

双金属条整温器的接触热应力

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摘 要

本文以整式得到了组成整温器的两金属条的接触面内的热应力, 包括剪应力和正应力. 应力集中分布在金属条的端部, 其分布的长度约为金属条的厚度.

一、引 言

双金属条整温器, 系由膨胀系数不同的两狭长的金属条焊合而成(图1). 当温度均匀地升高时, 在两金属条内将产生热应力, 并将发生弯曲.

Timoshenko教授用普通材料力学的方法, 就可计算双金属条离两端较远的横截面内的热应力. 但要分析两金属条的接触面内的热应力, 包括正应力与剪应力, 问题就不很容易了. 因接触面内的应力集中分布在金属条的端部, 其长度约为金属条的厚度. 因而接触应力的最大值, 可以与金属条内的最大应力同一量级. 在选择两金属条的弹性性质时, 以上这结论是有实际意义的.

二、双金属条整温器的热应力

以 α 与 α_1 各为两金属条A, B(图1)的膨胀系数, 并设 $\alpha_1 > \alpha$.

E 与 E_1 各为两金属条的弹性系数.

h 与 h_1 各为两金属条的厚度.

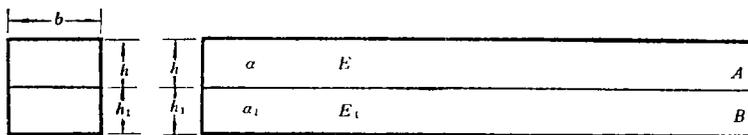


图 1

当温度均匀地自 t_0 增至 t 时, 由于两金属条的膨胀系数不同, 在这双金属条内将产生热应力, 并发生弯曲. 因在膨胀时两者互相牵制, 在接触面内将产生热应力. 一般地, 这接触应力将包括剪应力和正应力(图2).

以正弦级数及余弦级数各表示剪应力 τ_0 与正应力 σ_0 , 即

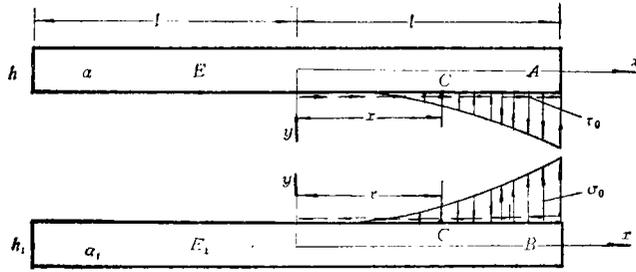


图 2

$$\left. \begin{aligned} \tau_0 &= \sum_{n=1} a_n \sin \frac{n\pi x}{l} \\ \sigma_0 &= \sum_{n=1} b_n \cos \frac{n\pi x}{l} \end{aligned} \right\} \quad (2.1)$$

式中的 a_n 与 b_n 均为未定系数。由于正应力将组成一力偶，故 $b_0=0$ 。

(A) 两金属条的内力分析

以相邻为 dx 的两横截面，自A这金属条截出一微段 dx (图3)。以 S, Q, M 分别为作用在横截面内的轴力，剪力与弯矩。由这微段的平衡，得：

$$\left. \begin{aligned} \frac{dS}{dx} &= -b\tau_0 \\ \frac{dQ}{dx} &= -b\sigma_0 \\ \frac{dM}{dx} &= Q - \frac{bh}{2}\tau_0 \end{aligned} \right\} \quad (2.2)$$

(2.2)式中的 b 为两金属条的宽度。由以上诸式，得到：

$$\left. \begin{aligned} S &= \frac{bl}{\pi} \sum_{n=1} \frac{a_n}{n} \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \\ M &= \frac{bl}{\pi} \sum_{n=1} \left(\frac{l}{\pi} \frac{b_n}{n^2} + \frac{h}{2} \cdot \frac{a_n}{n} \right) \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \end{aligned} \right\} \quad (2.3)$$

