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二锥形分离杯之间血液流动稳定性的窄间隙理论*

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摘 要

本文在获得血液分离器锥形分离杯内（其中一杯静止，另一杯以ω等角速旋转）血液流动的边界摄动解的基础上[1]，采用窄间隙稳定性理论，证明了带轴向流的二锥形分离杯（其中一杯静止，另一杯以ω等角速旋转）之间旋转密度分层血液流动的稳定性．

一、引 言

无轴向流的旋转流体流动的稳定性问题早已有严格的证明．L．N．Howard(1962)又进一步指出窄间圆柱体间旋转流体流动中，密度分层对流动起稳定性影响[2]．但当带了轴向流之后，旋转流体流动是否继续保持稳定，至今尚未获得完善的分析证明，而至于带轴向流的旋转圆锥体间流动（血液锥形分离杯之间血液流动属这一类型流动）的稳定性证明，比带轴向流的旋转圆柱体间流动稳定性的证明更为困难．

本文以窄间隙稳定性理论证明了带轴向流的二旋转圆柱体间和二旋转圆锥体间（其中一圆柱（圆锥）体为静止，另一圆柱（圆锥）以ω等角速旋转）均质血液流动的稳定性．而根据L.N.Howard理论，也就自然证明了带轴向流的密度分层旋转血液流动的稳定性．

二、带轴向流的旋转圆柱体间(其中一圆柱体静止，另一圆柱体以ω等角速度旋转)均质血液流动的稳定性

带轴流的旋转圆柱（圆锥）体间均质血液流动受下列一组方程控制(用柱坐标系表示)：

$$\frac{\partial V_r}{\partial t}+(\vec{V}\cdot\nabla)V_r-\frac{V_\varphi^2}{r}=-\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right)+\nu\left(\nabla^2V_r-\frac{2}{r}\frac{\partial V_\varphi}{\partial\varphi}-\frac{V_r}{r^2}\right) \tag{2.1a}$$

$$\frac{\partial V_\varphi}{\partial t}+(\vec{V}\cdot\nabla)V_\varphi+\frac{V_rV_\varphi}{r}=-\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{p}{\rho}\right)+\nu\left(\nabla^2V_\varphi+\frac{2}{r}\frac{\partial V_\varphi}{\partial\varphi}-\frac{V_\varphi}{r^2}\right) \tag{2.1b}$$

* 钱伟长推荐．

$$\frac{\partial V_z}{\partial t}+(\vec{V}\cdot\nabla)V_z=-\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)+\nu(\nabla^2 V_z) \tag{2.1c}$$

$$\frac{\partial V_r}{\partial r}+\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\varphi}{\partial \varphi}+\frac{\partial V_z}{\partial z}=0 \tag{2.1d}$$

上式中

$$\vec{V}\cdot\nabla=V_r\frac{\partial}{\partial r}+\frac{V_\varphi}{r}\frac{\partial}{\partial\varphi}+V_z\frac{\partial}{\partial z},\quad \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}$$

若在主流上加一轴对称扰动u_r，u_φ，u_z和p'，则速度和压强为：

$$V_r=V_{0r}+u_r,\quad V_\varphi=V_{0\varphi}+u_\varphi,\quad V_z=V_{0z}+u_z,\quad p=p_0+p' \tag{2.2}$$

式(2.2)中的$V_{0\varphi}$，V_{0r}，V_{0z}和p_0是我们所研究之血液流动的基本解（或平衡解），这已在"血液分离器锥形分离杯内血液流动的边界摄动解"一文中求出[1]。其解为：

$$V_{0r}=0 \tag{2.3a}$$

$$V_{0\varphi}=\frac{r_0^2\omega}{r_0^2-r_i^2}\left(r-\frac{r_i^2}{r}\right) \tag{2.3b}$$

$$V_{0z}=\frac{r_0^2-r_i^2}{4\mu(\ln r_0-\ln r_i)}\frac{\partial p_0}{\partial z}(\ln r_i-\ln r)+\frac{1}{4\mu}\frac{\partial p_0}{\partial z}(r^2-r_i^2) \tag{2.3c}$$

$$p_0=p_{0\lambda}+\frac{\rho r_0^4\omega^2}{(r_0^2-r_i^2)^2}\left(\frac{r^2}{2}-2r_i\ln r-\frac{r_i^4}{2r^2}\right)+\frac{Q}{C_1}z \tag{2.3d}$$

