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边界和算子双摄动的高阶椭圆型方程一般边值问题解的渐近式

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摘　要

本文在文[1]和[2]的基础上研究边界和算子双摄动的高阶椭圆型方程一般边值问题的奇摄动，建立含两参数的渐近解表达式，导出求渐近解的迭代过程，给出余项估计，改进和拓广了前文的工作.

一、前　言

文[1]和[2]作者曾研究过边界和算子摄动依赖于同一个参数时四阶椭圆型方程和更高阶的椭圆型方程的一般边值问题的奇摄动．本文将进一步研究边界与算子摄动依赖于不同的参数时高阶椭圆型方程一般边值问题的奇摄动，建立含两个参数的渐近解表达式，导出求形式渐近解的迭代过程，并对余项进行估计，改进和拓广了前文的工作.

设 Ω 是 n 维欧氏空间 R^n 的有界区域，Ω_μ 表示摄动区域，其边界是 $\partial\Omega_\mu$，Ω_0 为非摄动区域，其边界为 $\partial\Omega_0$，用 $X=(x_1,x_2,\cdots,x_n)$ 表示 R^n 内的任意点．下面所涉及的函数均假定是充分光滑的．在 Ω_μ 内研究如下的摄动问题 $A_{\varepsilon,\mu}$:

$$L_\varepsilon u_{\varepsilon,\mu}\equiv\varepsilon^{2l}L_1u_{\varepsilon,\mu}+L_0u_{\varepsilon,\mu}=f(x) \tag{1.1}$$

$$B_ju_{\varepsilon,\mu}|_{\partial\Omega_\mu}=g_j(\mu\alpha(\varphi),\varphi),\quad (j=0,1,\cdots,m+l-1) \tag{1.2}$$

其中 ε、μ 为正的小参数．L_0 表示 $2m$ 阶强椭圆型算子：

$$L_0u\equiv\sum_{|\beta|\leqslant 2m}C_\beta(x)D^\beta u\equiv\sum_{k=0}^{2m}\sum_{\beta_1+\cdots+\beta_n=k}C_{\beta_1\cdots\beta_n}(x)\,D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n}u \tag{1.3}$$

式中

$$(-1)^m\sum_{|\beta|=2m}C_\beta(x)\,\xi^\beta\geqslant a_0|\xi|^{2m} \tag{1.4}$$

$$\beta=(\beta_1,\cdots,\beta_n),\ |\beta|=\beta_1+\cdots+\beta_n,\ \xi=(\xi_1,\cdots,\xi_n),\ \xi^\beta=(\xi_1^{\beta_1},\cdots,\xi_n^{\beta_n}),\ D_x^{\beta_i}\equiv\frac{\partial^\beta}{\partial x_i^{\beta_i}};$$

L_1 表示 $2m+2l$ 阶强椭圆型算子：

$$L_1 u \equiv \sum_{|\beta| \leqslant 2(m+l)} a_\beta(x) D^\beta u \equiv \sum_{k=0}^{2(m+l)} \sum_{\beta_1+\cdots+\beta_n=k} a_{\beta_1\cdots\beta_n}(x) D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n} u \qquad (1.5)$$

式中

$$(-1)^{m+l} \sum_{|\beta|=2(m+l)} a_\beta(x)\xi^\beta \geqslant \alpha_1 |\xi|^{2(m+l)} \qquad (1.6)$$

又 $B_j(j=0,1,\cdots,m+l-1)$ 表示边界微分算子：

$$B_j u_{\varepsilon,\mu}|_{\partial\Omega_\mu} \equiv \sum_{h=0}^{m_j} b_h^{(j)}(x) D_\rho^h u_{\varepsilon,\mu}|_{\partial\Omega_\mu}, \quad 0 \leqslant m_j \leqslant 2(m+l)-1$$

当 $\varepsilon=0,\mu=0$ 时，摄动问题 $A_{\varepsilon,\mu}$ 退化为在区域 Ω_0 的非摄动问题 A_0：

$$L_0 u_{0,0} = f(x)$$
$$B_j u_{0,0}|_{\partial\Omega_0} = g_j(0,\varphi), \quad (j=0,1,\cdots,m-1)$$

二、形 式 渐 近 解

假设问题 $A_{\varepsilon,\mu}$ 存在唯一解 $u_{\varepsilon,\mu}\in C^{2(m+l)}(\bar{\Omega}_\mu)$ 和问题 A_0 对于任意充分光滑的函数 $f(x)$ 和 $g_j(0,\varphi)$ 存在唯一的充分光滑的解 $u_{0,0}(x)$. 在文[3]中讨论了这些解存在的条件.

1. 第一迭代过程

鉴于问题 $A_{\varepsilon,\mu}$ 含有两个参数 ε、μ, 因此，为了构造问题 $A_{\varepsilon,\mu}$ 的解 $u_{\varepsilon,\mu}(x)$, 我们先在区域 Ω_0 内求下列形式的解：

$$W_{0,0}(x) + \sum_{p=1}^{N} \sum_{i=0}^{p} \varepsilon^{p-i} \mu^i W_{p-i,i}(x) \qquad (2.1)$$

把(2.1)式代入方程(1.1)并比较 ε、μ 的各次幂的系数，得到关于求 $W_{0,0}(x)$, $W_{p-i,i}(x)$, $(p=1,2,\cdots,N;\ i=0,1,\cdots,p)$的递推方程：

$$L_0 W_{0,0} = f(x)$$
$$L_0 W_{p-i,i} = -L_1 W_{p-2l-i,i}, \quad (i=0,1,\cdots,p;\ p=1,2,\cdots,N) \qquad (2.3)$$

在上式以及以后的计算中，都将负下标的量取作零.

2. 第二迭代过程

由于(2.2), (2.3)是 $2m$ 阶椭圆型方程，所求得的解 $W_{0,0}(x)$, $W_{p-i,i}(x)$ 由 m 个边界条件完全确定，一般地，不能满足 $u_{\varepsilon,\mu}(x)$ 的全部 $(m+l)$ 个边界条件，为此，在边界邻域内，利用边界层函数 $V_{0,0}(t,\varphi)$, $V_{p-i,i}(t,\varphi)$ 来补足失去的部分边界条件. 为了构造边界层函数，我们先把算子 L_ε 再进行一次分解：

在边界 $\partial\Omega_0$ 的 η-邻域内引进局部坐标，$(\rho,\varphi)=(\rho,\varphi_1,\cdots,\varphi_{n-1})$, $\varphi=(\varphi_1,\cdots,\varphi_{n-1})$ 表示边界上的点坐标. ρ 表示其内法线上的点到边界的距离，$0<\rho<\eta$ 表示在邻接到边界的带形内的点的全体，$\rho=0$ 表示不摄动的边界 $\partial\Omega_0$, $\rho=\mu\alpha(\varphi)$ 定义了摄动的边界 $\partial\Omega_\mu$, 其中 $\alpha(\varphi)$ 是正的光滑函数.