由B这部分亦截出一微段 dx (图4)，并由其平衡得：

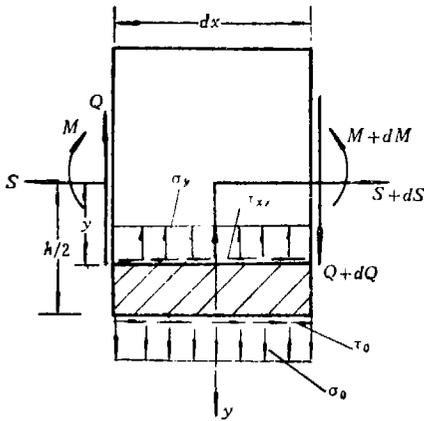


图 3

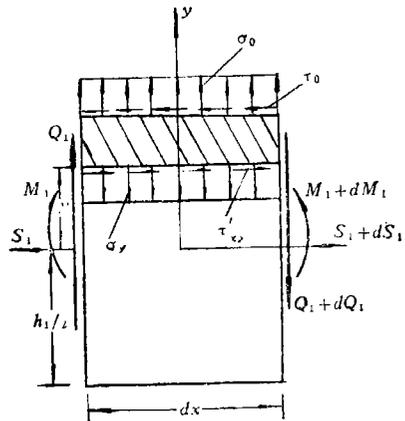


图 4

$$\left. \begin{aligned} \frac{dS_1}{dx} &= b\tau_0 \\ \frac{dQ_1}{dx} &= b\sigma_0 \\ \frac{dM_1}{dx} &= Q_1 - \frac{bh_1}{2}\tau_0 \end{aligned} \right\} \quad (2.4)$$

由以上这三式及(2.1)式, 得:

$$\left. \begin{aligned} S_1 &= -\frac{bl}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \\ M_1 &= \frac{bl}{\pi} \sum_{n=1}^{\infty} \left(-\frac{l}{\pi} \cdot \frac{b_n}{n^2} + \frac{h_1}{2} \frac{a_n}{n} \right) \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \end{aligned} \right\} \quad (2.5)$$

(B) 系数 a_n 与 b_n 的联系

由于A与B这两金属条弯曲后的曲率相等, 得:

$$\frac{M}{EI} = \frac{M_1}{E_1 I_1}$$

将方程(2.3)与(2.5)中的 M 与 M_1 代入上式, 得:

$$\frac{1}{EI} \sum_{n=1}^{\infty} \left(\frac{l}{\pi} \frac{b_n}{n^2} + \frac{h}{2} \frac{a_n}{n} \right) \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) = \frac{1}{E_1 I_1} \sum_{n=1}^{\infty} \left(-\frac{l}{\pi} \frac{b_n}{n^2} + \frac{h_1}{2} \frac{a_n}{n} \right) \left(\cos \frac{n\pi x}{l} - \cos n\pi \right)$$

这两级数的系数应一一相等, 于是得到:

$$\frac{1}{EI} \left(\frac{l}{\pi} \frac{b_n}{n^2} + \frac{h}{2} \frac{a_n}{n} \right) = \frac{1}{E_1 I_1} \left(-\frac{l}{\pi} \frac{b_n}{n^2} + \frac{h_1}{2} \frac{a_n}{n} \right)$$

$$b_n = \frac{\pi}{2l} \cdot \frac{\frac{h_1}{E_1 I_1} - \frac{h}{EI}}{\frac{1}{E_1 I_1} + \frac{1}{EI}} a_n n$$

或

$$b_n = k a_n n \quad (2.6)$$

式中的

$$k = \frac{\pi}{2l} \cdot \frac{\frac{h_1}{E_1 I_1} - \frac{h}{EI}}{\frac{1}{E_1 I_1} + \frac{1}{EI}} \quad (2.7)$$

从(2.7)式可知, 如果 $\frac{h_1}{E_1 I_1} = \frac{h}{EI}$, 则两金属条的接触面内无正应力, 而只有剪应力。由于

$$\frac{d\tau_0}{dx} = \frac{\pi}{l} \sum_{n=1}^{\infty} a_n n \cos \frac{n\pi x}{l}$$

并且由(2.1)与(2.6)式, 得:

$$\sigma_0 = \sum_{n=1} b_n \cos \frac{n\pi x}{l} = k \sum_{n=1} a_n n \cos \frac{n\pi x}{l}$$