如我们用内外圆柱体间的间距$d=r_0-r_i$来表示基本解V_{0z}的话，则

$$V_{0z}=-\frac{1}{4\mu}\frac{\partial p_0}{\partial z}\left[r_i^2-r^2+\frac{2r_i d+d^2}{\ln(1+d/r_i)}\ln\frac{r}{r_i}\right]$$

若间距d满足窄间隙条件，即

$$d\ll\frac{r_0+r_i}{2}$$

再令 $r=r_i+\zeta d$

把$r=r_i+\zeta d$代入V_{0z}中，经整理后可得V_{0z}为：

$$V_{0z}=6V_m\zeta(1-\zeta) \tag{2.4}$$

其中V_m为流道的平均速度（轴向），其值为：

$$V_m=\int_{r_i}^{r_0}V_{0z}(r)dr=6V_m\int_{r_i}^{r_0}\zeta(1-\zeta)dr=-\frac{d^2}{24\mu}\frac{\partial p_0}{\partial z}$$

把轴对称扰动后的速度（2.2）和（2.4）代入方程（2.1）及由于圆柱体的轴对称性即$\frac{\partial}{\partial\varphi}=0$，就可得扰动方程如下：

$$\frac{\partial u_r}{\partial t}+V_{0z}\frac{\partial u_r}{\partial z}-\frac{2V_{0\varphi}u_\varphi}{r}=-\frac{1}{\rho}\frac{\partial p'}{\partial r}+\nu\left(\nabla^2u_r-\frac{u_r}{r^2}\right) \tag{2.5a}$$

$$\frac{\partial u_\varphi}{\partial t}+V_{0z}\frac{\partial u_\varphi}{\partial z}+\frac{V_{0\varphi}}{r}u_r+\frac{dV_{0\varphi}}{dr}u_r=\nu\left(\nabla^2u_\varphi-\frac{u_\varphi}{r^2}\right) \tag{2.5b}$$

$$\frac{\partial u_z}{\partial t}+\frac{dV_{0z}}{dr}u_r+V_{0z}\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial z}+\nu\nabla^2u_z \tag{2.5c}$$

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}=0 \tag{2.5d}$$

式中 $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$

设扰动方程(2.5)的解为：

$$\left.\begin{array}{l} u_r=u(r)e^{i(qt+kz)},\ u_\varphi=v(r)e^{i(qt+kz)} \\ u_z=w(r)e^{i(qt+kz)},\ p'=\bar{p}(r)e^{i(qt+kz)} \end{array}\right\} \tag{2.6}$$

其中$q=q_r+iq_i$，q_r为角频率，q_i为扰动的放大系数，k为空间频率或波数．然后把解(2.6)代入扰动方程(2.5)可得如下方程：

$$D\frac{\bar{p}}{\rho}=\nu\left(DD_*-k^2-i\frac{q}{\nu}-i\frac{k}{\nu}V_{0z}\right)u+\frac{2V_{0\varphi}}{r}v \tag{2.7a}$$

$$\nu\left(DD_*-k^2-i\frac{q}{\nu}-\frac{ik}{\nu}V_{0z}\right)v=\frac{V_{0\varphi}}{r}u+\frac{dV_{0\varphi}}{dr}u \tag{2.7b}$$

$$ik\frac{\bar{p}}{\rho}=\nu\left(D_*D-k^2-\frac{iq}{\nu}-\frac{ik}{\nu}V_{0z}\right)w-uDV_{0z} \tag{2.7c}$$

$$D_*u=-ikw \tag{2.7d}$$

上式中

$$D=\frac{d}{dr},\qquad D_*=\frac{d}{dr}+\frac{1}{r}$$

所以符号算子∇^2为：

$$\nabla^2=\left(\frac{d}{dr}+\frac{1}{r}\right)\frac{d}{dr}-k^2=D_*D-k^2=DD_*+\frac{1}{r^2}-k^2$$

消去(2.7c)，(2.7d)中的w得：

$$\frac{\bar{p}}{\rho}=\frac{\nu}{k^2}\left(D_*D-k^2-\frac{iq}{\nu}-\frac{ik}{\nu}V_{0z}\right)D_*u+\frac{i}{k}(DV_{0z})u \tag{2.8}$$