在局部坐标系中，算子 L_ε 具有如下的形式：

$$L_\varepsilon \equiv \varepsilon^{2l} L_1 + L_0 \equiv \varepsilon^{2l} [a_{2(m+l)}(\rho, \varphi) D_\rho^{2(m+l)}$$

$$+ \sum_{\substack{\beta_1+\beta_n \leqslant 2(m+l) \\ \beta_1 \neq 2(m+l)}} a_{\beta_1 \cdots \beta_n}(\rho, \varphi) D_\rho^{\beta_1} D_{\varphi_1}^{\beta_2} \cdots D_{\varphi_{n-1}}^{\beta_n}] + C_{2m}(\rho, \varphi) D_\rho^{2m}$$

$$+ \sum_{\substack{\beta_1+\cdots+\beta_n \leqslant 2m \\ \beta_1 \neq 2m}} C_{\beta_1 \cdots \beta_n}(\rho, \varphi) D_\rho^{\beta_1} D_{\varphi_1}^{\beta_2} \cdots D_{\varphi_{n-1}}^{\beta_n} \tag{2.4}$$

其中 $a_{2(m+l)} = a_{2(m+l),0,\cdots,0}$； $C_{2m} = C_{2m,0,\cdots,0}$.

在 $\partial\Omega_0$ 的 η 邻域引进新变量 t，令

$$t = \frac{\rho - \mu\alpha(\varphi)}{\varepsilon} \tag{2.5}$$

则

$$D_\rho^s = \varepsilon^{-s} D_t^s \tag{2.6}$$

由于下面将要构造的边界层函数$V(t,\varphi) = V\left(\frac{\rho-\mu\alpha(\varphi)}{\varepsilon}, \varphi\right)$，(作为 $\rho, \varphi_1, \cdots, \varphi_{n-1}$ 的函数) 关于 φ_j 的偏导数有如下形式：

$$\frac{\partial V}{\partial \varphi_j} = -\left(\frac{\mu}{\varepsilon}\right) \cdot \frac{\partial \alpha(\varphi)}{\partial \varphi_j} \cdot \frac{\partial V}{\partial t} + \left(\frac{\partial}{\partial \varphi_j}\right) V, \quad (j=1,2,\cdots,n-1) \tag{2.7}$$

上式右边第二项表示 $V(t,\varphi)$ 作为 $t, \varphi_1, \cdots, \varphi_{n-1}$ 的函数对 φ_j 的偏导数，为了与左边的 $\frac{\partial V}{\partial \varphi_j}$ 区别，于是用另外记号 $\left(\frac{\partial}{\partial \varphi_j}\right) V$ 表示它．于是 (2.7) 式的微分算子可简记为

$$D_{\varphi_j} = -\left(\frac{\mu}{\varepsilon}\right) D_{\varphi_j} \alpha(\varphi) D_t + (D_{\varphi_j}) \tag{2.8}$$

这时有

$$D_{\varphi_j}^2 = \left(\frac{\mu}{\varepsilon}\right)^2 (D_{\varphi_j} \alpha(\varphi))^2 D_t^2 - \left(\frac{\mu}{\varepsilon}\right) [D_{\varphi_j}^2 \alpha(\varphi) D_t + 2 D_{\varphi_j} \alpha(\varphi) D_t (D_{\varphi_j})] + (D_{\varphi_j}^2) \tag{2.9}$$

....

$$D_{\varphi_j}^{\beta_j} = (-1)^{\beta_j} \left(\frac{\mu}{\varepsilon}\right)^{\beta_j} (D_{\varphi_j} \alpha(\varphi))^{\beta_j} D_t^{\beta_j} + \cdots + \left(\frac{\mu}{\varepsilon}\right)^{\beta_j - p} D_t^{\beta_j - p} A_{\varphi_j}^{\beta_j, p}$$

$$+ \cdots + (D_{\varphi_j}^{\beta_j}) \tag{2.10}$$

式中

$$A_{\varphi_j}^{\beta_j, p} \equiv \frac{(-1)^{\beta_j - p} \beta_j (\beta_j - 1) \cdots (\beta_j - p + 1)}{p!} (D_{\varphi_j}^{\alpha(\varphi)})^{\beta_j - p} (D_{\varphi_j}^p) + \cdots$$

上式右边的“…”表示关于 (D_{φ_j}) 的低于 p 阶的微分算子．

用(2.6)，(2.10)的微分算子替换(2.4)的右边，则得

$$L_\varepsilon \equiv \varepsilon^{-2m}\Big\{a_{2(m+l)}(\varepsilon t+\mu\alpha(\varphi),\varphi)\,D_t^{2(m+l)}$$

$$+\sum_{|k|\leqslant|\beta|}\ \sum_{\substack{|\beta|\leqslant 2(m+l)\\ \beta_1\neq 2(m+l)}} \varepsilon^{2(m+l)-|\beta|+|k|}\mu^{|\beta|-|k|}\tilde{a}_{\beta,k}(\varepsilon t+\mu\alpha(\varphi),\varphi)$$

$$\cdot D_t^{|\beta|-|k|}(D_{\varphi_1}^{k_1})\cdots(D_{\varphi_{n-1}}^{k_{n-1}})+C_{2m}(\varepsilon t+\mu\alpha(\varphi),\varphi)\,D_t^{2m}$$

$$+\sum_{|k|\leqslant|\beta|}\ \sum_{\substack{|\beta|\leqslant 2m\\ \beta_1\neq 2m}} \varepsilon^{2m-|\beta|+|k|}\mu^{|\beta|-|k|}\tilde{C}_{\beta,k}(\varepsilon t+\mu\alpha(\varphi),\varphi)D_t^{|\beta|-|k|}$$

$$\cdot(D_{\varphi_1}^{k_1})\cdots(D_{\varphi_{n-1}}^{k_{n-1}})\Big\} \tag{2.11}$$

式中 $k=(k_1,\cdots,k_{n-1})$，$|k|=k_1+\cdots+k_{n-1}$，$k_1,\cdots,k_{n-1}$为非负整数.

上式中的每一个系数在$\rho=0$的附近按Taylor公式展开，得到

$$a_{2(m+l)}(\varepsilon t+\mu\alpha(\varphi),\varphi)=a_{2(m+l)}(\varphi)+\sum_{j=1}^{n}(\varepsilon t+\mu\alpha(\varphi))^j a_{2(m+l),j}(\varphi)$$

$$+(\varepsilon t+\mu\alpha(\varphi))^{n+1}a_{2(m+l),n+1}[\theta(\varepsilon t+\mu\alpha(\varphi)),\varphi] \qquad 0<\theta<1 \tag{2.12}$$

其中

$$a_{2(m+l)}(\varphi)=a_{2(m+l)}(0,\varphi)$$

$$a_{2(m+l),j}(\varphi)=\frac{1}{j!}D_\rho^j a_{2(m+l)}(\rho,\varphi)|_{\rho=0} \qquad (j=1,2,\cdots,k)$$

$$a_{2(m+l),n+1}[\theta(\varepsilon t+\mu\alpha(\varphi)),\varphi]=\frac{1}{(n+1)!}D_\rho^{n+1}a_{2(m+l)}(\rho,\varphi)|_{\rho=\theta(\varepsilon t+\mu\alpha(\varphi))}$$

其余的系数类似地展开，将这些展开式代入(2.11)式得到

$$L_\varepsilon\equiv\varepsilon^{-2m}\Big(M_0+\sum_{p=1}^{n+1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i M_{p-i,i}\Big) \tag{2.13}$$

其中

$$M_0\equiv a_{2(m+l)}(\varphi)\,D_t^{2(m+l)}+C_{2m}(\varphi)\,D_t^{2m} \tag{2.14}$$