于是

$$\sigma_0 = k \frac{l}{\pi} \cdot \frac{d\tau_0}{dx} \quad (2.8)$$

(2.8)式表明: 正应力 σ_0 与 τ_0 曲线相应点的斜率成比例。

(C) A与B这两金属条的应力分析

既已有了轴力 S 与弯矩 M , 就可计算应力。

对于A这部分, 由方程(2.3)得到正应力 σ_x 为:

$$\begin{aligned} \sigma_x = \frac{S}{bh} + \frac{My}{I} = \frac{l}{h\pi} \sum_{n=1} \frac{a_n}{n} \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \\ + \frac{bl}{\pi} \cdot \frac{y}{I} \sum_{n=1} \left(\frac{l}{\pi} \cdot \frac{b_n}{n^2} + \frac{h}{2} \frac{a_n}{n} \right) \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \end{aligned}$$

由方程(2.7), 消去系数 b_n , 得到:

$$\sigma_x = \left\{ \frac{l}{\pi h} + \frac{y}{I} \cdot \frac{bl}{\pi} \left(k \frac{l}{\pi} + \frac{h}{2} \right) \right\} \sum_{n=1} \frac{a_n}{n} \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \quad (2.9)$$

将 σ_x 代入平衡方程:

$$\int_y^{h/2} \frac{\partial \sigma_x}{\partial x} dy + \int_{\tau_{xy}}^{\tau_0} \frac{\partial \tau_{xy}}{\partial y} dy = 0$$

得到剪应力 τ_{xy} :

$$\tau_{xy} = \left\{ \left(\frac{1}{2} + \frac{y}{h} \right) - \frac{b}{2I} \left(\frac{h}{2} + \frac{kl}{\pi} \right) (h^2 - y^2) \right\} \sum_{n=1} a_n \sin \frac{n\pi x}{l} \quad (2.10)$$

将 τ_{xy} 代入平衡方程:

$$\int_{\sigma_y}^{\sigma_0} \frac{\partial \sigma_y}{\partial y} dy + \int_y^{h/2} \frac{\partial \tau_{xy}}{\partial x} dy = 0$$

得正应力 σ_y :

$$\sigma_y = \left\{ k + \frac{\pi}{l} \left[\left(\frac{3}{8}h - \frac{y}{2} - \frac{y^2}{2h} \right) - \frac{b}{2I} \left(\frac{h}{2} + \frac{kl}{\pi} \right) \left(\frac{h^3}{12} - \frac{h^2y}{4} + \frac{y^3}{3} \right) \right] \right\} \sum_{n=1} a_n n \cos \frac{n\pi x}{l} \quad (2.11)$$

对于B这部分, 由方程(2.5), (2.6)得正应力 σ'_x 为:

$$\sigma'_x = \frac{S_1}{bh_1} - \frac{M_1 y}{I_1} = - \left\{ \frac{l}{h_1 \pi} + \frac{y}{I_1} \frac{bl}{\pi} \left(\frac{h_1}{2} - \frac{kl}{\pi} \right) \right\} \sum_{n=1} \frac{a_n}{n} \left(\cos \frac{n\pi x}{l} - \cos n\pi \right) \quad (2.12)$$

将 σ'_x 代入平衡方程:

$$\int_y^{h_1/2} \frac{\partial \sigma'_x}{\partial x} dy - \int_{\tau'_{xy}}^{\tau_0} \frac{\partial \tau'_{xy}}{\partial y} dy = 0$$

得剪应力 τ'_{xy} :

$$\tau'_{xy} = \left\{ \left(\frac{1}{2} + \frac{y}{h_1} \right) + \frac{b}{2I_1} \left(\frac{kl}{\pi} - \frac{h_1}{2} \right) \left(\frac{h_1^2}{4} - y^2 \right) \right\} \sum_{n=1} a_n \sin \frac{n\pi x}{l} \quad (2.13)$$

将 τ'_{xy} 代入平衡方程:

$$\int_{\sigma'_y}^{\sigma_0} \frac{\partial \sigma'_y}{\partial y} dy - \int_y^{h_1/2} \frac{\partial \tau'_{xy}}{\partial x} dy = 0$$

得正应力 σ'_y :

$$\sigma'_y = \left\{ k - \frac{\pi}{l} \left[\frac{3h_1}{8} - \frac{1}{2} \left(y + \frac{y^2}{h_1} \right) + \frac{b}{2I_1} \left(\frac{kl}{\pi} - \frac{h_1}{2} \right) \left(\frac{h_1^3}{12} - \frac{h_1^2 y}{4} + \frac{y^3}{3} \right) \right] \right\} \sum_{n=1} a_n n \cos \frac{n\pi x}{l} \quad (2.14)$$