再把所得之$\frac{\bar{p}}{\rho}$代入(2.7a)得：

$$\left[DD_*-k^2-\frac{i}{\nu}(q+kV_{0z})\right](DD_*-k^2)u+\frac{ik}{\nu}ru\left[DD_*\left(\frac{V_{0z}}{r}\right)\right]=\frac{2V_{0\varphi}k^2}{\nu r}v \tag{2.9}$$

上式简化过程中应用了下列等式：

$$D\left(D_*D-k^2-\frac{iq}{\nu}-i\frac{k}{\nu}V_{0z}\right)D_*u+\frac{ik}{\nu}D(uDV_{0z})$$

$$=\left(DD_*-k^2-\frac{iq}{\nu}-\frac{ik}{\nu}V_{0z}\right)(DD_*u)+\frac{ik}{\nu}ru\left[DD_*\left(\frac{V_{0\varphi}}{r}\right)\right]$$

式(2.7b)两边除以ν得：

$$\left[DD_*-k^2-\frac{i}{\nu}(q+kV_{0z})\right]v=\frac{V_{0\varphi}}{\nu r}u+\frac{dV_{0\varphi}}{dr}\frac{u}{\nu} \tag{2.10}$$

而扰动方程(2.9)和(2.10)的边界条件为：

$$r=r_i,\ r_0\text{处} \qquad u=v=w=0$$

由于方程(2.5d)，$D_*u=-ikw=0$，即有$D_*u=0$或$Du=0$，所以边界条件也可写为：

$$r=r_i,r_0\text{处}, \qquad u=v=Du=0$$

对$d \ll \frac{r_0+r_i}{2}$的窄间隙来说，R.C.Diprima (1960) 用数值计算方法指出，在证明小雷诺数窄间隙流动稳定性时，可用平均速度和平均角速度来代替速度和角速度。而血液分离器锥形分离杯内血液流动确属窄间隙小雷数流动。(如血液流量为40ml/m时，轴向雷诺数仅0.53)所以本文也应用了Diprima理论[3]。因此可应用如下窄间隙三近似等式：

$$\frac{ik}{\nu}d^4ru\left[D^2\left(\frac{V_{0z}}{r}\right)\right]\doteq-12iaRu$$

$$\frac{V_{0z}d}{\nu}=\frac{6V_md}{\nu}\zeta(1-\zeta)\doteq6R\zeta(1-\zeta)\text{和}D\doteq D_*$$

其中 $R=\frac{V_md}{\nu}$。那么方程(2.9)，(2.10)可简化为变量ζ的方程：

$$\begin{aligned}&\{(D^2-a^2)-i[\sigma+6Ra\zeta(1-\zeta)]\}(D^2-a^2)u-12iaRu\\&=\frac{2V_{0\varphi}d^2}{r}\frac{a^2}{\nu}v=wd^2\frac{a^2}{\nu}v\end{aligned} \tag{2.11}$$

$$\{(D^2-a^2)-i[\sigma+6Ra\zeta(1-\zeta)]\}v=\left(\frac{V_{0\varphi}}{r}+\frac{dV_{0\varphi}}{dr}\right)\frac{d^2}{\nu}u \tag{2.12}$$

式中$a=kd$，$\sigma=\frac{qd^2}{\nu}$，$D=\frac{d}{d\zeta}$以及应用了下述关系：

$$\frac{d}{d\zeta}=\frac{d}{dr}\frac{dr}{d\zeta}=d\frac{d}{dr},\quad \frac{d^2}{d\zeta^2}=d^2\frac{d^2}{dr^2}$$

如我们把ζ坐标的原点移到间距$d=r_0-r_i$的中点，那么方程(2.11)和(2.12)就变成：

$$\left\{\frac{d^2}{d\zeta^2}-a^2-i\sigma-i6Ra\left(\frac{1}{4}-\zeta^2\right)\right\}\left(\frac{d^2}{d\zeta^2}-a^2\right)u-12iaRu=\omega d^2\frac{a^2}{\nu}v \tag{2.13}$$

$$\left\{\frac{d^2}{d\zeta^2}-a^2-i\sigma-i6Ra\left(\frac{1}{4}-\zeta^2\right)\right\}v=\left(\frac{V_{0\varphi}}{r}+\frac{dV_{0\varphi}}{dr}\right)\frac{d^2}{\nu}u \tag{2.14}$$

