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扁壳压曲的蠕变效应*

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摘　要

本文讨论了混凝土扁壳压曲的蠕变效应.基于弹性薄壳的非线性理论，发现椭圆抛物面扁壳，其荷载-挠度曲线的上临界荷载将随时间而降低，而下临界荷载则随时间而上升.至于扁壳的局部失稳问题，其临界荷载仅取决于压曲发生瞬间材料的弹性模量.

一、引　言

建筑工程中的扁壳往往是整体或装配式钢筋混凝土结构.由于蠕变对于混凝土薄壁结构具有重要的影响，因而壳体结构的设计者对于了解蠕变的影响颇感兴趣.本文将导出蠕变对于扁壳承载能力影响的基本微分积分方程，并对四边简支椭圆抛物面双曲扁壳的情况进行研究.文中讨论了方程的解法，提供了一个简化计算方法，可以迅速求出扁壳压曲的蠕变效应，结果具有较好的精确度.

二、扁壳考虑蠕变的非线性方程

在处理壳体问题时，我们通常假定[1]

$$\nu_1(\tau)=\nu_2(t,\tau)=\nu=\text{const} \tag{2.1a}$$

正应力条件下的蠕变度$c(t,\tau)$与剪应力条件下的蠕变度$\omega(t,\tau)$成比例，可列出二者的关系式

$$\omega(t,\tau)=2(1+\nu)c(t,\tau) \tag{2.1b}$$

考虑蠕变时，距壳体中面某一距离z处的应力-应变关系为

$$\left.\begin{aligned}
\varepsilon_{1z}^*(t)=&\frac{1}{E(t)}[\sigma_{1z}^*(t)-\nu\sigma_{2z}^*(t)]\\
&-\int_{\tau_1}^{t}\{\sigma_{1z}^*(\tau)-\nu\sigma_{2z}^*(\tau)\}\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\\
\varepsilon_{2z}^*(t)=&\frac{1}{E(t)}[\sigma_{2z}^*(t)-\nu\sigma_{1z}^*(t)]
\end{aligned}\right\} \tag{2.2a,b,c}$$

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$$
\left.\begin{aligned}
&-\int_{\tau_1}^{t}\{\sigma_{2z}^{*}(\tau)-\nu\sigma_{1z}^{*}(\tau)\}\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\\
&\gamma_{z}^{*}(t)=\frac{2(1+\nu)}{E(t)}\tau_{z}^{*}(t)-\int_{\tau_1}^{t}2(1+\nu)\tau_{z}^{*}(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau
\end{aligned}\right|
$$

式中

$$
\delta(t,\tau)=\frac{1}{E(\tau)}+c(t,\tau) \tag{2.2d}
$$

对于大变形问题，应变与位移关系式为

$$
\left.\begin{aligned}
&\varepsilon_{1z}^{*}(t)=\frac{\partial u^{*}(t)}{\partial x}-k_1w^{*}(t)-z\frac{\partial^2w^{*}(t)}{\partial x^2}+\frac{1}{2}\left[\frac{\partial w^{*}(t)}{\partial x}\right]^2\\
&\varepsilon_{2z}^{*}(t)=\frac{\partial v^{*}(t)}{\partial y}-k_2w^{*}(t)-z\frac{\partial^2w^{*}(t)}{\partial y^2}+\frac{1}{2}\left[\frac{\partial w^{*}(t)}{\partial y}\right]^2\\
&\gamma_{z}^{*}(t)=\frac{\partial u^{*}(t)}{\partial y}+\frac{\partial v^{*}(t)}{\partial x}-2k_{12}w^{*}(t)-2z\frac{\partial^2w^{*}(t)}{\partial x\partial y}\\
&\qquad+\frac{\partial w^{*}(t)}{\partial x}\cdot\frac{\partial w^{*}(t)}{\partial y}
\end{aligned}\right\} \tag{2.3a,b,c}
$$

