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相似理论和泥沙的垂直分布

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摘　　要

本文利用相似性原理处理了二维渠道均匀定常挟沙水流，在湍流脉动速度和含沙量涨落值相似的条件下，我们得到了泥沙垂直于挟沙水流的流动方向的含沙量分布．这个含沙量分布和扩散理论得到的含沙量分布略有差别，而和重力理论得到的含沙量分布则差别较大．

一、前　　言

关于二维渠道的均匀定常挟沙水流的含沙量的垂直分布，过去讨论得很多了．有根据湍流扩散和泥沙在重力作用下沉降互相平衡的扩散理论[2]．也有从悬浮功等动力学考虑的重力理论[2]．还有一些从其他角度出发的泥沙理论．本文则是从流体动力学基本原理之一的相似理论[1]出发，在湍流脉动速度和含沙量涨落值相似的条件下来得到和水流相垂直的线上各点的平均含沙量分布．因为用的是相似理论，所以有很多常数无法从理论本身求出．但这个理论毕竟还能给出含沙量分布和坐标的函数关系式．所得到的结果在很大区域内是和扩散理论的结果一致的，而和重力理论的结果则相差很远．

二、二维渠道均匀定常流动的含沙量垂直分布

我们用d_1和d_2分别代表水和泥沙的密度，s代表体积含沙量．V_i和V_i'分别代表水和泥沙的速度．我们定义挟沙水流的平均密度D和平均速度U_i．

$$D=d_1(1-s)+d_2s=d_1+(d_2-d_1)s$$

$$DU_i=d_1(1-s)V_i+d_2sV_i'$$

令，$\alpha_i=V_i-V_i'$，则

$$U_i=V_i-\frac{d_2s\alpha_i}{d_1+(d_2-d_1)s}$$

D和U_i的湍流平均值为

$$\bar{D}=d_1+(d_2-d_1)\bar{s}$$

$$\bar{U}_i=\bar{V}_i-\overline{\left(\frac{d_2s\alpha_i}{d_1+(d_2-d_1)s}\right)}$$

脉动密度D'和脉动速度u_i为

$$D'=(d_2-d_1)s'$$

$$u_i=v_i-\frac{d_2 s\alpha_i}{d_1+(d_2-d_1)s}+\left(\overline{\frac{d_2 s\alpha_i}{d_1+(d_2-d_1)s}}\right)$$

式中$\bar{V}_i$为水的平均速度，v_i为水的脉动速度，包括没有含沙时候的脉动速度v_i^0和含沙以后脉动速度的改变量v_i^1．我们写出挟沙水流的运动方程式和连续方程式

$$\frac{\partial}{\partial t}(DU_i)+\frac{\partial}{\partial x_j}(DU_iU_j)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\sigma_{ij} \tag{2.1}$$

$$\frac{\partial}{\partial t}D+\frac{\partial}{\partial x_j}(DU_j)=0 \tag{2.2}$$

平均以后，我们得到

$$\frac{\partial}{\partial t}(\overline{DU_i})+\frac{\partial}{\partial x_j}(\overline{DU_iU_j})=-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\bar{\sigma}_{ij} \tag{2.3}$$

$$\frac{\partial}{\partial t}\bar{D}+\frac{\partial}{\partial x_j}(\overline{DU_j})=0 \tag{2.4}$$

(2.1)式减去(2.3)式和(2.2)式减去(2.4)式，我们就得到脉动密度和脉动速度所满足的方程式

$$\frac{\partial}{\partial t}(D'\bar{U}_i+\bar{D}u_i+D'u_i-\overline{D'u_i})+\frac{\partial}{\partial x_j}(D'\bar{U}_i\bar{U}_j+\bar{D}u_i\bar{U}_j+\bar{D}\bar{U}_iu_j+D'u_i\bar{U}_j$$

$$+D'u_j\bar{U}_i+\bar{D}u_iu_j+D'u_iu_j-\overline{D'u_i}\bar{U}_j-\overline{D'u_j}\bar{U}_i-\bar{D}\overline{u_iu_j}-\overline{D'u_iu_j})=-\frac{\partial p'}{\partial x_i}+\frac{\partial\sigma'_{ij}}{\partial x_j} \tag{2.5}$$

$$\frac{\partial}{\partial t}D'+\frac{\partial}{\partial x_j}(\bar{D}u_j+D'\bar{U}_j+D'u_j-\overline{D'u_j})=0 \tag{2.6}$$