是关于t的$2(m+l)$阶的常微分算子．而$M_{p-i,i}$ $(i=0,1,\cdots,p;\ p=1,2,\cdots,n)$为如下形式的微分算子：

$$P^{(1)}(t,\varphi)\,D_t^{2(m+l)}+\sum_{k\leqslant|\beta|}\ \sum_{\substack{|\beta|\leqslant 2(m+l)\\ \beta_1\neq 2(m+l)}}P_{\beta,k}^{(2)}(t,\varphi)\,D_t^{|\beta|-|k|}(D_{\varphi_1}^{k_1})\cdots(D_{\varphi_{n-1}}^{k_{n-1}})$$

$$+P^{(3)}(t,\varphi)D_t^{2m}+\sum_{k\leqslant|\beta|}\ \sum_{\substack{|\beta|\leqslant 2m\\ \beta_1\neq 2m}}P_{\beta,k}^{(4)}(t,\varphi)\,D_t^{|\beta|-|k|}(D_{\varphi_1}^{k_1})\cdots(D_{\varphi_{n-1}}^{k_{n-1}})$$

其中系数$P^{(1)}(t,\varphi)$，$P_{\beta,k}^{(2)}(t,\varphi)$，$P^{(3)}(t,\varphi)$，$P_{\beta,k}^{(4)}(t,\varphi)$为$t$的多项式，多项式的系数是$\varphi$的光滑函数．而$M_{n+1-i,i}(i=0,1,\cdots,n+1)$也为形式如上的微分算子，其系数是$\rho$，$\varphi$，$\varepsilon$，$\mu$的光滑函数.

由于经过以上变换，算子L_0，L_1的强椭圆性不变[4]，因此

$$(-1)^{m+l}a_{2(m+l)}(\varphi)>0 \tag{2.15}$$

$$(-1)^m C_{2m}(\varphi)>0 \tag{2.16}$$

并由条件(2.15)，(2.16)可知，常微分算子M_0的特征方程：

$$C_\varphi(\lambda)\equiv a_{2(m+l)}(\varphi)\lambda^{2(m+l)}+C_{2m}(\varphi)\lambda^{2m}=0 \tag{2.17}$$

存在l个具有负实部的根.

下面来构造边界层项，设它为如下形式：

$$\varepsilon^q\left[V_{0,0}(t,\varphi)+\sum_{p=1}^{N+m+l-1}\sum_{i=1}^{p}\varepsilon^{p-i}\mu^i V_{p-i,i}(t,\varphi)\right]$$

其中q为待定常数.

把(2.13)式的算子作用于上式的边界层函数，令ε、μ的各次幂的系数为零，得到关于求$V_{0,0}(t,\varphi)$，$V_{p-i,i}(t,\varphi)$的递推方程：

$$M_0V_{0,0}=0 \tag{2.18}$$

$$\left.\begin{aligned}
&M_0V_{1,0}=-M_{1,0}V_{0,0}\\
&M_0V_{0,1}=-M_{0,1}V_{0,0}\\
&M_0V_{n-k,k}=-M_{n-k,k}V_{0,0}-\sum_{r=0}^{k}\sum_{i=0}^{n-k-r}M_{i,r}V_{n-k-i,k-r}\\
&M_0V_{k,n-k}=-M_{k,n-k}V_{0,0}-\sum_{r=0}^{k}\sum_{i=0}^{n-k-r}M_{r,i}V_{k-r,n-k-i}
\end{aligned}\right\} \tag{2.19}$$

$$\left(n=2,3,\cdots,N+m+l-1;k=0,1,\cdots,\left[\frac{n}{2}\right]\right)$$

上式右边的$M_{0,0}$取恒等于零.

3. 形式渐近解的边界条件：

我们将边界条件(1.2)用局部坐标(ρ,φ)表示出：

$$B_ju_{\varepsilon,\mu}\Big|_{\partial\Omega_\mu}\equiv\sum_{h=0}^{m_j}b_h^{(j)}(\rho,\varphi)\,D_\rho^h u_{\varepsilon,\mu}\Big|_{\partial\Omega_\mu}=g_j(\mu\alpha(\varphi),\varphi) \tag{2.20}$$

$$(j=0,1,\cdots,m+l-1)$$

把函数$b_h^{(j)}(\rho,\varphi)$在$\rho=0$附近按Taylor公式展开：

$$b_h^{(j)}(\rho,\varphi)=b_h^{(j)}(0,\varphi)+\mu\alpha(\varphi)D_\rho b_h^{(j)}(0,\varphi)+\cdots+\frac{[\mu\alpha(\varphi)]^h}{h!}D_\rho^h b_h^{(j)}[\theta\mu\alpha(\varphi)]$$

$$0<\theta<1$$

考虑到(2.6)式，这时边界算子B_j可分解成如下形式：

$$B_j\equiv\varepsilon^{-m_j}\left(\sum_{p=0}^{m_j}\sum_{k=0}^{p}\varepsilon^{p-k}\mu^k H_{p-k,k}^{(j)}\right) \tag{2.21}$$

其中

$H_{0,0}^{(j)}=b_{m_j}^{(j)}(0,\varphi)D_t^{m_j}$，$H_{1,0}^{(j)}=b_{m_j-1}^{(j)}(0,\varphi)D_t^{m_j-1}$，$H_{0,1}^{(j)}=\alpha(\varphi)D_\rho b_m^{(j)}(0,\varphi)D_t^{m_j}$，

$\cdots$，$H_{m_j,0}^{(j)}=b_0^{(j)}(0,\varphi)$，$H_{0,m_j}^{(j)}=\dfrac{(\alpha(\varphi))^{m_j}}{m_j!}D_\rho^m b_{m_j}^{(j)}(\theta\mu\alpha(\varphi),\varphi)D_t^{m_j}$

我们取具有余项的有限和：

$$u_{\varepsilon,\mu}=\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i W_{p-i,i}+\varepsilon^q\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i V_{p-i,i}+Z_N \tag{2.22}$$

做为问题 $A_{\varepsilon,\mu}$ 的渐近展开式.

将展开式(2.22)代入边界条件(2.20)，考虑到(2.21)，得

$$\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i B_j W_{p-i,i}\big|_{\rho=\mu\alpha(\varphi)}$$

$$+\varepsilon^{q-m_j}\left(\sum_{p=0}^{m_j}\sum_{k=0}^{p}\varepsilon^{p-k}\mu^k H_{p-k,k}^{(j)}\right)\left(\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i V_{p-i,i}\right)\Big|_{t=0}$$

$$+B_j Z_N\big|_{\rho=\mu\alpha(\varphi)}=g_j(\mu\alpha(\varphi),\varphi),\quad (j=0,1,\cdots,m+l-1) \tag{2.23}$$

作为一个例子，考察 $m_0=1$，$m_j=j+1$，$j=1,2,\cdots$，$m+l-1$) 和 $m=l$ 的情形（关于其它情形可以类似地建立边值条件的递推公式），这时(2.23)式具有形式：

$$B_0\left(\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i W_{p-i,i}\right)\Big|_{\rho=\mu\alpha(\varphi)}+\varepsilon^{q-1}(H_{0,0}^{(0)}+\varepsilon H_{1,0}^{(0)}+\mu H_{0,1}^{(0)})\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i V_{p-i,i}\big|_{t=0}$$

$$+B_0 Z_N\big|_{\rho=\mu\alpha(\varphi)}=g_0(\mu\alpha(\varphi),\varphi)$$

$$\cdots\cdots\cdots\cdots$$

$$B_m\left(\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i W_{p-i,i}\right)\Big|_{\rho=\mu\alpha(\varphi)}$$

$$+\varepsilon^{q-(m+1)}\left(\sum_{p=0}^{m+1}\sum_{k=0}^{p}H_{p-k,k}^{(j)}\varepsilon^{p-k}\mu^k\right)\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i V_{p-i,i}\big|_{t=0}$$

$$+B_m Z_N\big|_{\rho=\mu\alpha(\varphi)}=g_m(\mu\alpha(\varphi),\varphi)$$

$$\cdots\cdots\cdots\cdots$$

$$B_{m+l-1}\left(\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i W_{p-i,i}\right)\Big|_{\rho=\mu\alpha(\varphi)}$$

$$+\varepsilon^{q-(m+l)}\left(\sum_{p=0}^{m+l}\varepsilon^{p-k}\mu^k H_{p-k,k}^{(j)}\right)\left(\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i V_{p-i,i}\right)\Big|_{t=0}$$

$$+B_{m+l-1}Z_N\big|_{\rho=\mu\alpha(\varphi)}=g_{m+l-1}(\mu\alpha(\varphi),\varphi)$$