由此，可导出壳体的积分-微分方程如下

$$
\left.\begin{aligned}
&D(t)\Delta^2w^{*}(t)-\left[\Delta_k\varphi^{*}(t)-E(t)\int_{\tau_1}^{t}\Delta_k\varphi^{*}(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\right]\\
&\quad-\left\{L[w^{*}(t),\varphi^{*}(t)]-E(t)\int_{\tau_1}^{t}L[w^{*}(\tau),\varphi^{*}(\tau)]\right.\\
&\quad\left.\cdot\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\right\}\\
&=q(t)-E(t)\int_{\tau_1}^{t}q(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\\
&\Delta^2\varphi^{*}(t)-E(t)\int_{\tau_1}^{t}\Delta^2\varphi^{*}(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau+E(t)\delta\Delta_kw^{*}(t)\\
&\quad+\frac{1}{2}E(t)\delta L[w^{*}(t),w^{*}(t)]=0
\end{aligned}\right\} \tag{2.4a,b}
$$

式中

$$
\left.\begin{aligned}
&L[w^{*}(t),\varphi^{*}(t)]=\frac{\partial^2w^{*}(t)}{\partial x^2}\frac{\partial^2\varphi^{*}(t)}{\partial y^2}+\frac{\partial^2w^{*}(t)}{\partial y^2}\\
&\quad\cdot\frac{\partial^2\varphi^{*}(t)}{\partial x^2}-2\frac{\partial^2w^{*}(t)}{\partial x\partial y}\frac{\partial^2\varphi^{*}(t)}{\partial x\partial y}\\
&\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\\
&\Delta_k=k_1\frac{\partial^2}{\partial y^2}+k_2\frac{\partial^2}{\partial x^2}-2k_{12}\frac{\partial^2}{\partial x\partial y}
\end{aligned}\right\} \tag{2.5a,b,c}
$$

$\varphi^{*}(t)$——应力函数

为使方程(2.4a,b)无量纲化，令

$$\left.\begin{array}{l}\xi=\dfrac{x}{a},\eta=\dfrac{y}{b},\quad \Phi^*(t)=\dfrac{\varphi^*(t)}{E(t)\delta^3},\quad W^*(t)=\dfrac{w^*(t)}{\delta}\\ \alpha_1=\dfrac{k_1a^2}{\delta},\alpha_2=\dfrac{k_2b^2}{\delta},\quad \alpha_{12}=\dfrac{k_{12}ab}{\delta}\\ D=\dfrac{1}{12(1-\nu^2)},\quad \lambda=\dfrac{b}{a}\end{array}\right\}\tag{2.6 a～i}$$

$$\left.\begin{array}{l}\Delta^{0^2}=\lambda^4\dfrac{\partial^4}{\partial\xi^4}+2\lambda^2\dfrac{\partial^4}{\partial\xi^2\partial\eta^2}+\dfrac{\partial^4}{\partial\eta^4}\\ \Delta_k^0=\alpha_1\dfrac{\partial^2}{\partial\eta^2}+\alpha_2\dfrac{\partial^2}{\partial\xi^2}-2\alpha_{12}\dfrac{\partial^2}{\partial\xi\partial\eta}\\ L^0=\dfrac{\partial^2}{\partial\xi^2}\cdot\dfrac{\partial^2}{\partial\eta^2}+\dfrac{\partial^2}{\partial\eta^2}\cdot\dfrac{\partial^2}{\partial\xi^2}-2\left(\dfrac{\partial^2}{\partial\xi\partial\eta}\right)^2\end{array}\right\}\tag{2.7a,b,c}$$

于是，可得

$$\left.\begin{array}{l}D\Delta^{0^2}W^*(t)-\lambda^2\left[\Delta_k^0\Phi^*(t)-\displaystyle\int_{\tau_1}^t E(\tau)\Delta_k^0\Phi^*(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\right]\\ \quad-\left\{L^0[W^*(t),\Phi^*(t)]-\displaystyle\int_{\tau_1}^t E(\tau)L^0[W^*(\tau),\Phi^*(\tau)]\right.\\ \quad\left.\cdot\dfrac{\partial\delta(t,\tau)}{\partial\tau}d\tau\right\}=q^*(t)E(t)\delta(t,\tau_1)\\ \Delta^{0^2}\Phi^*(t)-\displaystyle\int_{\tau_1}^t E(\tau)\Delta^{0^2}\Phi^*(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau+\lambda^2\Delta_k^0W^*(t)\\ \quad+\dfrac{1}{2}\lambda^2L^0[W^*(t),W^*(t)]=0\end{array}\right\}\tag{2.8a,b}$$