式中p'为脉动压力，σ'_{ij}为压力以外的脉动应力，我们消去脉动压力，得到涡量涨落所满足的方程式．然后令

$$\bar{U}_i=U_i^0+U_i^1,\qquad u_i=u_i^0+u_i^1$$

指标“0”为不含泥沙时候的零级项．指标“1”为含泥沙以后的一级修正项．我们把涡量涨落方程分为不含泥沙的零级近似和与含沙量成正比的一级修正项．这样就得到两个方程式．

$$\frac{\partial}{\partial t}\omega_{ik}^0+U_j^0\frac{\partial}{\partial x_j}\omega_{ik}^0+u_j^0\frac{\partial}{\partial x_j}\Omega_{ik}^0+u_j^0\frac{\partial}{\partial x_j}\omega_{ik}^0+\frac{\partial u_i^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_k}-\frac{\partial u_k^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_i}$$

$$+\frac{\partial U_i^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}-\frac{\partial U_k^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_i}+\frac{\partial u_i^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}-\frac{\partial u_k^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_i}=\frac{\partial^2}{\partial x_j\partial x_k}\overline{u_i^0u_j^0}$$

$$-\frac{\partial^2}{\partial x_i\partial x_j}\overline{u_j^0u_k^0}+\frac{1}{d_1}\left(\frac{\partial^2\sigma_{ij}^0}{\partial x_j\partial x_k}-\frac{\partial^2\sigma_{kj}^0}{\partial x_i\partial x_j}\right) \tag{2.7}$$

$$\bar{s}\frac{\partial}{\partial t}\omega_{ik}^0+\sigma\frac{\partial}{\partial t}\omega_{ik}^1+s'\frac{\partial}{\partial t}\Omega_{ik}^0+\left[\left(\frac{\partial u_i^0}{\partial t}\frac{\partial\bar{s}}{\partial x_k}-\frac{\partial u_k^0}{\partial t}\frac{\partial\bar{s}}{\partial x_i}\right)\right.$$

$$+\left(u_i^0\frac{\partial^2\bar{s}}{\partial t\partial x_k}-u_k^0\frac{\partial^2\bar{s}}{\partial t\partial x_i}\right)+\frac{\partial U_i^0}{\partial t}\frac{\partial s'}{\partial x_k}-\frac{\partial U_k^0}{\partial t}\frac{\partial s'}{\partial x_i}+U_i^0\frac{\partial^2 s'}{\partial x_k\partial t}-U_k^0\frac{\partial^2 s'}{\partial x_i\partial t}$$

$$+\frac{\partial^2}{\partial x_k\partial t}(s'u_i^0)-\frac{\partial^2}{\partial x_i\partial t}(s'u_k^0)-\frac{\partial^2}{\partial x_k\partial t}(\overline{s'u_i^0})+\frac{\partial^2}{\partial x_i\partial t}(\overline{s'u_k^0})\Big]$$

$$-\omega_{ik}^0\frac{\partial}{\partial x_j}\overline{s'u_j^0}+\bar{s}U_j^0\frac{\partial}{\partial x_j}\omega_{ik}^0+\sigma U_j^1\frac{\partial}{\partial x_j}\omega_{ik}^0+\sigma U_j^0\frac{\partial}{\partial x_j}\omega_{ik}^1+\bar{s}u_j^0\frac{\partial}{\partial x_j}\Omega_{ik}^0$$

$$+\sigma u_j^1\frac{\partial}{\partial x_j}\Omega_{ik}^0+\sigma u_j^0\frac{\partial}{\partial x_j}\Omega_{ik}^1+s'U_j^0\frac{\partial}{\partial x_j}\Omega_{ik}^0+s'u_j^0\frac{\partial}{\partial x_j}\Omega_{ik}^0-\overline{s'u_j^0}\frac{\partial}{\partial x_j}\Omega_{ik}^0$$

$$+U_j^0\left(\frac{\partial^2 s'}{\partial x_j\partial x_k}U_i^0-\frac{\partial^2 s'}{\partial x_i\partial x_j}U_k^0\right)+U_j^0\left(\frac{\partial s'}{\partial x_k}\frac{\partial U_i^0}{\partial x_j}-\frac{\partial s'}{\partial x_i}\frac{\partial U_k^0}{\partial x_j}\right)$$

$$+\frac{\partial s'}{\partial x_j}\left(U_i^0\frac{\partial U_j^0}{\partial x_k}-U_k^0\frac{\partial U_j^0}{\partial x_i}\right)+s'\left(\frac{\partial U_i^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_k}-\frac{\partial U_k^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_i}\right)+U_j^0\left(\frac{\partial u_i^0}{\partial x_j}\frac{\partial\bar{s}}{\partial x_k}\right.$$

$$\left.-\frac{\partial u_k^0}{\partial x_j}\frac{\partial\bar{s}}{\partial x_i}\right)+U_j^0\left(u_i^0\frac{\partial^2\bar{s}}{\partial x_k\partial x_j}-u_k^0\frac{\partial^2\bar{s}}{\partial x_i\partial x_j}\right)+\frac{\partial\bar{s}}{\partial x_j}\left(u_i^0\frac{\partial U_j^0}{\partial x_k}-u_k^0\frac{\partial U_j^0}{\partial x_i}\right)$$

