2}{5(1-\nu)}\left(\frac{m\pi}{a}\right)^2\tag{3.8}$$

$$\Delta_1=B^2\,\mathrm{sh}^2\alpha_m-\alpha_m^2\tag{3.9}$$

3. 若 $y=0$ 及 $y=b$ 两边自由，则有条件：

$$M_y|_{y=0,b}=0,\quad \left(Q_y-\frac{\partial M_{xy}}{\partial x}\right)\bigg|_{y=0,b}=0\tag{3.10}$$

由(2.24)及(2.25)，得到关于 A_{im} 的线性方程组：

$$
\left.\begin{aligned}
&\sum_{i=1}^{4}A_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(0)-Y''_{im}(0)\right]\\
&\quad-\frac{2h^2}{5}[A_{3m}Y_{2m}(0)+A_{4m}Y_{1m}(0)]\left(\frac{m\pi}{a}\right)^4=0\\
&\sum_{i=1}^{4}A_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(b)-Y''_{im}(b)\right]\\
&\quad-\frac{2h^2}{5}[A_{3m}Y_{2m}(b)+A_{4m}Y_{1m}(b)]\left(\frac{m\pi}{a}\right)^4\\
&=-\sum_{i=1}^{4}g_{im}\left[\nu\left(\frac{m\pi}{a}\right)^2Y_{im}(b)-Y''_{im}(b)\right]\\
&\quad+\frac{2h^2}{5}[g_{3m}Y_{2m}(b)+g_{4m}Y_{1m}(b)]\left(\frac{m\pi}{a}\right)^4\\
&\sum_{i=1}^{4}A_{im}Y'_{im}(0)-\frac{2\bar{B}}{1-\nu}[A_{3m}Y_{1m}(0)+A_{4m}Y_{2m}(0)]\left(\frac{m\pi}{a}\right)=0\\
&\sum_{i=1}^{4}A_{im}Y'_{im}(b)-\frac{2\bar{B}}{1-\nu}[A_{3m}Y_{1m}(b)+A_{4m}Y_{2m}(b)]\left(\frac{m\pi}{a}\right)\\
&=-\sum_{i=1}^{4}g_{im}Y'_{im}(b)+\frac{2\bar{B}}{1-\nu}[g_{3m}Y_{1m}(b)+g_{4m}Y_{2m}(b)]\left(\frac{m\pi}{a}\right)
\end{aligned}\right\}\tag{3.11}
$$

式中

$$\bar{B}=1-\frac{h^2}{5}\left(\frac{m\pi}{a}\right)^2 \tag{3.12}$$

由此得到：

$$\left.\begin{aligned}
A_{1m}&=-\frac{2C}{1-\nu}A_{4m}\\
A_{2m}&=\left(\frac{2}{1-\nu}-B\right)A_{3m}\\
A_{3m}&=\frac{1}{\Delta_2}\Big\{g_{1m}\left(\frac{3+\nu}{1-\nu}\mathrm{ch}\alpha_m\cdot\mathrm{sh}\alpha_m-\alpha_m\right)+g_{2m}\frac{3+\nu}{1-\nu}\mathrm{sh}^2\alpha_m\\
&\quad+g_{3m}\left(\frac{2C(3+\nu)}{(1-\nu)^2}\mathrm{sh}^2\alpha_m-\alpha_m^2\right)+g_{4m}\Big(\frac{2C(3+\nu)}{(1-\nu)^2}\mathrm{sh}\alpha_m\cdot\mathrm{ch}\alpha_m\\
&\quad-\frac{2C}{1-\nu}\alpha_m\Big)\Big\}\\
A_{4m}&=-\frac{1}{\Delta_2}\Big\{g_{1m}\frac{3+\nu}{1-\nu}\mathrm{sh}^2\alpha_m+g_{2m}\Big(\frac{3+\nu}{1-\nu}\mathrm{sh}\alpha_m\cdot\mathrm{ch}\alpha_m\\
&\quad+\alpha_m\Big)+g_{3m}\left(B-\frac{2}{1-\nu}\right)\left(\frac{3+\nu}{1-\nu}\mathrm{ch}\alpha_m\cdot\mathrm{sh}\alpha_m+\alpha_m\right)\\
&\quad+g_{4m}\left[\alpha_m^2+\frac{3+\nu}{1-\nu}\left(B-\frac{2}{1-\nu}\right)\mathrm{sh}^2\alpha_m\right]\Big\}
\end{aligned}\right\} \tag{3.13}$$

$$(m=1,2\cdots)$$

式中

$$C=1+\frac{h^2}{5}\left(\frac{m\pi}{a}\right)^2 \tag{3.14}$$

$$\Delta_2=\alpha_m^2-\left(\frac{3+\nu}{1-\nu}\right)^2\mathrm{sh}^2\alpha_m \tag{3.15}$$

四、在线分布荷载作用下的矩形板

设在$y=\eta$上作用有任意分布的横向线荷载：

$$p=p(x) \tag{4.1}$$

于是，作用于板的横向荷载 q 可表示为：

$$q=q(x,y;\eta)=\delta(y-\eta)p(x) \tag{4.2}$$

仍设$x=0$及$x=a$两边简支，则为了满足此两边的边界条件，仍可设：

$$w=w(x,y;\eta)=\sum_{m=1}^{\infty}Y_m(y)\sin\frac{m\pi x}{a} \tag{4.3}$$

设 q 可展成：

$$q=\delta(y-\eta)\sum_{m=1}^{\infty}p_m\sin\frac{m\pi x}{a} \tag{4.4}$$

式中

$$p_m=\frac{2}{a}\int_0^a p(x)\sin\frac{m\pi x}{a}dx \tag{4.5}$$

于是，将w和q代入(1.1)，则对任何的m都有：

$$Y_m^{(4)}(y)-2\left(\frac{m\pi}{a}\right)^2 Y_m''(y)+\left(\frac{m\pi}{a}\right)^4 Y_m(y)=h_0\delta(y-\eta)+H_0\delta''(y-\eta) \tag{4.6}$$

式中

$$\left.\begin{aligned} h_0&=\frac{p_m}{D}\left[1+\frac{h^2(2-\nu)}{10(1-\nu)}\left(\frac{m\pi}{a}\right)^2\right] \\ H_0&=-\frac{h^2(2-\nu)}{10D(1-\nu)}p_m \end{aligned}\right\} \tag{4.7}$$

因此，若在前面所得的公式中，分别以h_0和H_0代替$\bar{h}(m\xi)$及$\bar{H}(m\xi)$，则所有公式仍然成立．但此时这些公式中都不出现ξ．于是得到在线分布荷载作用下板的弯曲问题的解．

最后指出，根据集中力及线分布荷载作用下的结果，容易得到在任意横向分布荷载$q(x, y)$作用下两对边简支的矩形板的弯曲问题的解．

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Bending of Thick Plates with a Concentrated Load

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Abstract

In this paper according to the simplified theory of[1] the bending problem of rectangular plates with two opposite edges simply supported and other two opposite edges being arbitrary under the action of a concentrated load is treated by means of properties of two-variable δ-function and the method of series[2]. The effect of transverse shearing forces on the bending of plates is considered. When the thickness h of plates is small, the terms, where orders are more than the order of h^2, are neglected, then the results agree with the solutions corresponding to the problem of thin plates[3]. At the end, the solutions of the bending problem of plates with arbitrary linear distributed load are also obtained.