扰动方程(2.13)和(2.14)的边界条件为：

$$\zeta=\pm\frac{1}{2}, \qquad u=v=Du=0 \tag{2.15}$$

由于

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}V_{0z}d\zeta=\int_{-\frac{1}{2}}^{\frac{1}{2}}6V_m\left(\frac{1}{4}-\zeta^2\right)d\zeta=V_m$$

所以$6\left(\frac{1}{4}-\zeta^2\right)=1$。以及再应用式(2.3b)，方程(2.13)和(2.14)可进一步简化为：

$$\left(\frac{d^2}{d\zeta^2}-a^2-i\sigma-iaR\right)\left(\frac{d^2}{d\zeta^2}-a^2\right)u-i12aRu=\omega d^2\frac{a^2}{\nu}v \tag{2.16}$$

$$\left(\frac{d^2}{d\zeta^2}-a^2-i\sigma-iaR\right)v=b\omega\frac{d^2}{\nu}u \tag{2.17}$$

式中$b=\dfrac{2r_0^2}{r_0^2-r_i^2}$，以及边界条件

$$\zeta=\pm\frac{1}{2}\text{处,}\qquad u=v=Du=0 \tag{2.18}$$

我们的稳定性分析，就从方程(2.16),(2.17)和(2.18)出发.

取式(2.17)的共轭式

$$\frac{d^2}{d\zeta^2}\tilde{v}-a^2\tilde{v}+i\tilde{\sigma}\tilde{v}+iaR\tilde{v}=b\frac{d^2}{\nu}\omega\tilde{u} \tag{2.19}$$

方程(2.19)的边界条件为:

$$\zeta=\pm\frac{1}{2}\text{处,}\qquad \tilde{u}=\tilde{v}=0 \tag{2.20}$$

式(2.19)乘以v得:

$$v\frac{d^2\tilde{v}}{d\zeta^2}-a^2v\tilde{v}+i\tilde{\sigma}v\tilde{v}+iaRv\tilde{v}=b\frac{d^2}{\nu}\omega\tilde{u}v \tag{2.21}$$

式(2.21)从$-\frac{1}{2}$到$+\frac{1}{2}$对ζ积分:

$$-\int_{-\frac{1}{2}}^{\frac{1}{2}}|v'|^2d\zeta-a^2\int_{-\frac{1}{2}}^{\frac{1}{2}}|v|^2d\zeta+i\tilde{\sigma}\int_{-\frac{1}{2}}^{\frac{1}{2}}|v|^2d\zeta+iaR\int_{-\frac{1}{2}}^{\frac{1}{2}}|v|^2d\zeta=b\omega\frac{d^2}{\nu}\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}vd\zeta \tag{2.22}$$

上式已利用公式

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}v\frac{d^2\tilde{v}}{d\zeta^2}d\zeta=-\int_{-\frac{1}{2}}^{\frac{1}{2}}|v'|^2d\zeta$$

方程(2.16)展开得:

$$\left(\frac{d^4}{d\zeta^4}-2a^2\frac{d^2}{d\zeta^2}-i\sigma\frac{d^2}{d\zeta^2}-iaR\frac{d^2}{d\zeta^2}+a^4+i\sigma a^2+ia^3R\right)u-12iRau=\frac{\omega a^2d^2}{\nu}v \tag{2.23}$$

式(2.23)乘u的共轭速度$\tilde{u}$，并从$-\frac{1}{2}$到$+\frac{1}{2}$对ζ积分得:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\{|u''|^2+2a^2|u'|^2+i\sigma|u'|^2+iaR|u'|^2+a^4|u|^2$$
$$+i\sigma a^2|u|^2+ia^3R|u|^2-12iRa|u|^2\}d\zeta=\frac{\omega a^2d^2}{\nu}\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}vd\zeta \tag{2.24}$$

上式整理过程中已利用下列各式

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}\frac{d^4u}{d\zeta^4}d\zeta=\int_{-\frac{1}{2}}^{\frac{1}{2}}|u''|^2d\zeta\qquad ('')=\frac{d^2}{d\zeta^2}$$

$$-2a^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}\frac{d^2u}{d\zeta^2}d\zeta=2a^2\int_{-\frac{1}{2}}^{\frac{1}{2}}|u'|^2d\zeta\qquad (')=\frac{d}{d\zeta}$$

$$-i\sigma\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}\frac{d^2u}{d\zeta^2}d\zeta=i\sigma\int_{-\frac{1}{2}}^{\frac{1}{2}}|u'|^2d\zeta\qquad (')=\frac{d}{d\zeta}$$

$$-iaR\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}\frac{d^2u}{d\zeta^2}d\zeta=iaR\int_{-\frac{1}{2}}^{\frac{1}{2}}|u'|^2d\zeta\qquad (')=\frac{d}{d\zeta}$$