$$+\bar{s}\left(\frac{\partial u_i^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_k}-\frac{\partial u_k^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_i}\right)+\sigma\left(\frac{\partial u_i^1}{\partial x_j}\frac{\partial U_j^0}{\partial x_k}-\frac{\partial u_k^1}{\partial x_j}\frac{\partial U_j^0}{\partial x_i}\right)+\sigma\left(\frac{\partial u_i^0}{\partial x_j}\frac{\partial U_j^1}{\partial x_k}\right.$$

$$\left.-\frac{\partial u_k^0}{\partial x_j}\frac{\partial U_j^1}{\partial x_i}\right)+\sigma\left(u_i^0\frac{\partial^2 U_j^1}{\partial x_j\partial x_k}-u_k^0\frac{\partial^2 U_j^1}{\partial x_i\partial x_j}\right)+u_j^0\left(\frac{\partial\bar{s}}{\partial x_k}\frac{\partial U_i^0}{\partial x_j}-\frac{\partial\bar{s}}{\partial x_i}\frac{\partial U_k^0}{\partial x_j}\right)$$

$$+\frac{\partial\bar{s}}{\partial x_j}\left(U_i^0\frac{\partial u_j^0}{\partial x_k}-U_k^0\frac{\partial u_j^0}{\partial x_i}\right)+\bar{s}\left(\frac{\partial U_i^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}-\frac{\partial U_k^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_i}\right)+\sigma\left(\frac{\partial U_i^1}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}\right.$$

$$\left.-\frac{\partial U_k^1}{\partial x_j}\frac{\partial u_j^0}{\partial x_i}\right)+\sigma\left(\frac{\partial U_i^0}{\partial x_j}\frac{\partial u_j^1}{\partial x_k}-\frac{\partial U_k^0}{\partial x_j}\frac{\partial u_j^1}{\partial x_i}\right)+\sigma\left(U_i^0\frac{\partial^2 u_j^1}{\partial x_j\partial x_k}-U_k^0\frac{\partial^2 u_j^1}{\partial x_i\partial x_j}\right)$$

$$+U_j^0\left(\frac{\partial^2 s'}{\partial x_j\partial x_k}u_i^0-\frac{\partial^2 s'}{\partial x_i\partial x_j}u_k^0\right)+U_j^0\left(\frac{\partial s'}{\partial x_k}\frac{\partial u_i^0}{\partial x_j}-\frac{\partial s'}{\partial x_i}\frac{\partial u_k^0}{\partial x_j}\right)+\frac{\partial s'}{\partial x_j}U_j^0\omega_{ik}^0$$

$$+s'U_j^0\frac{\partial\omega_{ik}^0}{\partial x_j}+\frac{\partial s'}{\partial x_j}\left(u_i^0\frac{\partial U_j^0}{\partial x_k}-u_k^0\frac{\partial U_j^0}{\partial x_i}\right)+s'\left(\frac{\partial u_i^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_k}-\frac{\partial u_k^0}{\partial x_j}\frac{\partial U_j^0}{\partial x_i}\right)$$

$$+u_j^0\left(\frac{\partial^2 s'}{\partial x_j\partial x_k}U_i^0-\frac{\partial^2 s'}{\partial x_i\partial x_j}U_k^0\right)+u_j^0\left(\frac{\partial s'}{\partial x_k}\frac{\partial U_i^0}{\partial x_j}-\frac{\partial s'}{\partial x_i}\frac{\partial U_k^0}{\partial x_j}\right)+\frac{\partial s'}{\partial x_j}\left(U_i^0\frac{\partial u_j^0}{\partial x_k}\right.$$

$$\left.-U_k^0\frac{\partial u_j^0}{\partial x_i}\right)+s'\left(\frac{\partial U_i^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}-\frac{\partial U_k^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}\right)+u_j^0\left(\frac{\partial^2\bar{s}}{\partial x_k\partial x_j}u_i^0-\frac{\partial^2\bar{s}}{\partial x_i\partial x_j}u_k^0\right)$$

$$+u_j^0\left(\frac{\partial\bar{s}}{\partial x_k}\frac{\partial u_i^0}{\partial x_j}-\frac{\partial\bar{s}}{\partial x_i}\frac{\partial u_k^0}{\partial x_j}\right)+\frac{\partial\bar{s}}{\partial x_j}\omega_{ik}^0u_j^0+\bar{s}\frac{\partial\omega_{ik}^0}{\partial x_j}u_j^0+\sigma\frac{\partial}{\partial x_j}\omega_{ik}^0u_j^1$$