式中

$$q^*(t)=\frac{qb^4}{E(t)\delta^4}\tag{2.9}$$

q 考虑为一常量参数。

三、四边简支椭圆抛物面扁壳考虑蠕变时的荷载-位移关系

本节讨论第一类压曲的上、下临界荷载。

壳体的边界条件为[2]

$$\left.\begin{array}{l}\varphi^*(t)=\dfrac{xy}{ab}\cdot\dfrac{E(t)\delta}{1+\nu}k_{12}\displaystyle\int_0^b\int_0^a w^*(t)dxdy=0\\ w^*(t)=\Delta w^*(t)=\Delta\varphi^*(t)=0\end{array}\right\}\tag{3.1a,b}$$

所以可得

$$W^*(t)=\Delta^0W^*(t)=\Phi^*(t)=\Delta^0\Phi^*(t)=0\tag{3.2}$$

令

$$\Psi^*(t)=\Phi^*(t)-\int_{\tau_1}^t E(\tau)\Phi^*(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\tag{3.3}$$

于是，$\Psi^*(t)$ 的边界条件将与式(3.2)给出的 $\Phi^*(t)$ 的相同。

式(2.8a,b)可重新改写为

$$\left.\begin{aligned}&\bar{D}\Delta^{0^2}W^*(t)-\lambda^2\Delta_k^0\Psi^*(t)-\lambda^2L^0[W^*(t),\Psi^*(t)]\\&\quad=q^*(t)E(t)\delta(t,\tau_1)+\lambda^2\int_{\tau_1}^{t}E(\tau)L^0[W^*(t)\\&\qquad-W^*(\tau),\Phi^*(\tau)]\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\\&\Delta^{0^2}\Psi^*(t)+\lambda^2\Delta_k^0W^*(t)+\frac{\lambda^2}{2}L^0[W^*(t),W^*(t)]=0\end{aligned}\right\}\tag{3.4a,b}$$

将 $W^*(t),\Psi^*(t),\Phi^*(t)$ 均表示成下述重三角级数，满足已知边界条件

$$\left.\begin{aligned}W^*(t)&=\sum_{m,n}A_{mn}(t)\sin m\pi\xi\sin n\pi\eta\\\Psi^*(t)&=\sum_{m,n}B_{mn}(t)\sin m\pi\xi\sin n\pi\eta\\\Phi^*(t)&=\sum_{m,n}C_{mn}(t)\sin m\pi\xi\sin n\pi\eta\end{aligned}\right\}\quad(m,n=1,3,5,\cdots)\tag{3.5a,b}$$

将它们代入方程(3.4a,b)，并利用伽辽金方法，可得

$$\left.\begin{aligned}&\pi^4A_{mn}(t)\bar{D}(\lambda^4m^4+2m^2n^2\lambda^2+n^4)+\pi^2\lambda^2B_{mn}(t)(\alpha_1n^2\\&\quad+\alpha_2m^2)-\lambda^2\pi^2L_1[A_{ij}(t)B_{kl}(t)]\\&\quad=\frac{16}{mn\pi^2}q^*(t)E(t)\ \delta(t,\tau_1)+q_m^*\\&B_{mn}(t)(\lambda^4m^4+2\lambda^2m^2n^2+n^4)-\frac{\lambda^2}{\pi^2}A_{mn}(t)\ (\alpha_1n^2+\alpha_2m^2)\\&\quad-\frac{\lambda^2}{\pi^2}L_2[A_{ij}(t)\cdot A_{kl}(t)]=0\end{aligned}\right\}\tag{3.6a,b}$$

式中：

$$\left.\begin{aligned}&L_1[A_{ij}(t)B_{kl}(t)]=\sum_{k,l,i,j}(i^2l^2+j^2k^2)\left(\frac{1}{i-k+m}+\frac{1}{k+m-i}\right.\\&\quad\left.+\frac{1}{i+k-m}-\frac{1}{i+k+m}\right)\left(\frac{1}{j-l+n}+\frac{1}{l+n-j}+\frac{1}{j+l-n}-\frac{1}{j+l+n}\right)\\&\quad-2ijkl\left(\frac{1}{m+i+k}-\frac{1}{i+k-m}+\frac{1}{m+i-k}+\frac{1}{m+k-i}\right)\\&\quad\cdot\left(\frac{1}{j+n+l}-\frac{1}{j+l-n}+\frac{1}{n+j-l}+\frac{1}{n+l-j}\right)A_{ij}(t)B_{kl}(t)\\&L_2[A_{ij}(t)\cdot B_{kl}(t)]=\sum_{k,l,i,j}\left[klij\left(\frac{1}{i+m+k}-\frac{1}{i+k-m}\right.\right.\\&\quad\left.+\frac{1}{m+i-k}+\frac{1}{m+k-i}\right)\left(\frac{1}{n+j+l}-\frac{1}{j+l-n}+\frac{1}{n+j-l}\right.\\&\quad\left.+\frac{1}{n+l-j}\right)-k^2j^2\left(\frac{1}{m-k+i}+\frac{1}{k+i-m}+\frac{1}{m+k-i}\right.\end{aligned}\right\}\tag{3.6c,d,e}$$