若把式(2.22)和(2.24)中的$\int_{-\frac{1}{2}}^{\frac{1}{2}}\tilde{u}v d\zeta$消去可得：

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\{|u''|^2+2a^2|u'|^2+iaR|u'|^2+a^4|u|^2+ia^3R|u|^2-12iRa|u|^2\}d\zeta$$

$$+i\ (\sigma_r+i\sigma_i)\int_{-\frac{1}{2}}^{\frac{1}{2}}\{|u'|^2+a^2|u|^2\}d\zeta$$

$$=\frac{a^2}{b}\ \{-\int_{-\frac{1}{2}}^{\frac{1}{2}}[|v'|^2+a^2|v|^2-iaR|v|^2]d\zeta$$

$$+i\ (\sigma_r-i\sigma_i)\int_{-\frac{1}{2}}^{\frac{1}{2}}|v|^2d\zeta \tag{2.25}$$

式中已应用$\sigma=\sigma_r+i\sigma_i$，$\tilde{\sigma}=\sigma_r-i\sigma_i$。将式(2.25)取实部得：

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\{|u''|^2+2a^2|u'|^2+a^4|u|^2\}\,d\zeta+\frac{a^2}{b}\int_{-\frac{1}{2}}^{\frac{1}{2}}[|v|^2+a^2|v|^2]d\zeta$$

$$-\sigma_i\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}}[|u'|^2+a^2|u|^2]d\zeta+\frac{a^2}{b}\int_{-\frac{1}{2}}^{\frac{1}{2}}|v|^2d\zeta\right\}=0 \tag{2.26}$$

由于$b=\dfrac{2r_0^2}{r_0^2-r_i^2}>0$，所以要使式(2.26)成立，必须满足下式：

$$\sigma_i>0 \tag{2.27}$$

而σ为

$$\sigma=\sigma_r+i\sigma_i=\frac{qd}{\nu}=\frac{d^2}{\nu}(q_r+iq_i)$$

因此要满足式(2.27)，则下式必成立：

$$\sigma_i=\frac{d^2}{\nu}q_i>0\qquad 或\quad q_i>0$$

我们从前面知$e^{iqt}=e^{i(q_r+iq_i)t}=e^{iq_rt-q_it}$，而$q_i$是扰动放大系数。显然，当放大系数$q_i>0$时，轴对称扰动将随$e^{-q_it}$呈指数衰减，那么也就证明了带轴向流旋转圆柱体间的血液流动是稳定的。当然，根据L.N.Howard理论，带轴向流旋转圆柱体间的密度分层流体流动（如血液流动）更是稳定的。虽然本文是针对血液流动的，但证明对任何窄间隙小雷诺数流体流动都是适用的。

三、带轴向流的旋转圆锥体间(一圆锥体静止，另一圆锥体以ω等角速旋转)均质血液流动稳定性

我们仍采用窄间隙小雷诺数稳定性理论，来证明带轴向流的旋转圆锥体间血液流动的稳定性。

设轴对称扰动后的速度与压强为：

$$V_r=-\varepsilon A(r)+u_r \tag{3.1a}$$

$$V_\varphi=V_{0\varphi}+\varepsilon C(r,z)+u_\varphi \tag{3.1b}$$

$$V_z=B_1(r)+\varepsilon B_2(r)z+u_z \tag{3.1c}$$

$$p=p_0+\varepsilon p_1+p' \tag{3.1d}$$

上式中$-\varepsilon A(r)$，$V_{0\varphi}+\varepsilon C(r,z)$，$B_1(r)+\varepsilon B_2(r)z$及$p_0+\varepsilon p_1$是带轴向流的旋转圆锥体间血液流动的基本解（或平衡解），这已在"血液分离器锥形分离杯内血液流动边界摄动解"一文中求得[1]．其中：