取 $q=m+1$，并对函数 $g_j(\mu\alpha(\varphi),\varphi)$，$W_{p-i,i}(\mu\alpha(\varphi),\varphi)$，$(i=0,1,\cdots,p;p=0,1,\cdots,N)$ 及它们的导数在 $\rho=0$ 附近按 Taylor 公式展开，代入上式，由比较系数得：

$$
\left.\begin{aligned}
&B_0W_{0,0}(0,\varphi)=g_0(0,\varphi)\\
&B_0W_{1,0}(0,\varphi)=0\\
&B_0W_{0,1}(0,\varphi)=\alpha(\varphi)D_\rho g_0(0,\varphi)-\alpha(\varphi)b_1^{(0)}(0,\varphi)D_\rho^2W_{0,0}(0,\varphi)\\
&\qquad\qquad -\alpha(\varphi)D_\rho b_1^{(0)}(0,\varphi)D_\rho W_{0,0}(0,\varphi)\\
&B_0W_{n,0}(0,\varphi)=-H_{0,0}^{(0)}V_{n-m,0}(0,\varphi)-H_{1,0}^{(0)}V_{n-m-1,0}(0,\varphi)\\
&B_0W_{0,n}(0,\varphi)=\delta_{0,n}^{(0)}\\
&B_0W_{n-k,k}(0,\varphi)=\delta_{n-k,k}^{(0)}\qquad(n=2,3,\cdots,N;k=1,2,\cdots,n-1)
\end{aligned}\right\}\qquad (B^0)
$$

式中

$$
\begin{aligned}
\delta_{0,n}^{(0)}=&(\alpha(\varphi))^n\Big[\frac{D_\rho^n g_0(0,\varphi)}{n!}-\sum_{j=0}^{n}\frac{D_\rho^j b_1^{(0)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{n+1-j}W_{0,0}(0,\varphi)}{(n-j)!}\\
&-\sum_{j=0}^{n}\frac{D_\rho^j b_0^{(0)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{n-j}W_{0,0}(0,\varphi)}{(n-j)!}\Big]\\
&-(\alpha(\varphi))^{n-1}\Big[\sum_{j=0}^{n-1}\frac{D_\rho^j b_1^{(0)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{n-j}W_{0,1}(0,\varphi)}{(n-1-j)!}\\
&-\sum_{j=0}^{n-1}\frac{D_\rho^j b_0^{(0)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{n-1-j}W_{0,1}(0,\varphi)}{(n-1-j)!}\Big]-\cdots\\
&-\alpha(\varphi)\,[b_1^{(0)}(0,\varphi)D_\rho^2W_{0,n-1}(0,\varphi)+D_\rho b_1^{(0)}(0,\varphi)D_\rho W_{0,n-1}(0,\varphi)]
\end{aligned}
$$

$$
\begin{aligned}
\delta_{n-k,k}^{(0)}=&-(\alpha(\varphi))^k\Big[\sum_{j=0}^{k}\frac{D_\rho^j b_1^{(0)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{k+1-j}W_{n-k,0}(0,\varphi)}{(k-j)!}\\
&+\sum_{j=0}^{k}\frac{D_\rho^j b_0^{(0)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{k-j}W_{n-k,0}(0,\varphi)}{(k-j)!}\Big]-\cdots\\
&-\alpha(\varphi)[b_1^{(0)}(0,\varphi)D_\rho^2W_{n-k,k-1}(0,\varphi)+D_\rho b_1^{(0)}(0,\varphi)D_\rho W_{n-k,k-1}(0,\varphi)]\\
&-[H_{0,0}^{(0)}V_{n-m-k,k}(0,\varphi)+H_{1,0}^{(0)}V_{n-m-k-1,k}(0,\varphi)+H_{0,1}^{(0)}V_{n-m-k,k-1}(0,\varphi)]
\end{aligned}
$$

$$
\begin{aligned}
B_0Z_N|_{\rho=\mu\alpha(\varphi)}=&\sum_{j=1}^{N+1}\varepsilon^{N+1-j}\mu^j\overset{(0)}{\Phi}_{N+1-j,j}(\varepsilon,\mu,\varphi)\\
&+\varepsilon^m\sum_{p=N+1-m}^{N+m+1-1}\sum_{j=0}^{p}\varepsilon^{p-j}\mu^j(H_{0,0}^{(0)}+\varepsilon H_{1,0}^{(0)}+\mu H_{0,1}^{(0)})V_{p-j,j}(0,\varphi)\\
\equiv&\gamma_0(\varepsilon,\mu,\varphi)
\end{aligned}\qquad (R_0)
$$

这里函数$\overset{(0)}{\Phi}_{N+1-j,j}(\varepsilon,\mu,\varphi)$及其对$\mu$的足够高阶导数具有$D_\mu^k\overset{(0)}{\Phi}_{N+1-j,j}(\varepsilon,\mu,\varphi)=O(1)$性质，以下类似函数均具有此性质.

$$
\left.\begin{aligned}
&B_{m-1}W_{0,0}(0,\varphi)=g_{m-1}(0,\varphi)\\
&B_{m-1}W_{1,0}(0,\varphi)=-H_{0,0}^{(m-1)}V_{0,0}(0,\varphi)\\
&B_{m-1}W_{0,1}(0,\varphi)=\alpha(\varphi)D_\rho g_{m-1}(0,\varphi)-\alpha(\varphi)\sum_{j=0}^{m}b_j^{(m-1)}(0,\varphi)D_\rho^{j+1}W_{0,0}(0,\varphi)\\
&\qquad\qquad -\alpha(\varphi)\sum_{j=0}^{m}D_\rho b_j^{(m-1)}(0,\varphi)D_\rho^{j}W_{0,0}(0,\varphi)\\
&B_{m-1}W_{n,0}(0,\varphi)=-\sum_{j=0}^{m}H_{j,0}^{(m-1)}V_{n-1-j,0}(0,\varphi)\\
&B_{m-1}W_{0,n}(0,\varphi)=\delta_{0,n}^{(m-1)}\\
&B_{m-1}W_{n-k,k}(0,\varphi)=\delta_{n-k,k}^{(m-1)},\quad (n=2,3,\cdots,N;k=1,2,\cdots,n-1)
\end{aligned}\right\}(B^{m-1})
$$