$$
-\frac{1}{m+k+i}\Big)\Big(\frac{1}{n-l+j}+\frac{1}{l+j-n}+\frac{1}{n+l-j}-\frac{1}{n+l+j}\Big)\Big]
$$

$$
\cdot A_{ij}(t)A_{kl}(t)
$$

$$
q_m^* = \lambda^2\pi^2\int_{\tau_1}^{t} E(\tau)L_1[A_{ij}(t)-A_{ij}(\tau)C_{kl}(\tau)]\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau
$$

$$
(m,n,i,j,k,l=1,3,5,7,\cdots\cdots)
$$

从方程(3.3)，我们有 B_{mn} 及 C_{mn} 之间的关系为

$$
B_{mn}(t)=C_{mn}(t)-\int_{\tau_1}^{t} E(\tau)C_{mn}(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau \tag{3.7}
$$

取$t=\tau_1$，则$E(\tau_1)\delta(\tau_1,\tau_1)=1$，求解这组非线性代数方程组，可求得$A_{mn}(\tau_1)$，$B_{mn}(\tau_1)=C_{mn}(\tau_1)$。有了这些，就不难绘出荷载-位移 $(q^*(\tau_1)\text{-}W_0^*(\tau_1))$ 曲线，$W_0^*(\tau_1)$ 为壳体中心点的垂直位移。这一曲线相应于时刻 $t=\tau_1$ 时壳体的性状。由此曲线可迅速找出其上临界荷载$q^*_{\max}(\tau_1)$ 及下临界荷载 $q^*_{\min}(\tau_1)$。然而随着时间的延续，上临界荷载将降低，而下临界荷载将上升，为了说明这一点，我们求解方程(3.6a,b)及(3.7)。

(1) 逐次迭代修正法

由于q_m^* 值一般相对较小，首先取 $q_m^*=0$。从非线性代数方程组(3.6a,b)求出 $A_{mn}(t)$ 及 $B_{mn}(t)$，$C_{mn}(t)$值可由下述公式求得

$$
\left.\begin{aligned}
B_{mn}(\tau_p)=C_{mn}(\tau_p)-[&E(\tau_1)C_{mn}(\tau_1)K(\tau_p,\tau_1)(\tau_2-\tau_1)\\
&+E(\tau_2)C_{mn}(\tau_2)K(\tau_p,\tau_2)(\tau_3-\tau_2)+\cdots\\
&+E(\tau_{p-1})C_{mn}(\tau_{p-1})K(\tau_p,\tau_{p-1})(\tau_p-\tau_{p-1})]\\
B_{mn}(\tau_1)=C_{mn}(\tau_1)&
\end{aligned}\right\}\tag{3.8a,b}
$$

式中

$$
K(t,\tau)=\frac{\partial\delta(t,\tau)}{\partial\tau}\tag{3.8c}
$$

将 $A_{mn}(t)$ 及 $C_{mn}(t)$ 的值代入下式以求 q_m^* 作为第一步近似

$$
\begin{aligned}
q_m^* =\lambda^2\pi^2[&E(\tau_1)(A_{ij}(t)-A_{ij}(\tau_1))C_{kl}(\tau_1)K(t,\tau_1)(\tau_2-\tau_1)\\
&+\cdots+E(\tau_{p-1})(A_{ij}(t)-A_{ij}(\tau_{p-1}))C_{kl}(\tau_{p-1})(\tau_p-\tau_{p-1})+\cdots]
\end{aligned}\tag{3.9}
$$

再回到方程(3.6a)，我们可得到方程右端自由项考虑蠕变影响的修正值。重复上述步骤，直至方程(3.6a)的右端自由项值稳定为止。计算经验表明两轮循环已可获得较为满意的结果。