$$+\sigma\frac{\partial}{\partial x_j}\omega_{ik}^1u_j^0+\sigma\omega_{ik}^0\frac{\partial u_j^1}{\partial x_j}+\frac{\partial\bar{s}}{\partial x_j}\left(u_i^0\frac{\partial u_j^0}{\partial x_k}-u_k^0\frac{\partial u_j^0}{\partial x_i}\right)+\bar{s}\left(\frac{\partial u_i^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}-\frac{\partial u_k^0}{\partial x_j}\frac{\partial u_j^0}{\partial x_i}\right)$$

$$+\sigma\left(\frac{\partial u_i^1}{\partial x_j}\frac{\partial u_j^0}{\partial x_k}-\frac{\partial u_k^1}{\partial x_j}\frac{\partial u_j^0}{\partial x_i}\right)+\sigma\left(\frac{\partial u_i^0}{\partial x_j}\frac{\partial u_j^1}{\partial x_k}-\frac{\partial u_k^0}{\partial x_j}\frac{\partial u_j^1}{\partial x_i}\right)+\sigma\left(u_i^0\frac{\partial^2 u_j^1}{\partial x_j\partial x_k}\right.$$

$$-u_k^0\frac{\partial^2 u_j^1}{\partial x_i\partial x_i}\Big)+\frac{\partial^2}{\partial x_i\partial x_k}(s'u_i^0u_j^0)-\frac{\partial^2}{\partial x_i\partial x_j}(s'u_i^0u_k^0)-U_j^0\Big(\frac{\partial^2}{\partial x_i\partial x_k}\overline{s'u_i^0}$$

$$-\frac{\partial^2}{\partial x_i\partial x_j}\overline{s'u_k^0}\Big)-\Big(\frac{\partial}{\partial x_j}\overline{s'u_i^0}\frac{\partial U_j^0}{\partial x_k}-\frac{\partial}{\partial x_j}\overline{s'u_k^0}\frac{\partial U_j^0}{\partial x_i}\Big)-\Big(\frac{\partial^2}{\partial x_i\partial x_k}\overline{s'u_j^0}U_i^0$$

$$-\frac{\partial^2}{\partial x_i\partial x_j}\overline{s'u_j^0}U_k^0\Big)-\Big(\frac{\partial}{\partial x_k}\overline{s'u_i^0}\frac{\partial U_i^0}{\partial x_j}-\frac{\partial}{\partial x_i}\overline{s'u_j^0}\frac{\partial U_k^0}{\partial x_i}\Big)-\Big(\frac{\partial^2\bar{s}}{\partial x_i\partial x_k}\overline{u_i^0u_j^0}$$

$$-\frac{\partial^2\bar{s}}{\partial x_i\partial x_j}\overline{u_i^0u_k^0}\Big)-\Big(\frac{\partial\bar{s}}{\partial x_k}\frac{\partial\overline{u_i^0u_j^0}}{\partial x_j}-\frac{\partial\bar{s}}{\partial x_i}\frac{\partial\overline{u_j^0u_k^0}}{\partial x_j}\Big)-\frac{\partial\bar{s}}{\partial x_i}\Big(\frac{\partial\overline{u_i^0u_j^0}}{\partial x_k}$$

$$-\frac{\partial\overline{u_j^0u_k^0}}{\partial x_i}\Big)-\bar{s}\Big(\frac{\partial^2\overline{u_i^0u_j^0}}{\partial x_j\partial x_k}-\frac{\partial^2\overline{u_j^0u_k^0}}{\partial x_i\partial x_j}\Big)-\sigma\Big(\frac{\partial^2\overline{u_i^0u_j^1}}{\partial x_j\partial x_k}-\frac{\partial^2\overline{u_k^0u_j^1}}{\partial x_i\partial x_j}\Big)$$

$$-\Big(\frac{\partial^2}{\partial x_j\partial x_k}\overline{s'u_i^0u_j^0}-\frac{\partial^2}{\partial x_i\partial x_j}\overline{s'u_k^0u_j^0}\Big)=\frac{1}{d_2-d_1}\Big(\frac{\partial^2\sigma_{ij}^1}{\partial x_j\partial x_k}-\frac{\partial^2\sigma_{kj}^1}{\partial x_i\partial x_j}\Big)\qquad(2.8)$$

式中 $\sigma=\frac{d_1}{d_2-d_1}$，$\sigma_{ij}^0$ 和 σ_{ij}^1 各自为压力从外的脉动应力的零级项和一级项.