式中

$$
\begin{aligned}
\delta_{0,n}^{(m-1)}=[\alpha(\varphi)]^n\Big[&\frac{D_\rho^n g_{m-1}(0,\varphi)}{n!}-\sum_{j=0}^{n}\frac{D_\rho^j b_m^{(m-1)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{m+n-j}W_{0,0}(0,\varphi)}{(n-j)!}\\
&-\cdots-\sum_{j=0}^{n}\frac{D_\rho^j b_0^{(m-1)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{n-j}W_{0,0}(0,\varphi)}{(n-j)!}\Big]-\cdots\\
&-\alpha(\varphi)\Big[b_m^{(m-1)}(0,\varphi)D_\rho^{m+1}W_{0,n-1}(0,\varphi)\\
&+D_\rho b_m^{(m-1)}(0,\varphi)D_\rho^m W_{0,n-1}(0,\varphi)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\delta_{n-k,k}^{(m-1)}=-[\alpha(\varphi)]^k&\sum_{j=0}^{k}\frac{D_\rho^j b_m^{(m-1)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{m+k-j}W_{n-k,0}(0,\varphi)}{(m+k-j)!}-\cdots\\
&-\alpha(\varphi)\Big[b_m^{(m-1)}(0,\varphi)D_\rho^{m+1}W_{n-k,k-1}(0,\varphi)-D_\rho b_m^{(m-1)}(0,\varphi)D_\rho^m\\
&\cdot W_{n-k,k-1}(0,\varphi)\Big]-\sum_{p=0}^{m}\sum_{i=0}^{p}H_{p-i,i}V_{n-k-1-p+i,k-i}
\end{aligned}
$$

$$
B_{m-1}Z_N\Big|_{\rho=\mu\alpha(\varphi)}=\sum_{j=1}^{N+1}\varepsilon^{N+1-j}\mu^j\overset{(m-1)}{\Phi}_{N+1-j,j}(\varepsilon,\mu,\varphi)
$$

$$+\varepsilon\sum_{p=N}^{N+m+l-1}\sum_{j=0}^{p}\varepsilon^{p-j}\mu^{j}\left(\sum_{p=0}^{m}\sum_{i=0}^{p}\varepsilon^{p-i}\mu H_{p-i,i}^{(m-1)}\right)V_{p-j,j}(0,\varphi)$$

$$=\gamma_{m-1}(\varepsilon,\mu,\varphi) \qquad (R_{m-1})$$

$$\left.\begin{aligned}
&H_{0,0}^{(m)}V_{0,0}(0,\varphi)=g_m(0,\varphi)-B_mW_{0,0}(0,\varphi)\\
&H_{0,0}^{(m)}V_{n,0}(0,\varphi)=\begin{cases}-B_mW_{n,0}(0,\varphi)-\sum\limits_{i=1}^{n}H_{i,0}^{(m)}V_{n-i,0}(0,\varphi),\ (n=1,2,\cdots,N)\\ 0,\ (n=N+1,\cdots,N+m+l-1)\end{cases}\\
&H_{0,0}^{(m)}V_{0,n}(0,\varphi)=\begin{cases}\delta_{0,n}^{(m)},\ (n=1,2,\cdots,N)\\ 0,\ (n=N+1,\cdots,N+m+l-1)\end{cases}\\
&H_{0,0}^{(m)}V_{n-k,k}(0,\varphi)=\begin{cases}\delta_{n-k,k}^{(m)},\ (n=2,3,\cdots,N;k=1,2,\cdots,n-1)\\ 0,\ (n=N+1,\cdots,N+m+l-1;k=1,2,\cdots,n-1)\end{cases}
\end{aligned}\right\} \qquad (B^m)$$

式中

$$\delta_{0,n}^{(m)}=[\alpha(\varphi)]^n\left[\frac{D_\rho^n g_m(0,\varphi)}{n!}-\sum_{j=0}^{n}\frac{D_\rho^j b_{m-1}^{(m)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{m+1+n-j}W_{0,0}(0,\varphi)}{(n-j)!}-\cdots\right.$$

$$\left.-\sum_{j=0}^{n}\frac{D_\rho^j b_0^{(m)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{n-j}W_{0,0}(0,\varphi)}{(n-j)!}\right]-\cdots-\alpha(\varphi)\Big[b_{m+1}^{(m)}(0,\varphi)D_\rho^{m+2}$$

$$\cdot W_{0,n-1}(0,\varphi)+D_\rho b_{m+1}^{(m)}(0,\varphi)D_\rho^{m+1}W_{0,n-1}(0,\varphi)\Big]-\cdots$$

$$-\Big[b_0^{(m)}(0,\varphi)D_\rho W_{0,n-1}(0,\varphi)+D_\rho b_0^{(m)}(0,\varphi)W_{0,n-1}(0,\varphi)\Big]$$

$$-\sum_{j=1}^{n}H_{0,j}^{(m)}V_{0,n-j}(0,\varphi)$$

$$\delta_{n-k,k}^{(m)}=-[\alpha(\varphi)]^k\left[\frac{D_\rho^j b_{m+1}^{(j)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{m+1+k-j}W_{n-k,0}(0,\varphi)}{(n-j)!}+\cdots\right.$$

$$\left.+\sum_{j=0}^{k}\frac{D_\rho^j b_0^{(m)}(0,\varphi)}{j!}\cdot\frac{D_\rho^{k-j}W_{n-k,0}(0,\varphi)}{(k-j)!}\right]-\cdots$$

$$-\alpha(\varphi)\Big[b_{m+1}^{(m)}(0,\varphi)D_\rho^{m+2}W_{n-k,k-1}(0,\varphi)+D_\rho b_{m+1}^{(m)}(0,\varphi)D_\rho^{m+1}W_{n-k,k-1}(0,\varphi)$$

$$-\sum_{p=1}^{m+1}\sum_{i=0}^{p}H_{p-i,i}V_{n-k-p+i,k-i}$$

$$B_mZ_N\Big|_{\rho=\mu\alpha(\varphi)}=\sum_{j=1}^{N+1}\varepsilon^{N+1-j}\mu^j\overset{(m)}{\Phi}_{N+1-j,j}(\varepsilon,\mu,\varphi)=\gamma_m(\varepsilon,\mu,\varphi) \qquad (R_m)$$