$$A(r)=V_{0z},\quad C(r,z)=V_{1\varphi},\quad B_1(r)=V_{0z}+\varepsilon f(r)$$

而$f(r)$是V_{1z}中只是r的函数那部份，这些都是已知量．见[1]．

若把式(3.1)代入方程(2.1)，并注意到圆锥体的轴对称性，即$\frac{\partial}{\partial\varphi}=0$，就可得扰动方程如下：

$$\frac{\partial u_r}{\partial t}-\varepsilon A\frac{\partial u_r}{\partial r}-u_r\frac{\partial \varepsilon A}{\partial r}+(B_1+\varepsilon B_2 z)\frac{\partial u_r}{\partial z}-\frac{2}{r}(V_{0\varphi}+\varepsilon C)u_\varphi$$

$$=-\frac{1}{\rho}\frac{\partial p'}{\partial r}+\nu\left(\nabla^2 u_r-\frac{u_r}{r^2}\right) \tag{3.2a}$$

$$\frac{\partial u_\varphi}{\partial t}+u_r\left(\frac{dV_{0\varphi}}{dr}+\varepsilon\frac{dC}{dr}\right)-\varepsilon A\frac{\partial u_\varphi}{\partial r}+(B_1+\varepsilon B_2 z)\frac{\partial u_\varphi}{\partial z}+\frac{1}{r}(-\varepsilon A)u_\varphi$$

$$+\frac{1}{r}u_r(V_{0\varphi}+\varepsilon C)=\nu\left(\nabla^2 u_\varphi-\frac{1}{r^2}u_\varphi\right) \tag{3.2b}$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial(B_1+\varepsilon B_2 z)}{\partial r}-\varepsilon A\frac{\partial u_z}{\partial r}+\varepsilon B_2 u_z+(B_1+\varepsilon B_2 z)\frac{\partial u_z}{\partial z}$$

$$=-\frac{1}{\rho}\frac{\partial p'}{\partial z}+\nu\nabla^2 u_z \tag{3.2c}$$

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}=0 \tag{3.2d}$$

设扰动方程的解为：

$$\left.\begin{aligned}&u_r=u(r)e^{i(qt+kz)},\quad u_\varphi=v(r)e^{i(qt+kz)}\\&u_z=w(r)e^{i(qt+kz)},\quad p'=\bar{p}(r)e^{i(qt+kz)}\end{aligned}\right\} \tag{3.3}$$

式中$q=q_r+iq_i$，q_r和q_i的意义同上一节．如把解(3.3)代入扰动方程(3.2)得如下方程：

$$\nu\left[DD_*-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}+\frac{\varepsilon dA/dr}{\nu}-\frac{(B_1+\varepsilon B_2 z)ik}{\nu}\right]u+\frac{2}{r}(V_{0\varphi}+\varepsilon C)v=D\frac{\bar{p}}{\rho} \tag{3.4a}$$

$$\nu\left[DD_*-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}+\frac{\varepsilon A}{\nu r}-\frac{(B_1+\varepsilon B_2 z)ik}{\nu}\right]v=\left(\frac{V_{0\varphi}}{r}+\frac{dV_{0\varphi}}{dr}+\frac{\varepsilon C}{r}+\varepsilon\frac{dC}{dr}\right)u \tag{3.4b}$$

$$\nu\left[D_*D-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}-\frac{\varepsilon B_2}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]w-uD(B_1+\varepsilon B_2 z)=ik\frac{\bar{p}}{\rho} \tag{3.4c}$$

$$D_*u=-ikw \tag{3.4d}$$

式中D，D_*的意义与上节相同．

消去(3.4c)和(3.4d)中的w得：

$$\frac{\nu}{k^2}\left[D_*D-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}-\frac{\varepsilon B_2}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]D_*u$$

$$+\frac{i}{k}uD(B_1+\varepsilon B_2 z)=\frac{\bar{p}}{\rho}$$

然后将上式所得之$\frac{\bar{p}}{\rho}$代入(3.4d)式中的$\frac{\bar{p}}{\rho}$得

$$D\left[D_*D-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}-\frac{\varepsilon B_2}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]D_*u+\frac{ik}{\nu}D[uD(B_1+\varepsilon B_2 z)]$$

$$=k^2\left[DD_*-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}+\frac{\varepsilon dA/dr}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]u+\frac{2k^2}{\nu r}(V_{0\varphi}+\varepsilon C)v \quad (3.5a)$$