我们现在把周培源老师的涡量相似性观点[1]用到泥沙运动问题上来. 经过坐标变换，在某一运动点 P^0 附近来讨论涨落速度的相似性. 因为泥沙浓度是小量，水和泥沙的相对速度也是小量，我们在 $\bar{U}_i$ 和 u_i 的表达式中略去含沙量和相对速度的相乘积各项. 这样 $U_i^1\approx 0$，$u_i^1\approx v_i^1$. 假设 u_i^1，s' 和 u_i^0 都具有相似性结构. 和周培源老师的原文一样，我们舍去平均流速不带微分的各项[1]. 并且令

$$\xi_i=\frac{x_i'}{\Lambda},\qquad u_i^0=q\phi_i^0,\qquad u_i^1=\frac{d_2-d_1}{d_1}q\bar{s}\phi_i^1,\qquad s'=\bar{s}\phi'$$

式中 x_i' 为运动坐标系的坐标，q 和 Λ 各自为涡旋的特征速度和特征长度. 我们现在来讨论二维渠道的均匀定常流动. 我们用 U 代表平均流速 U_x^0，写出零级近似的方程式如下.

$$\left.\begin{aligned}
&i=1,\quad k=2\\
&\frac{d^2U}{dy^2}q\phi_2^0+\frac{q^2}{\Lambda^2}\phi_j^0\frac{\partial}{\partial\xi_j}\Big(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\Big)+\frac{dU}{dy}\frac{q}{\Lambda}\frac{\partial\phi_1^0}{\partial\xi_1}\\
&\qquad+\frac{dU}{dy}\frac{q}{\Lambda}\frac{\partial\phi_2^0}{\partial\xi_2}+\frac{q^2}{\Lambda^2}\Big(\frac{\partial\phi_1^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_1}\Big)\\
&\qquad=\frac{d^2}{dy^2}\overline{u_x^0u_y^0}+\nu\frac{q}{\Lambda^3}\nabla_\xi^2\Big(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\Big)\\
&i=3,\quad k=2\\
&\frac{q^2}{\Lambda^2}\phi_j^0\frac{\partial}{\partial\xi_j}\Big(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\Big)+\frac{dU}{dy}\frac{q}{\Lambda}\frac{\partial\phi_3^0}{\partial\xi_1}\\
&\qquad+\frac{q^2}{\Lambda^2}\Big(\frac{\partial\phi_3^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_3}\Big)=\frac{\nu q}{\Lambda^3}\nabla_\xi^2\Big(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\Big)\\
&i=1,\quad k=3\\
&\frac{q^2}{\Lambda^2}\phi_j^0\frac{\partial}{\partial\xi_j}\Big(\frac{\partial\phi_1^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_1}\Big)+\frac{q^2}{\Lambda^2}\Big(\frac{\partial\phi_1^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_1}\Big)\\
&\qquad=\nu\frac{q}{\Lambda^3}\nabla_\xi^2\Big(\frac{\partial\phi_1^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_1}\Big)
\end{aligned}\right\}\qquad(2.9)$$

因为零级方程在周培源老师原文中已经详细地讨论过了，我们在这里就不再重复.

我们再写出一级的三个方程式. 略去粘性应力项以后，我们有

$i=1, \quad k=2$

$$
\begin{aligned}
&-\frac{q}{\Lambda}\frac{d}{dy}\overline{s'u_y^0}\left(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\right)+\bar{s}q\frac{d^2U}{dy^2}\phi_2^0+\bar{s}q\frac{d^2U}{dy^2}\phi_2^1+\bar{s}q\frac{d^2U}{dy^2}\phi'\phi_2^0-\frac{d^2U}{dy^2}\overline{s'u_y^0}\\
&+\bar{s}\frac{q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_1^0}{\partial\xi_1}+\bar{s}\frac{q}{\Lambda}\frac{\partial\phi_1^1}{\partial\xi_1}\frac{dU}{dy}+q\frac{d\bar{s}}{dy}\frac{dU}{dy}\phi_2^0+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_2^0}{\partial\xi_2}\\
&+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_2^1}{\partial\xi_2}+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\phi_1^0\frac{\partial\phi'}{\partial\xi_1}+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\phi'\frac{\partial\phi_1^0}{\partial\xi_1}+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\phi_2^0\frac{\partial\phi'}{\partial\xi_2}\\
&+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_2^0}{\partial\xi_2}+q^2\frac{d^2\bar{s}}{dy^2}\phi_1^0\phi_2^0+\frac{q^2}{\Lambda}\frac{d\bar{s}}{dy}\phi_i^0\frac{\partial\phi_1^0}{\partial\xi_i}+\frac{q^2}{\Lambda}\frac{d\bar{s}}{dy}\left(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\right)\phi_2^0\\
&+\bar{s}\frac{q^2}{\Lambda^2}\phi_i^0\frac{\partial}{\partial\xi_i}\left(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\phi_i^1\frac{\partial}{\partial\xi_i}\left(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\phi_i^0\frac{\partial}{\partial\xi_i}\left(\frac{\partial\phi_1^1}{\partial\xi_2}\right.\\
&\left.-\frac{\partial\phi_2^1}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial\phi_i^1}{\partial\xi_i}\left(\frac{\partial\phi_1^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_1}\right)+\frac{q^2}{\Lambda}\frac{d\bar{s}}{dy}\left(\phi_1^0\frac{\partial\phi_2^0}{\partial\xi_2}-\phi_2^0\frac{\partial\phi_2^0}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\left(\frac{\partial\phi_1^0}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_2}\right.\\
&\left.-\frac{\partial\phi_2^0}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\left(\frac{\partial\phi_1^1}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_2}-\frac{\partial\phi_2^1}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\left(\frac{\partial\phi_1^0}{\partial\xi_i}\frac{\partial\phi_i^1}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_i}\frac{\partial\phi_i^1}{\partial\xi_1}\right)\\
&+\frac{\bar{s}q^2}{\Lambda^2}\left(\phi_1^0\frac{\partial^2\phi_i^1}{\partial\xi_2\partial\xi_i}-\phi_2^0\frac{\partial^2\phi_i^1}{\partial\xi_1\partial\xi_i}\right)+\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial^2}{\partial\xi_2\partial\xi_i}(\phi'\phi_1^0\phi_i^0)-\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial^2}{\partial\xi_1\partial\xi_i}(\phi'\phi_2^0\phi_i^0)\\
&-\frac{d}{dy}\overline{s'u_y^0}\frac{dU}{dy}-\frac{d^2\bar{s}}{dy^2}\overline{u_x^0u_y^0}-2\frac{d\bar{s}}{dy}\frac{d}{dy}\overline{u_x^0u_y^0}-\bar{s}\frac{d^2}{dy^2}\overline{u_x^0u_y^0}-\sigma\frac{d^2}{dy^2}\overline{u_x^1u_y^0}\\
&-\sigma\frac{d^2}{dy^2}\overline{u_x^0u_y^1}-\frac{d^2}{dy^2}\overline{s'u_x^0u_y^0}=0
\end{aligned}
\tag{2.10a}
$$