$$
\left.
\begin{array}{l}
H_{0,0}^{(m+l-1)} V_{0,0}(0,\varphi)=0 \\[2ex]
H_{0,0}^{(m+l-1)} V_{n,0}(0,\varphi)=\begin{cases}
-\sum\limits_{i=1}^{n} H_{i,0}^{(m+l-1)} V_{n-i,0}(0,\varphi), \quad (n=1,2,\cdots,l-2) \\
g_{m+l-1}(0,\varphi)-B_{m+l-1} W_{0,0}(0,\varphi) \\
\quad -\sum\limits_{i=1}^{n} H_{i,0}^{(m+l-1)} V_{n-i,0}(0,\varphi) \qquad (n=l-1) \\
-B_{m+l-1} W_{n-l+1}(0,\varphi)-\sum\limits_{i=0}^{n} H_{i,0}^{(m+l-1)} V_{n-i,0}(0,\varphi) \\
\qquad\qquad (n=l,l+1,\cdots,N+l-1) \\
0, \quad (n=N+l,\cdots,N+m+l-1)
\end{cases} \\[2ex]
H_{0,0}^{(m+l-1)} V_{0,n}(0,\varphi)=-\sum\limits_{i=1}^{n} H_{0,i}^{(m+l-1)} V_{0,n-i}(0,\varphi), (n=1,2,\cdots,N+m+l-1) \\[2ex]
H_{0,0}^{(m+l-1)} V_{l-1,n}(0,\varphi)=\begin{cases}
\delta_{l-1,n}^{(m+l-1)}, \quad (n=1,2,\cdots,N) \\
0, \quad (n=N+1,\cdots,N+m+l-1)
\end{cases} \\[2ex]
H_{0,0}^{(m+l-1)} V_{n-k,k}(0,\varphi)=\begin{cases}
-H_{1,0}^{(m+l-1)} V_{n-k-1,k}(0,\varphi)-H_{0,1}^{(m+l-1)} V_{n-k,k-1}(0,\varphi)-\cdots \\
\quad -H_{n-k,k}^{(m+l-1)} V_{0,0}(0,\varphi) \\
\qquad\qquad (n=2,3,\cdots,l-1;k=1,2,\cdots,n-1) \\
\delta_{n-k,k}^{(m+l-1)} \quad (n=l+1,\cdots,N+l-1;\ k=1,2,\cdots,n-l) \\
-H_{1,0}^{(m+l-1)} V_{n-k-1,k}(0,\varphi)-H_{0,1}^{(m+l-1)} V_{n-k,k-1}(0,\varphi) \\
\quad -\cdots-H_{n-k,k}^{(m+l-1)} V_{0,0}(0,\varphi) \\
\qquad (n=l+1,\cdots,N+l-1;k=n-l+2,\cdots,n-1) \\
0, \quad (n=N+l,\cdots,N+m+l-1;\ k=1,2,\cdots,n-1)
\end{cases}
\end{array}
\right\} (B^{m+l-1})
$$

式中

$$
\begin{aligned}
\delta_{l-1,n}^{(m+l-1)} = & \frac{[\alpha(\varphi)]^n}{n!} D_\rho^n g_n(0,\varphi)-[\alpha(\varphi)]^n\Big[\sum_{j=0}^{n} \frac{D_\rho^j b_{m+l}^{(m+l-1)}(0,\varphi)}{j!} \\
& \cdot \frac{D_\rho^{(m+l+n-j)} W_{0,0}(0,\varphi)}{(n-j)!}+\cdots+\sum_{j=0}^{n} \frac{D_\rho^j b_0^{(m+l-1)}(0,\varphi)}{j!} \\
& \cdot \frac{D_\rho^{n-j} W_{0,0}(0,\varphi)}{(n-j)!}\Big]-\cdots-\alpha(\varphi)\Big[b_{m+l}^{(m+l-1)}(0,\varphi) D_\rho^{m+l-1} W_{0,n-1}(0,\varphi) \\
& +D_\rho b_{m+l}^{(m+l-1)}(0,\varphi) D_\rho^{m+l} W_{0,n-1}(0,\varphi)+\cdots+b_0^{(m+l-1)}(0,\varphi) D_\rho W_{0,n-1}(0,\varphi) \\
& +D_\rho b_0^{(m+l-1)}(0,\varphi) W_{0,n-1}(0,\varphi)\Big]-B_{m+l-1} W_{0,n}(0,\varphi) \\
& -H_{1,0}^{(m+l-1)} V_{l-2,n}(0,\varphi)-H_{0,1}^{(m+l-1)} V_{l-1,n-1}(0,\varphi)-\cdots-H_{l-1,n}^{(m+l-1)} V_{0,0}(0,\varphi)
\end{aligned}
$$

$$\delta_{n-k,k}^{(m+l-1)}=\frac{[\alpha(\varphi)]^k}{k!}D_\rho^k g_k(0,\varphi)-[\alpha(\varphi)]^k\Big[\sum_{j=0}^{k}\frac{D_\rho^j b_{m+l}^{(m+l-1)}(0,\varphi)}{j!}$$

$$\cdot\frac{D_\rho^{m+l+k-j}W_{n-k-l+1,0}(0,\varphi)}{(n-j)!}+\cdots+\sum_{j=0}^{k}\frac{D_\rho^j b_0^{(m+l-1)}(0,\varphi)}{j!}$$

$$\cdot\frac{D_\rho^{k-j}W_{n-k-l+1,0}(0,\varphi)}{(k-j)!}\Big]-\cdots-\alpha(\varphi)\Big[b_{m+l}^{(m+l-1)}(0,\varphi)D_\rho^{m+l-1}W_{n-k-l+1,k-1}(0,\varphi)$$

$$-D_\rho b_{m+l}^{(m+l-1)}(0,\varphi)D_\rho^{m+l}W_{n-k-l+1,k-1}(0,\varphi)+\cdots$$

$$+b_0^{(m+l-1)}(0,\varphi)D_\rho W_{n-k-l+1,k-1}(0,\varphi)-D_\rho b_0^{(m+l-1)}(0,\varphi)W_{n-k-l-1,k-1}(0,\varphi)\Big]$$

$$-B_{m+l-1}W_{n-k-l+1,k}(0,\varphi)-H_{1,0}^{(m+l-1)}V_{n-k-1,k}(0,\varphi)$$

$$-H_{0,1}^{(m+l-1)}V_{n-k,k-1}(0,\varphi)-\cdots-H_{n-k,k}V_{0,0}(0,\varphi)$$

$$B_{m+l-1}Z_N|_{\rho=\mu\alpha(\varphi)}=\sum_{j=1}^{N+1}\varepsilon^{N+1-j}\mu^j\Phi_{N+1-j,j}^{(m+l-1)}(\varepsilon,\mu,\varphi)=\gamma_{m+l-1}(\varepsilon,\mu,\varphi)\qquad(R_{m+l-1})$$

4. 求解程序

关于渐近展开式 (2.22) 中的 $W_{0,0}(x)$、$W_{i,j}(x)$ 和 $V_{0,0}(t,\varphi)$、$V_{i,j}(t,\varphi)$，由递推方程 (2.2)、(2.3)和(2.18)、(2.19)及边界（或初始）条件(B^0)—(B^{m+l-1})知道，它们分别由求 $2m$ 阶椭圆型方程的一般边值问题和 $2(m+l)$阶常微分方程满足 l 个初值条件的解而得到．其求解程序规定如下：

首先，求出 $W_{0,0}(x)$，接着求$V_{0,0}(t,\varphi)$．而 $W_{i,j}(x)$和 $V_{i,j}(t,\varphi)$的求法是：1)二者均为按其下标和$(i+j)$的大小，依次从小到大求解；2) 在求相同的下标和 $(i+j)$时，先求第二个下标为零的，即 $W_{i,0}(x)$和 $V_{i,0}(t,\varphi)$，次之，求第一个下标为零的，即 $W_{0,j}(x)$和 $V_{0,j}(t,\varphi)$，然后按第一个下标依次递减求解；3)在程序2)求解过程中，按 $W_{i,j}(x)$和 $V_{i,j}(t,\varphi)$的下标数相同的交替进行，先求 $W_{i,j}(x)$，后求 $V_{i,j}(t,\varphi)$当下标之和$(i+j)>N$时，转为求解 $V_{i,j}(t,\varphi)$，此时认为 $W_{i,j}(x)\equiv 0$．

上面求得的 $V_{i,j}(t,\varphi)$，$(i,j=0,1,\cdots,N+m+l-1)$只在边界 $\partial\Omega_\mu$ 的 η 邻域有定义，为了得出在整个区域 Ω_μ，有定义的边界层型函数，可引进光滑函数$\Psi(\rho-\mu\alpha(\varphi))\in C^\infty(\bar{\Omega}_\mu)$，使在边界的 η 邻域之外取零值，当$0\leqslant\rho-\mu\alpha(\varphi)\leqslant\frac{1}{3}\eta$时，$\Psi\equiv1$，且