式(3.4b)除以ν得：

$$\left[DD_*-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}+\frac{\varepsilon A}{r\nu}\right]v=\frac{V_{0\varphi}/r+dV_{0\varphi}/dr+\varepsilon C/r+\varepsilon dC/dr}{\nu}u \quad (3.5b)$$

方程(3.5)的边界条件为：

$$r=r_0-\varepsilon z,\ r=r_i-\varepsilon z\text{处}, \qquad u=v=Du=0 \quad (3.5c)$$

如为窄间隙即$d\ll\frac{r_0+r_i}{2}$，则有如下近似式

$$D_*\approx D$$

那么方程(3.5)可简化为：

$$D\left[D^2-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}-\frac{\varepsilon B_2}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]Du+\frac{ik}{\nu}D[uD(B_1+\varepsilon B_2 z)]$$

$$=k^2\left[D^2-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}+\frac{\varepsilon dA/dr}{\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]u+\frac{2k^2}{\nu r}(V_{0\varphi}+\varepsilon C)v \quad (3.6a)$$

$$\left[D^2-k^2-\frac{iq}{\nu}+\frac{\varepsilon AD}{\nu}+\frac{\varepsilon A}{r\nu}-\frac{ik(B_1+\varepsilon B_2 z)}{\nu}\right]v=\frac{V_{0\varphi}/r+dV_{0\varphi}/dr+\varepsilon C/r+\varepsilon dC/dr}{\nu}u \quad (3.6b)$$

展开方程(3.6)可得：

$$D^4u-2k^2D^2u+\frac{q_i}{\nu}D^2u+\frac{\varepsilon A}{\nu}D^3u+\frac{\varepsilon}{\nu}DA\,D^2u-\frac{\varepsilon}{\nu}DB_2D\,u-\frac{\varepsilon B_2}{\nu}D^2u$$

$$-\frac{ik}{\nu}(B_1+\varepsilon B_2 z)D^2u+k^4u-\frac{q_i}{\nu}k^2u-\frac{\varepsilon Ak^2}{\nu}Du-\frac{\varepsilon dA/dr}{\nu}k^2u-\frac{iq_r}{\nu}D^2u$$

$$+\frac{ik}{\nu}uD^2(B_1+\varepsilon B_2 z)+\frac{iq_rk^2}{\nu}u+\frac{ik^3(B_1+\varepsilon B_2 z)}{\nu}u=\frac{2k^2}{\nu r}(V_{0\varphi}+\varepsilon C)v \quad (3.7a)$$

$$D^2v-k^2v+\frac{q_i}{\nu}v+\frac{\varepsilon A}{\nu}Dv-\frac{\varepsilon A}{\nu r}v-\frac{iq_r}{\nu}v-\frac{ik}{\nu}(B_1+\varepsilon B_2 z)v$$

$$=\frac{V_{0\varphi}/r+dV_{0\varphi}/dr+\varepsilon C/r+\varepsilon dC/dr}{\nu}u \quad (3.7b)$$

根据R. C. Diprima理论，对窄间隙，小雷诺数血液流动来说，可用平均速度（角速度）代替速度(角速度)[2]，即

$$\frac{1}{d}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z} A(r)\,dr=\bar{A}(z)\text{代替}A(r) \tag{3.8a}$$

$$\frac{1}{d}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z} C(r,z)\,dr=\bar{C}(z)\text{代替}C(r,z) \tag{3.8b}$$

$$\frac{1}{d}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z} [B_1(r)+\varepsilon B_2(r)z]dr=\bar{B}_1(z)+\varepsilon\bar{B}_2(z)z\text{代替}B_1(r)+\varepsilon B_2(r)z \tag{3.8c}$$

若把式(3.7a)两边乘以u的共轭速度$\tilde{u}$，并从$r_i-\varepsilon z$到$r_0-\varepsilon z$对r积分，以及在整理过程中应用式(3.8)，那么可得下式：

$$\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\{|u''|^2+2k^2|u'|^2-\frac{q_i}{\nu}|u'|^2+k^4|u|^2-\frac{q_ik^2}{\nu}|u|^2\}dr+iF$$

$$=\frac{2k^2}{\nu}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\left(\frac{V_{0\varphi}}{r}+\frac{\varepsilon\bar{C}}{r}\right)\tilde{u}v\,dr \tag{3.9}$$