(2) 数值积分法

为了直接得到 $A_{mn}(t)$ 及 $C_{mn}(t)$，我们对方程(2.8a,b)施用伽辽金法，可得

$$
\left.\begin{aligned}
&\pi^4 DA_{mn}(t)(\lambda^4m^4+2\lambda^2m^2n^2+n^4)+\pi^2\lambda^2C_{mn}(t)(a_1n^2+a_2m^2)\\
&\quad-\lambda^2\pi^2L_1[A_{ij}(t)C_{kl}(t)]\\
&=\frac{16}{mn\pi^2}q^*(t)E(t)\delta(t,\tau_1)+\lambda^2\pi^2(a_1n^2+a_2m^2)\int_{\tau_1}^{t}E(\tau)C_{mn}(\tau)\\
&\quad\cdot K(t,\tau)d\tau-\lambda^2\pi^2\int_{\tau_1}^{t}E(\tau)L_1[A_{ij}(\tau)C_{kl}(\tau)]K(t,\tau)d\tau\\
&C_{mn}(t)(\lambda^4m^4+2\lambda^2m^2n^2+n^4)-\frac{\lambda^2}{\pi^2}(a_1n^2+a_2m^2)A_{mn}(t)
\end{aligned}\right\}\tag{3.10a,b}
$$

$$
\left.\begin{aligned}
&-\frac{\lambda^2}{\pi^2}L_2[A_{ij}(t)\cdot A_{kl}(t)]\\
&=(m^4\lambda^4+2\lambda^2m^2n^2+n^4)\int_{\tau_1}^{t}E(\tau)C_{mn}(\tau)K(t,\tau)d\tau
\end{aligned}\right)
$$

如果我们将积分代换为数值计和，方程 (3.10a,b) 即改变为下述的非线性代数方程组

$$
\left.\begin{aligned}
&\pi^4A_{mn}(\tau_p)\bar{D}(\lambda^4m^4+2m^2n^2\lambda^2+n^4)+\pi^2\lambda^2C_{mn}(\tau_p)(\alpha_1n^2+\alpha_2m^2)\\
&\quad-\lambda^2\pi^2L_1[A_{ij}(\tau_p)C_{kl}(\tau_p)]=\frac{16}{mn\pi^2}q^*(t)E(t)\delta(\tau_p,\tau_1)\\
&\quad+\lambda^2\pi^2[E(\tau_1)C_{mn}(\tau_1)K(\tau_p,\tau_1)(\tau_2-\tau_1)+\cdots\\
&\quad+E(\tau_{p-1})C_{mn}(\tau_{p-1})K(\tau_p,\tau_{p-1})(\tau_p-\tau_{p-1})]\\
&\quad-\lambda^2\pi^2\{E(\tau_1)L_1[A_{ij}(\tau_1)C_{kl}(\tau_1)]K(\tau_p,\tau_1)(\tau_2-\tau_1)+\cdots\\
&\quad+E(\tau_{p-1})L_1[A_{ij}(\tau_{p-1})C_{kl}(\tau_{p-1})K(\tau_p,\tau_{p-1})(\tau_p-\tau_{p-1})\}\\
&C_{mn}(\tau_p)(\lambda^4m^4+2\lambda^2m^2n^2+n^4)-\frac{\lambda^2}{\pi^2}(\alpha_1n^2+\alpha_2m^2)A_{mn}(\tau_p)\\
&\quad-\frac{\lambda^2}{\pi^2}L_2[A_{ij}(\tau_p)A_{kl}(\tau_p)]=(\lambda^4m^4+2m^2n^2\lambda^2+n^4)\\
&\quad\cdot[E(\tau_1)C_{mn}(\tau_1)K(\tau_p,\tau_1)(\tau_2-\tau_1)+\cdots\\
&\quad+E(\tau_{p-1})C_{mn}(\tau_{p-1})(\tau_p-\tau_{p-1})
\end{aligned}\right\}\quad(3.11\text{a,b})
$$