$0\leqslant\Psi(\rho-\mu\alpha(\varphi))\leqslant1$．作函数

$$\tilde{V}_{i,j}(t,\varphi)=\Psi(\rho-\mu\alpha(\varphi))V_{i,j}(t,\varphi),\quad(i,j=0,1,\cdots,N+m+l-1)$$

则函数 $\tilde{V}_{i,j}(t,\varphi)$在整个区域$\Omega_\mu$有定义且在边界 $\partial\Omega_\mu$ 的$\frac{1}{3}\eta$ 邻域内 $\tilde{V}_{i,j}(t,\varphi)=V_{i,j}(t,\varphi)$．

因此，可以证明函数：

$$U_N(\varepsilon,\mu,x)=\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i W_{p-i,i}+\varepsilon^{m+l}\sum_{p=0}^{N+m+l-1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i\tilde{V}_{p-i,i}$$

是摄动问题(1.1)—(1.2)的形式渐近解．兹证明如下：

以 Ω^{η} 表示边界 $\partial\Omega_\mu$ 的 η 邻域．当 $x\in\Omega_\mu\backslash\Omega_\mu^\eta$ 时 $\tilde{V}_{i,j}\equiv 0$，所以

$$L_{\varepsilon,\mu}U_N=f(x)+\varepsilon^{2l}L_1\Big(\sum_{p=N+1-2l}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i W_{p-i,i}\Big)$$
$$=f(x)+\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}\Phi_1(x)$$

其中$\Phi_1(x)=O(1)$；又当$x\in\Omega_\mu^\eta\backslash\Omega_\mu^{\frac{1}{3}\eta}$ 时，因

$$L_\varepsilon\Big(\varepsilon^{m+1}\sum_{p=0}^{N+m+l-1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i\tilde{V}_{p-i,i}\Big)$$

$$=\varepsilon^{-2m}\Big(M_0+\sum_{p=1}^{N+m+}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^iM_{p-i,i}\Big)\Big(\varepsilon^{m+1}\sum_{p=0}^{N+m+l-1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i\tilde{V}_{p-i,i}\Big)$$
$$=\varepsilon^M\Phi_2(x)$$

其中 M为任意正整数和 $\Phi_2(x)=O(1)$，所以

$$L_\varepsilon U_N=f(x)+\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}\Phi_3(x)$$

其中$\Phi_3(x)=O(1)$；又当$x\in\Omega_\mu^{\frac{1}{3}\eta}$ 时，$\tilde{V}_{i,j}=V_{i,j}$，因

$$L_\varepsilon\Big(\varepsilon^{m+1}\sum_{p=0}^{N+m+l-1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^iV_{p-i,i}\Big)$$

$$=\varepsilon^{-2m}\Big(M_0+\sum_{p=1}^{N+m+}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^iM_{p-i,i}\Big)\Big(\varepsilon^{m+1}\sum_{p=0}^{N+m+l-1}\sum_{j=0}^{p}\varepsilon^{p-i}\mu^iV_{p-i,i}\Big)$$

$$=\varepsilon^{-m+1}\sum_{k=1}^{N+m+}\Big[\Big(\sum_{i=0}^{k}\varepsilon^{k-i}\mu^iM_{k-i,i}\Big)\Big(\sum_{p=N+m+l-k}^{N+m+l-1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^iV_{p-i,i}\Big)$$
$$=\varepsilon^{-m+1}(\varepsilon+\mu)^{N+m+l}\Phi_4(x)$$

其中$\Phi_4(x)=O(1)$，所以

$$L_{\varepsilon,\mu}U_N=f(x)+[\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}+\varepsilon^{-m+1}(\varepsilon+\mu)^{N+m+l}]\Phi(x)$$

其中$\Phi(x)=O(1)$．又由边界条件的关系式$(B^0)-(B^{m+l-1})$可以证明

$$B_jU_N|_{\partial\Omega_\mu}=g_j(\mu\alpha(\varphi),\varphi)-\gamma_j(\varepsilon,\mu,\varphi)$$
$$=\begin{cases}g_j[\mu\alpha(\varphi),\varphi]+[\mu(\varepsilon+\mu)^N+\varepsilon^{m-j}(\varepsilon+\mu)^{N+1+j-m}]G_j(\varphi)\\ \qquad (j=0,1,\cdots,m-1)\\ g_j[\mu\alpha(\varphi),\varphi]+\mu(\varepsilon+\mu)^NG_j(\varphi)\\ \qquad (j=m,m+1,\cdots,m+l-1)\end{cases}$$

其中$G_j(\varphi)=O(1)$，所以$U_N(\varepsilon,\mu,x)$是摄动问题的形式渐近解．

三、余 项 估 计

下面将导出摄动问题的解$u_{\varepsilon,\mu}$与$U_N(\varepsilon,\mu,x)$的余项的估计，以Z_N表示余项，即

$$Z_N = u_{\varepsilon,\mu} - U_N \tag{3.1}$$

将$u_{\varepsilon,\mu}=U_N+Z_N$代入边值问题(1.1)—(1.2)得到关于$Z_N$的边值问题：

$$L_\varepsilon Z_N = [\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}+\varepsilon^{-m+1}(\varepsilon+\mu)^{N+m+l}]\Phi(x),\ x\in\Omega_\mu \tag{3.2}$$

$$B_j Z_N|_{\partial\Omega_\mu} = \gamma_j(\varepsilon,\mu,\varphi),\ \ (j=0,1,\cdots,m+l-1) \tag{3.3}$$

以 $\tilde{Z}_N$ 表示在 $\partial\Omega_\varepsilon$ 上满足边值条件 (3.2) 的 $C^{2(m+l)}(\bar{\Omega}_\mu)$ 中的函数和成立：$\tilde{Z}_N = O[\mu(\varepsilon+\mu)^N + \sum_{j=1}^{m}\varepsilon^j(\varepsilon+\mu)^{N+1-j}]P(x)$，其中$P(x)=O(1)$，作函数

$$\bar{Z}_N = Z_N - \tilde{Z}_N \tag{3.4}$$

则$\bar{Z}_N$确定于下面的齐次边值问题：

$$L_\varepsilon \bar{Z}_N = [\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}+\varepsilon^{-m+1}(\varepsilon+\mu)^{N+m+l}+\mu(\varepsilon+\mu)^N + \sum_{j=1}^{m}\varepsilon^j(\varepsilon+\mu)^{N+1-j}]\Phi(x) \quad x\in\Omega_\mu \tag{3.5}$$

$$B_j\bar{Z}_N|_{\partial\Omega_\mu} = 0,\ \ (j=0,1,\cdots,m+l-1) \tag{3.6}$$

考虑齐次边值问题：

$$L_{\varepsilon_1}u \equiv (\varepsilon_1 L_1 + L_0)u = f(x),\ x\in\Omega_\mu \tag{3.7}$$

$$B_j u|_{\partial\Omega_\mu} = 0,\ \ (j=0,1,\cdots,m+l-1) \tag{3.8}$$

式中 $\varepsilon_1=\varepsilon^{2l}$，以 $\hat{C}^{2(m+l)}(\bar{\Omega}_\mu)$ 表示 $C^{2(m+l)}(\bar{\Omega}_\mu)$ 中满足边值条件(3.8)的函数集合．假设算子L_0在$\hat{C}^{2(m+l)}(\bar{\Omega}_\mu)$中按$L_2$范数是正定的，即成立

$$(L_0u,\ u) \geqslant \delta_0\|u\|^2_{L_2},\ \ u\in\hat{C}^{2(m+l)}(\bar{\Omega}_\mu) \tag{3.9}$$