取式(3.7b)的共轭方程并乘以v，再从$r_i-\varepsilon z$到$r_0-\varepsilon z$对r积分，和应用式(3.8)，经整理后可得：

$$\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\left\{-|v'|^2-k^2|v|^2+\frac{q_i}{\nu}|v|^2+\frac{\varepsilon\bar{A}}{\nu}\frac{|v|^2}{r}\right\}dr+i\Phi$$

$$=\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\frac{1}{\nu}[V_{0\varphi}/r+dV_{0\varphi}/dr+\varepsilon\bar{C}/r+\varepsilon d\bar{C}/dr]\tilde{u}v\,dr$$

$$=\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\frac{1}{\nu}\left[\frac{2V_{0\varphi}}{r}+\varepsilon\frac{\bar{C}}{r}\right]\tilde{u}v\,dr \tag{3.10}$$

上式已利用$\frac{V_{0\varphi}}{r}=\frac{dV_{0\varphi}}{dr}$，$\frac{d\bar{C}}{dr}=0$ $(\bar{C}=\bar{C}(z))$。而式(3.9)和(3.10)中的F和Φ是所有虚数式之和：

$$F=i\Big[\frac{q_r}{\nu}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}|u'|^2dr+\frac{k}{\nu}(\bar{B}_1+\varepsilon\bar{B}_2z)\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}|u'|^2dr$$

$$+\frac{q_rk^2}{\nu}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}|u|^2dr+\frac{k^3(\bar{B}_1+\varepsilon\bar{B}_2z)}{\nu}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}|v|^2dr\Big]$$

$$\Phi=i\Big[q_r/\nu\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}|v|^2dr+\frac{k}{\nu}(\bar{B}_1+\varepsilon\bar{B}_2z)\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}|v|^2dr\Big]$$

其中带有DA，DB_2，$D^2(B_1+\varepsilon B_2z)$以及$\frac{dA}{dr}$各项，由于$A,B_1,B_2$用平均值$\bar{A}(z)$，$\bar{B}_1(z)$，$\bar{B}_2(z)$代替之后都为零，所以其积分也为零。其次是$\frac{\varepsilon A}{\nu}D^3u$，$\frac{\varepsilon Ak^2}{\nu}Du$乘上$\tilde{u}$的积分和$\frac{\varepsilon A}{\nu r}vD\tilde{v}$的积分的实部为零。因此在式(3.9)和式(3.10)中就不出现这些项的积分式了。

若我们把式(3.9)和(3.10)中相对小量$\frac{\varepsilon\bar{C}}{r}$项加以忽略，然后把方程(3.9)，(3.10)中

公共项$\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\tilde{u}vdr$消去、重新得到实部：

$$\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\left\{|u''|^2+2k^2|u'|^2+k^4|u|^2\right\}dr+k^2\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\left\{|v'|^2+k^2|v|^2+\frac{\varepsilon\bar{A}}{\nu}\frac{|v|^2}{r}\right\}dr$$

$$-\frac{q_i}{\nu}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}\{|u'|^2+k^2|u|^2+|v|^2\}dr=0 \tag{3.11}$$

由于$r_0-\varepsilon z>r_i-\varepsilon z>0$，及$\bar{A}(z)=\frac{1}{d}\int_{r_i-\varepsilon z}^{r_0-\varepsilon z}A(r)dr>0$，所以式(3.11)前二个积分为正值，因此为使式(3.11)成立，必须有

$$q_i>0$$

这样，轴对称扰动将随e^{-q_it}呈指数衰减，从而证明了带轴向流旋转圆锥体间均质血液流动是稳定的。当然带轴向流旋转圆锥体间密度分层血液流动更是稳定的。虽然我们的证明是针对血液分离器锥形分离杯内血液流动的，但只要是窄间小雷诺数任何流动，证明同样是适用的。

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Narrow Gap Stability Theory of Blood Flow between Two Relatively Rotating Concentric Cones

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Abstract

The boundary perturbation solutions for the blood flow between two relatively rotating conceneric cones (one of them is stationary, the other is rotating with constant angle velocity ω) have been obtained[1]. Then on basis of the solutions obtained, using theory of narrow gap stability, the stability of the density stratified blood flow between relatively rotating concentric cones with an axial flow is demonstrated.