逐次令 $p=1,2,3\cdots$，我们得到相应的非线性代数方程组，可逐次解得 $A_{mn}(\tau_p)$ 及 $C_{mn}(\tau_p)$。

（3）　**计算蠕变效应的简化方法**

上述两个方法可以给出问题的较精确的解，并可在实际中应用，但此二法涉及的计算工作量很大，因此寻找一个简化的计算方法是需要的．如果压曲问题的不考虑蠕变的解已经求得，那么下面建议的计算蠕变效应的方法只需很少量的计算工作．

a)　令 q_m^* 等于零，$t=\tau_1$，则式 (3.6a,b) 简化为不考虑蠕变效应的通常压曲问题，它代表壳体在 $t=\tau_1$ 时刻的实际状态．这时方程 (3.6a) 右端自由项为 $q^*(\tau_1)E(\tau_1)\delta(\tau_1,\tau_1)=\dfrac{qb^4}{E(\tau_1)\delta^4}$，我们可用现行方法求得 $A_{mn}(\tau_1)$ 及 $B_{mn}(\tau_1)=C_{mn}(\tau_1)$．利用这些值，可迅速绘出 $q^*(\tau_1)$-$W_0^*(\tau_1)$ 曲线，它通常是一条三次抛物线，见图 1 曲线 1($\tau_1=14$)。

b)　作为壳体蠕变效应的第一次近似，仍然取 $q_m^*=0$，但令方程 (3.6a) 右端自由项等于 $q^*(t)E(t)\delta(t,\tau_1)$．如时间 t 值已知，右端自由项再是一个数值，我们可利用已有的解从 $A_{mn}(\tau_1)$ 及 $B_{mn}(\tau_1)$ 值求得 $A_{mn}(t)$ 及 $B_{mn}(t)$ 值。相应的 $C_{mn}(t)$ 值可由式 (3.8) 得到。知道了 $A_{mn}(t)$，我们可再次绘出 $q^*(t)$-$W_0^*(t)$ 曲线，见图 1 曲线 2($t=374$)。

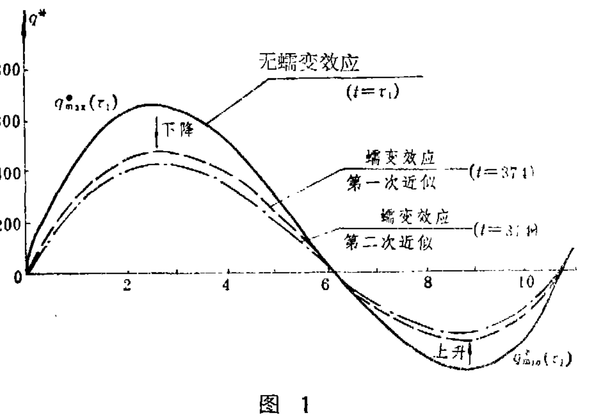

图　1

c) 作为第二次近似，我们仅仅取前面近似解中的 $A_{11}(t)$ 及 $C_{11}(t)$，将它们代入方程(3.6e)去求出 q_m^* 的近似值。其余的 $A_{mn}(t)$ 及 $C_{mn}(t)$ 值一般在数值上与 $A_{11}(t)$ 及 $C_{11}(t)$ 相比居于次要予以忽略。于是，方程 (3.6a) 的新的自由项成为 $q^*(t)E(t)\delta(t,\tau_1)+q_m^*$，其余计算步骤则与第一次近似时相同。一个新的 $q^*(t)$-$W_0^*(t)$ 曲线求得如图 1 的曲线 3。

d) 重复同样步骤以进一步改进计算结果，直至 $q^*(t)E(t)\delta(t,\tau_1)+q_m^*$ 稳定为止。采用这一计算方法的经验表明，q_m^* 一般仅仅是 $q^*(t)E(t)\delta(t,\tau_1)$ 的一个小的分数(通常小于5～10%)，所以实际上往往只要一次循环(即做到第二次近似)即可获得较好的结果，并且偏于安全方面。这个方法可以应用于工程设计。从图 1 我们可以推知的一般结论是，壳体压曲的上临界荷载将随时间而降低，在一年的时间内可降至 $\tau_1=14$ 天时的约 2/3，但壳体的下临界荷载将随时间而有所提高。