δ_0是正的常数，又假设对于算子L_1成立关系式：

$$(L_1u,\ u) \geqslant -k_0\|u\|^2_{L_2},\ \ u\in\hat{C}^{2(m+l)}(\bar{\Omega}_\mu) \tag{3.10}$$

k_0是正的常数．

在条件(3.9)和(3.10)下，有

$$(L_{\varepsilon_1}u,\ u) \geqslant (\delta_0-\varepsilon_1k_0)\|u\|^2_{L_2}$$

因

$$|(L_{\varepsilon_1}u,u)| \leqslant \frac{1}{\lambda^2}\|L_{\varepsilon_1}u\|^2_{L_2} + \frac{\lambda^2}{4}\|u\|^2_{L_2}$$

λ是任意常数，所以

$$\left(\delta_0-\varepsilon_1k_0-\frac{\lambda^2}{4}\right)\|u\|^2_{L_2} \leqslant \frac{1}{\lambda^2}\|L_{\varepsilon_1}u\|^2_{L_2}$$

当ε_1充分小时，假如$\varepsilon_1<\frac{\delta_0}{4k_0}$，若取$\lambda^2=2\delta_0$，则

$$\frac{\delta_0}{4}\|u\|^2_{L_2} \leqslant \frac{1}{2\delta_0}\|L_{\varepsilon_1}u\|^2_{L_2}$$

即

$$\|u\|^2_{L_2} \leqslant C\|L_{\varepsilon_1}u\|^2_{L_2} \tag{3.11}$$

其中C是与ε_1无关的常数．

从(3.11)知

$$\left\|\tilde{Z}_N\right\|_{L_2}=O\left[\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}+\varepsilon^{-m+1}(\varepsilon+\mu)^{N+m+l}+\mu(\varepsilon+\mu)^N\right.$$
$$\left.+\sum_{j=1}^{m}\varepsilon^j(\varepsilon+\mu)^{N+1-j}\right]$$

所以

$$\left\|Z_N\right\|_{L_2}\leqslant\left\|\tilde{Z}_N\right\|_{L_2}+\left\|Z_N-\tilde{Z}_N\right\|_{L_2}$$
$$=O\left[\varepsilon^{2l}(\varepsilon+\mu)^{N+1-2l}+\varepsilon^{-m+l}(\varepsilon+\mu)^{N+m+l}+\mu(\varepsilon+\mu)^N\right.$$
$$\left.+\sum_{j=1}^{m}\varepsilon^j(\varepsilon+\mu)^{N+1-j}\right] \tag{3.12}$$

四、结　　论

综合前面各个部分的结果，我们得到如下几个定理：

定理1. 假设成立如下条件：

1) 算子L_0和L_1分别为$2m$和$2(m+l)$阶的强椭圆型算子，

2) 问题$A_{\varepsilon,\mu}$和A_0的参数，即算子L_0和L_ε的系数、右边函数$f(x)$、边界$\partial\Omega_\mu$和$\partial\Omega_0$都是充分光滑的，

3) 问题A_0和$A_{\varepsilon,\eta}$的解存在且唯一，

4) 对于算子L_0，L_1成立关系式

$$(L_0u,u)\geqslant\delta_0\left\|u\right\|^2_{L_2}$$
$$(L_1u,u)\geqslant-k_0\left\|u\right\|^2_{L_2},\quad u\in\hat{C}^{2(m+l)}(\bar{\Omega}_\mu)$$

则问题$A_{\varepsilon,\mu}$的解$u_{\varepsilon,\mu}$有渐近式

$$u_{\varepsilon,\mu}=\sum_{p=0}^{N}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^iW_{p-i,i}+\varepsilon^{m+1}\sum_{p=0}^{N+m+l-1}\sum_{i=0}^{p}\varepsilon^{p-i}\mu^i\tilde{V}_{p-i,i}+Z_N \tag{4.1}$$

其中$W_{0,0}(x)$是退化问题A_0的解，$W_{p-i,i}(x)$，$(p=1,2,\cdots,N;\ i=0,1,\cdots,p)$由递推方程(2.2)、(2.3)和边界条件$(B^0)-(B^{m-1})$确定．$\tilde{V}_{p-i,i}(t,\varphi)=\Psi(\rho-\mu\alpha(\varphi))V_{p-i,i}(t,\varphi)$，$(p=0,1,\cdots,N+m+l-1;\ i=0,1,\cdots,p)$，$\Psi(\rho-\mu\alpha(\varphi))$为光滑函数，$V_{p-i,i}$是边界层函数，由递推方程(2.18)，(2.19)和初值条件$(B^m)-(B^{m+l-1})$确定；余项Z_N满足估计式(3.12).

从上面过程中可以看到：1)若区域边界不摄动，即$\mu=0$的情形，问题变成含单参数ε的摄动问题$A_{\varepsilon,0}$，其解$u_{\varepsilon,0}(x)$有渐近式：

$$u_{\varepsilon,0}(x)=\sum_{p=0}^{N}\varepsilon^pW_{p,0}+\varepsilon^{m+1}\sum_{p=0}^{N+m+l-1}\varepsilon^p\tilde{V}_{p,0}+Z_N \tag{4.2}$$

此结果与文[5]相同.

2) 若区域边界的摄动和方程的摄动依赖于同一参数ε时，其解与文[2]相同．在这两种特殊情形下，渐近解的余项估计为：

$$\left\|Z_N\right\|_{L_2}=O(\varepsilon^{N+1}) \tag{4.3}$$

3) 若参数μ依赖于参数ε，并写成$\mu(\varepsilon)$，于是由定理 1 立即得到如下的结果：

定理2. 在定理 1 的条件下，若$\lim\limits_{\varepsilon\to 0}\mu(\varepsilon)=0$且$\lim\limits_{\varepsilon\to 0}\dfrac{\mu(\varepsilon)}{\varepsilon}=\delta\ (0\leqslant\delta\leqslant\infty)$，则渐近解的余项估计由(4.3)给出.

定理3. 在定理 1 的条件下，若$\lim\limits_{\varepsilon\to 0}\mu(\varepsilon)=0$且$\lim\limits_{\varepsilon\to 0}\dfrac{[\mu(\varepsilon)]^k}{\varepsilon}=\delta_1\ (k>1,\ 0<\delta_1<\infty)$，则渐近解的余项有估计式

$$\left\|Z_N\right\|_{L_2}=O\left(\varepsilon^{\frac{N+1+\alpha}{k}}\right)$$

其中 $\alpha=\min\ (0,\ l-1-k(m-1)+m)$.

定理 3 说明，若边界较大摄动，当m,l给定时，一般地要使$\left\|Z_N\right\|_{L_2}=O(\varepsilon^p)$，($p$为某正数)，必须取$N$足够大.

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Asymptotic Expression of the Solution of General Boundary Value Problem for Higher Order Elliptic Equation with Perturbation Both in Boundary and in Operator

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Abstract

In this paper based on [1] and [2], we study the singular perturbation of general boundary value problem for higher order elliptic equation with perturbation both in boundary and in operator, so as to establish the asymptotic expression involving two parameters. Thus derive the iterative process of finding the asymptotic solution and give out the estimation of the remainder term, we extend and improve the previously published papers.