$i=3, \quad k=2$

$$
\begin{aligned}
&-\frac{q}{\Lambda}\frac{d}{dy}\overline{s'u_y^0}\left(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\right)+\bar{s}\frac{q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_3^0}{\partial\xi_1}+\bar{s}\frac{q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_3^1}{\partial\xi_1}+\bar{s}\frac{q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi'}{\partial\xi_1}\phi_3^0\\
&+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\phi'\frac{\partial\phi_3^0}{\partial\xi_1}+q^2\phi_3^0\phi_2^0\frac{d^2\bar{s}}{dy^2}+\frac{q^2}{\Lambda}\frac{d\bar{s}}{dy}\phi_i^0\frac{\partial\phi_3^0}{\partial\xi_i}+\frac{q^2}{\Lambda}\frac{d\bar{s}}{dy}\phi_2^0\left(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\right)\\
&+\frac{q^2}{\Lambda^2}\bar{s}\frac{\partial}{\partial\xi_i}\left(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\right)\phi_i^0+\frac{q^2}{\Lambda}\bar{s}\frac{\partial}{\partial\xi_i}\left(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\right)\phi_i^1+\frac{q^2}{\Lambda^2}\bar{s}\phi_i^0\frac{\partial}{\partial\xi_i}\left(\frac{\partial\phi_3^1}{\partial\xi_2}\right.\\
&\left.-\frac{\partial\phi_2^1}{\partial\xi_3}\right)+\frac{q^2}{\Lambda^2}\bar{s}\left(\frac{\partial\phi_3^0}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_3}\right)\frac{\partial\phi_i^1}{\partial\xi_i}+\frac{d\bar{s}}{dy}\frac{q^2}{\Lambda}\left(\phi_3^0\frac{\partial\phi_2^0}{\partial\xi_2}-\phi_2^0\frac{\partial\phi_2^0}{\partial\xi_3}\right)+\bar{s}\frac{q^2}{\Lambda^2}\left(\frac{\partial\phi_3^0}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_2}\right.\\
&\left.-\frac{\partial\phi_2^0}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_3}\right)+\bar{s}\frac{q^2}{\Lambda^2}\left(\frac{\partial\phi_3^1}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_2}-\frac{\partial\phi_2^1}{\partial\xi_i}\frac{\partial\phi_i^0}{\partial\xi_3}\right)+\frac{\bar{s}q^2}{\Lambda^2}\left(\frac{\partial\phi_3^0}{\partial\xi_i}\frac{\partial\phi_i^1}{\partial\xi_2}-\frac{\partial\phi_2^0}{\partial\xi_i}\frac{\partial\phi_i^1}{\partial\xi_3}\right)\\
&+\frac{\bar{s}q^2}{\Lambda^2}\left(\phi_3^0\frac{\partial^2\phi_i^1}{\partial\xi_i\partial\xi_2}-\phi_2^0\frac{\partial^2\phi_i^1}{\partial\xi_i\partial\xi_3}\right)+\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial^2}{\partial\xi_i\partial\xi_2}(\phi'\phi_3^0\phi_y^0)
\end{aligned}
$$

$$-\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial^2}{\partial\xi_j\partial\xi_3}(\phi'\phi_j^0\phi_2^0)=0 \tag{2.10b}$$