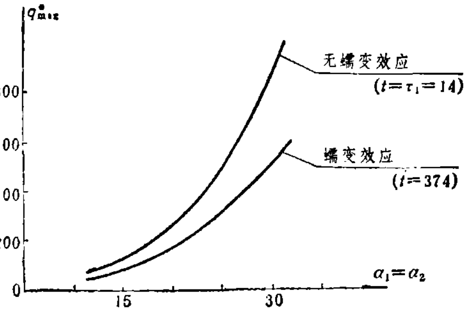

图 2

图 2 表示壳体上临界荷载随 α 值的变化情况，考虑了蠕变效应，$\tau_1=14$天，t 为 374 天。

四、椭圆抛物面扁壳考虑蠕变时的局部失稳问题

我们将局部压曲称为第二类压曲问题。假设壳体的局部失稳发生于某一时刻 t。在壳体局部压曲以前，下述积分-微分方程应该得到满足。

$$\left.\begin{aligned}
&D(t)\Delta^2 w_1^*(t)-\left[\Delta_k\varphi_1^*(t)-E(t)\int_{\tau_1}^{t}\Delta_k\varphi^*(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\right]\\
&\quad-\left\{L[w_1^*(t),\varphi_1^*(t)]-E(t)\int_{\tau_1}^{t}L[w_1^*(\tau),\varphi_1^*(\tau)]\right.\\
&\quad\left.\cdot\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau\right\}=qE(t)\delta(t,\tau_1)\\
&\Delta^2\varphi_1^*(t)-E(t)\int_{\tau_1}^{t}\Delta^2\varphi_1^*(\tau)\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau+E(t)\,\delta\Delta_k w_1^*(t)\\
&\quad+\frac{1}{2}E(t)\delta L[w_1^*(t),w_1^*(t)]=0
\end{aligned}\right\}\qquad(4.1\text{a, b})$$

在局部失稳发生的时刻，壳面将局部向内发生凹陷，如图 3 所示。于是壳体的性状将满足下述方程组：

$$\left.\begin{aligned}
&D(t)\Delta^2[w_1^*(t)+w_2^*(t)]-\Delta_k[\varphi_1^*(t)+\varphi_2^*(t)+E(t)\int_{\tau_1}^{t}\Delta_k[\varphi_1^*(\tau)\\
&\quad+\varphi_2^*(\tau)H(\tau-t)]\cdot\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau-L[w_1^*(t)+w_2^*(t),\varphi_1^*(t)\\
&\quad+\varphi_2^*(t)]+E(t)\int_{\tau_1}^{t}L[w_1^*(\tau)+w_2^*(\tau)H(\tau-t),\varphi_1^*(\tau)\\
&\quad+\varphi_2^*(\tau)H(\tau-t)]\frac{\partial\delta(t,\tau)}{\partial\tau}d\tau=qE(t)\delta(t,\tau_1)\\
&\Delta^2[\varphi_1^*(t)+\varphi_2^*(t)]-E(t)\int_{\tau_1}^{t}\Delta^2[\varphi_1^*(\tau)+\varphi_2^*(\tau)H(\tau-t)]
\end{aligned}\right\}\qquad(4.2\text{a, b})$$

$$\cdot \frac{\partial \delta(t,\tau)}{\partial \tau} d\tau + E(t)\,\delta \Delta_k [w_1^*(t) + w_2^*(t)]$$

$$+ \frac{E(t)\delta}{2} L[w_1^*(t) + w_2^*(t), w_1^*(t) + w_2^*(t)] = 0$$

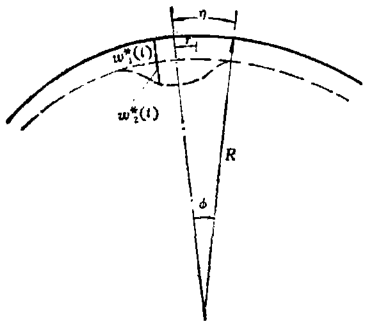

图 3

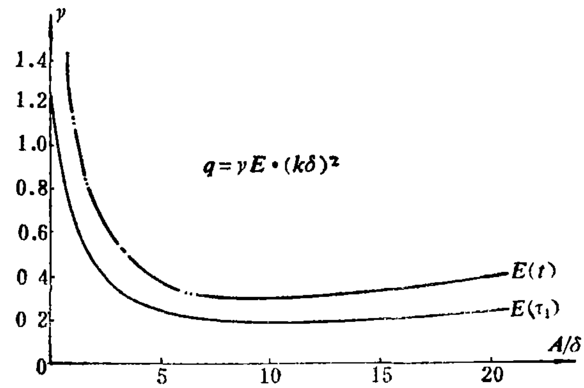

图 4

式中 $H(\tau - t)$ 为阶梯函数.

由于壳体局部压曲面积相对较小，而且主要是发生在壳的中央部分，我们有

$$\left.\begin{aligned}
&\frac{\partial^2 w_1^*(t)}{\partial x^2} \approx \frac{\partial^2 w_1^*(t)}{\partial y^2} \approx \frac{\partial^2 w_1^*(t)}{\partial x \partial y} = 0 \\
&T_1^0 = \frac{\partial^2 \varphi_1^*(t)}{\partial y^2},\ T_2^0 = \frac{\partial^2 \varphi_1^*(t)}{\partial x^2},\ S^0 = -\frac{\partial^2 \varphi_1^*(t)}{\partial x \partial y} = 0
\end{aligned}\right\} \tag{4.2c, d}$$

从方程 (4.2a, b) 减去方程 (4.1a, b) 中的相应方程，可得

$$\left.\begin{aligned}
&D(t)\Delta^2 w_2^*(t) - \Delta_k \varphi_2^*(t) - L[w_2^*(t), \varphi_2^*(t)] = T_1^0 \frac{\partial^2 w_2^*(t)}{\partial x^2} + T_2^0 \frac{\partial^2 w_2^*(t)}{\partial y^2} \\
&\Delta^2 \varphi_2^*(t) + E(t)\delta \Delta_k w_2^*(t) + \frac{1}{2} E(t)\delta L[w_2^*(t), w_2^*(t)] = 0
\end{aligned}\right\} \tag{4.3a, b}$$

与通常的局部失稳问题相比较，可以容易地注意到压曲荷载是 $E(t)$ 的一个函数. 作者曾对等曲率双曲扁壳局部失稳的临界荷载用能量法求得了一个解. 我们取失稳曲面方程为

$$\left.\begin{aligned}
&w_2 = A\left(1 - \frac{r^2}{\eta^2}\right)^2 \\
&u = B_1 r\left(1 - \frac{r^2}{\eta^2}\right) + B_2 r^3\left(1 - \frac{r^2}{\eta^2}\right) + B_3 r^5\left(1 - \frac{r^2}{\eta^2}\right)
\end{aligned}\right\} \tag{4.4a, b}$$

临界荷载的公式为

$$q_{cr} = 0.288 E(t)(k\delta)^2 \tag{4.5}$$

结果绘于图4.

鉴于 $t = \tau_1$ 时 $E(t)$ 取最小值，这就告诉我们壳体局部失稳的临界荷载发生于壳体开始受荷的时刻.

同样，对于不等曲率的扁壳，作者取局部失稳时的曲面方程为

$$
\left.\begin{aligned}
w&=A\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\\
u&=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(B_1\frac{x}{a}+B_2\frac{x^3}{a^3}+B_3\frac{xy^2}{ab^2}+B_4\frac{x^5}{a^5}+2B_5\frac{x^3y^2}{a^3b^2}+B_6\frac{xy^4}{ab^4}\right)\\
v&=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(C_1\frac{y}{b}+C_2\frac{y^3}{b^3}+C_3\frac{x^2y}{a^2b}+C_4\frac{y^5}{b^5}+2C_5\frac{x^2y^3}{a^2b^3}+C_6\frac{x^4y}{a^4b}\right)
\end{aligned}\right\}\quad(4.6\mathrm{a,b,c})
$$

用能量法及广义伽辽金法求得其临界荷载的公式为

$$q_{cr}=(0.288+0.013\lambda+0.03\lambda^2)E(t)k_xk_y\delta^2 \qquad (4.7)$$

式中

$$\lambda=k_x/k_y-1 \qquad (0\leqslant\lambda\leqslant1)$$

k_x, k_y——x及y方向壳面之原始曲率。

与等曲率扁壳的情况相同，壳体局部失稳的最小临界荷载发生在壳体开始受荷的时刻。

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The Creep Effects on the Buckling of Shallow Shells

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Abstract

An analysis for the creep-effects on the buckling of concrete shallow shells is presented in the paper. Based on the non-linear theory of thin elastic shells, it is found that for those of the elliptical paraboloidal type, the upper critical load of the load-deflection curve will decrease while the lower limit will increase with time. As to the problem of local buckling of the shell, the critical load is dependent only upon the modulus of elasticity at the instant when it occurs.