$i=1,\quad k=3$

$$-\frac{q}{\Lambda}\left(\frac{\partial\phi_1^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_1}\right)\frac{d}{dy}\overline{s'u_y^0}+\bar{s}\frac{q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_2^0}{\partial\xi_3}+\bar{s}\frac{q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_2^1}{\partial\xi_3}+\bar{s}\frac{q}{\Lambda}\phi_2^0\frac{\partial\phi'}{\partial\xi_3}\frac{dU}{dy}$$
$$+\frac{\bar{s}q}{\Lambda}\frac{dU}{dy}\frac{\partial\phi_2^0}{\partial\xi_3}+\frac{d\bar{s}}{dy}\frac{q^2}{\Lambda}\phi_2^0\left(\frac{\partial\phi_1^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_1}\right)+\bar{s}\frac{q^2}{\Lambda^2}\frac{\partial}{\partial\xi_j}\left(\frac{\partial\phi_1^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_1}\right)\phi_j^0$$
$$+\bar{s}\frac{q^2}{\Lambda^2}\frac{\partial}{\partial\xi_j}\left(\frac{\partial\phi_1^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_1}\right)\phi_j^1+\bar{s}\frac{q^2}{\Lambda^2}\frac{\partial}{\partial\xi_j}\left(\frac{\partial\phi_1^1}{\partial\xi_3}-\frac{\partial\phi_3^1}{\partial\xi_1}\right)\phi_j^0+\bar{s}\frac{q^2}{\Lambda^2}\left(\frac{\partial\phi_1^0}{\partial\xi_3}\right.$$
$$\left.-\frac{\partial\phi_3^0}{\partial\xi_1}\right)\frac{\partial\phi_j^1}{\partial\xi_j}+\frac{d\bar{s}}{dy}\frac{q^2}{\Lambda}\left(\phi_1^0\frac{\partial\phi_2^0}{\partial\xi_3}-\phi_3^0\frac{\partial\phi_2^0}{\partial\xi_1}\right)+\bar{s}\frac{q^2}{\Lambda^2}\left(\frac{\partial\phi_1^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_1}\right)$$
$$+\frac{\bar{s}q^2}{\Lambda^2}\left(\frac{\partial\phi_1^1}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_3}-\frac{\partial\phi_3^1}{\partial\xi_j}\frac{\partial\phi_j^0}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\left(\frac{\partial\phi_1^0}{\partial\xi_j}\frac{\partial\phi_j^1}{\partial\xi_3}-\frac{\partial\phi_3^0}{\partial\xi_j}\frac{\partial\phi_j^1}{\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\left(\phi_1^0\frac{\partial^2\phi_j^1}{\partial\xi_j\partial\xi_3}\right.$$
$$\left.-\phi_3^0\frac{\partial^2\phi_j^1}{\partial\xi_j\partial\xi_1}\right)+\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial^2}{\partial\xi_j\partial\xi_3}(\phi'\phi_1^0\phi_j^0)-\frac{\bar{s}q^2}{\Lambda^2}\frac{\partial^2}{\partial\xi_j\partial\xi_1}(\phi'\phi_j^0\phi_3^0)=0 \tag{2.10c}$$

从这三个方程式，我们得到相似条件为

$$\frac{\bar{s}q\dfrac{d^2U}{dy^2}}{q\dfrac{d\bar{s}}{dy}\dfrac{dU}{dy}}=\frac{\bar{s}\dfrac{d^2U}{dy^2}}{\dfrac{d\bar{s}}{dy}\dfrac{dU}{dy}}=C_1,\quad 即\ \frac{\dfrac{d^2U}{dy^2}}{\dfrac{dU}{dy}}=C_1\frac{\dfrac{d\bar{s}}{dy}}{\bar{s}} \tag{2.11}$$

$$\frac{\bar{s}q\dfrac{d^2U}{dy^2}}{\bar{s}\dfrac{q}{\Lambda}\dfrac{dU}{dy}}=\Lambda\frac{\dfrac{d^2U}{dy^2}}{\dfrac{dU}{dy}}=C_2,\quad 即\ \frac{\dfrac{d^2U}{dy^2}}{\dfrac{dU}{dy}}=\frac{C_2}{\Lambda} \tag{2.12}$$

$$\frac{q^2\dfrac{d^2\bar{s}}{dy^2}}{\dfrac{q^2}{\Lambda}\dfrac{d\bar{s}}{dy}}=\Lambda\frac{\dfrac{d^2\bar{s}}{dy^2}}{\dfrac{d\bar{s}}{dy}}=C_3,\quad 即\ \frac{\dfrac{d^2\bar{s}}{dy^2}}{\dfrac{d\bar{s}}{dy}}=\frac{C_3}{\Lambda} \tag{2.13}$$

所以，从(2.12)和(2.13)消去Λ，得到

$$\frac{\dfrac{d^2U}{dy^2}}{\dfrac{dU}{dy}}=\frac{C_2}{C_3}\frac{\dfrac{d^2\bar{s}}{dy^2}}{\dfrac{d\bar{s}}{dy}} \tag{2.14}$$

现在化成无量纲量．设 s_0 为某一点 y_0 上的含沙量，$\bar{V}$ 为平均流速，H 为水深．令

$$\theta=\frac{\bar{s}}{s_0},\qquad \xi=\frac{U}{\bar{V}},\qquad \eta=\frac{y}{H}$$

于是方程(2.11)和(2.14)化成

$$\frac{\dfrac{d^2\xi}{d\eta^2}}{\dfrac{d\xi}{d\eta}}=C_1\frac{\dfrac{d\theta}{d\eta}}{\theta} \tag{2.11$'$}$$

$$\frac{\dfrac{d^2\xi}{d\eta^2}}{\dfrac{d\xi}{d\eta}}=\frac{C_2}{C_3}\frac{\dfrac{d^2\theta}{d\eta^2}}{\dfrac{d\theta}{d\eta}} \tag{2.14'}$$

于是有

$$C_1\frac{\dfrac{d\theta}{d\eta}}{\theta}=\frac{C_2}{C_3}\frac{\dfrac{d^2\theta}{d\eta^2}}{\dfrac{d\theta}{d\eta}}$$

即

$$\frac{\dfrac{d^2\theta}{d\eta^2}}{\dfrac{d\theta}{d\eta}}=\frac{C_1C_3}{C_2}\frac{\dfrac{d\theta}{d\eta}}{\theta} \tag{2.15}$$

令$\dfrac{C_1C_3}{C_2}=\varepsilon+1$，积分以后，就有

$$\ln\frac{d\theta}{d\eta}=(\varepsilon+1)\ln\theta+\ln K \tag{2.16}$$

式中 $\ln K$ 为积分常数．化简以后

$$\frac{d\theta}{d\eta}=K\theta^{\varepsilon+1} \tag{2.16}$$

$$\frac{d\theta}{\theta^{\varepsilon+1}}=Kd\eta$$

积分以后

$$\frac{\theta^{-\varepsilon}}{-\varepsilon}=K\eta+C$$

再用边界条件，$y=y_0$，$\theta=1$． 令 $\eta_0=\dfrac{y_0}{H}$

$$\frac{1}{-\varepsilon}=K\eta_0+C$$

则有

$$-\frac{1}{\varepsilon}(\theta^{-\varepsilon}-1)=K(\eta-\eta_0)$$

化简就有

$$\theta=[-\varepsilon K(\eta-\eta_0)+1]^{\frac{1}{-\varepsilon}}=A(\eta+B)^{-n} \tag{2.17}$$

式中

$$A=(-\varepsilon K)^{-\frac{1}{\varepsilon}},\qquad B=\frac{1}{-\varepsilon K}-\eta_0,\qquad n=\frac{1}{\varepsilon}$$

写成无量纲的量，就成

$$\bar{s}=s_0A\left(\frac{y}{H}+B\right)^{-n} \tag{2.17'}$$

含沙量垂直分布公式(2.17)或(2.17)′和扩散理论的含沙量分布公式[2]

$$\bar{s}=s_0\left(\frac{1-\eta}{\eta}\ \frac{\eta_0}{1-\eta_0}\right)^{\frac{w}{KV_*}}=s_0\left(\frac{\eta_0}{1-\eta_0}\right)^{\frac{w}{KV_*}}\left(\frac{\eta}{1-\eta}\right)^{-\frac{w}{KV_*}}$$

比较接近．而和重力理论的含沙量分布公式[2]

$$\bar{s}=s_0[\Phi(\eta)]^{\beta},\quad \Phi(\eta)=\exp\left(-\int_{\delta}^{\eta}\frac{d\eta}{(1-\eta)\ln\left(1+\frac{\eta}{\alpha}\right)}\right),\quad \beta=\frac{aKw}{i\sqrt{gHi}}$$

相差较大．

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The Theory of Similarity and the Profile of Vertical Distribution of Concentration of Sand Particles

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Abstract

In this article, we treat the problem of two-dimensional uniform steady channel flow of turbid water with the theory of similarity. Under the condition of similarity of turbulent fluctuation velocity and fluctuation of concentration of sand particles, we obtain the profile of the vertical distribution of concentration of sand particles. This profile of vertical distribution of concentration of sand particles is slightly different from that obtained by diffusion theory, and it departs from that obtained by gravitational theory.