(D) 以最小应变能原理求解

A与B这两金属条的应变能各为 U_1 与 U_2 。

$$\left. \begin{aligned} U_1 &= \int_0^l \int_{-h/2}^{h/2} \left\{ \frac{1}{2E} (\sigma_x^2 + \sigma_y^2) - \frac{\mu}{E} \sigma_x \sigma_y + \frac{1}{2G} \tau_{xy}^2 \right\} b dx dy \\ U_2 &= \int_0^l \int_{-h_1/2}^{h_1/2} \left\{ \frac{1}{2E_1} (\sigma_x'^2 + \sigma_y'^2) - \frac{\mu_1}{E_1} \sigma_x' \sigma_y' + \frac{\tau_{xy}'^2}{2G_1} \right\} b dx dy \end{aligned} \right\} \quad (2.15)$$

以 $f(x)$ 表示对于A这金属条在分布剪应力 τ_0 与正应力 σ_0 作用下(图2), 离原点为 x 的C点的水平位移。并由于温度上升 $(t-t_0)$, C点的总水平位移为:

$$f(x) + \alpha(t-t_0)x \quad (2.16)$$

同样的, 对于膨胀系数为 α_1 的金属条B, C点的水平位移为:

$$-f_1(x) + \alpha_1(t-t_0)x \quad (2.17)$$

其中 $-f_1(x)$ 为在 τ_0 与 σ_0 作用下C点的位移。由于 τ_0 为负, 故 $f_1(x)$ 前有一负号。由于上下两点C的相对位移不应存在, 故

$$f(x) + f_1(x) - (\alpha_1 - \alpha)(t-t_0)x = 0$$

将上式展成正弦级数, 其系数应均等于零, 即

$$\int_0^l \{ f(x) + f_1(x) - (\alpha_1 - \alpha)(t-t_0)x \} \sin \frac{n\pi x}{l} dx = 0 \quad (2.18)$$

由卡氏定理(Castigliano), 得:

$$\left. \begin{aligned} \frac{1}{b} \frac{\partial U_1}{\partial a_n} &= \int_0^l f(x) \sin \frac{n\pi x}{l} dx \\ \frac{1}{b} \frac{\partial U_2}{\partial a_n} &= \int_0^l f_1(x) \sin \frac{n\pi x}{l} dx \end{aligned} \right\} \quad (2.19)$$

并且

$$\int_0^l x \sin \frac{n\pi x}{l} dx = -\frac{l^2}{n\pi} \cos n\pi \quad (2.20)$$

将方程(2.19)与(2.20)代入(2.18)式, 得到:

$$\frac{\partial U_1}{\partial a_n} + \frac{\partial U_2}{\partial a_n} + (\alpha_1 - \alpha)(t-t_0) \frac{bl^2}{n\pi} \cos n\pi = 0 \quad (2.21)$$

或

$$\frac{\partial U}{\partial a_n} + (\alpha_1 - \alpha)(t-t_0) \frac{bl^2}{n\pi} \cos n\pi = 0 \quad (2.22)$$

式中的 $U = U_1 + U_2$ 为两金属条的总应变能。将方程(2.15)代入(2.21)式, 得计算系数 a_n 的

方程为:

$$\int_0^l \int_{-h/2}^{h/2} \left\{ \frac{1}{E} \left[\sigma_x \frac{\partial \sigma_x}{\partial a_n} + \sigma_y \frac{\partial \sigma_y}{\partial a_n} - \mu \left(\sigma_x \frac{\partial \sigma_y}{\partial a_n} + \sigma_y \frac{\partial \sigma_x}{\partial a_n} \right) \right] + \frac{\tau_{xy}}{G} \frac{\partial \tau_{xy}}{\partial a_n} \right\} b dx dy$$

$$+ \int_0^l \int_{-h_1/2}^{h_1/2} \left\{ \frac{1}{E_1} \left[\sigma'_x \frac{\partial \sigma'_x}{\partial a_n} + \sigma'_y \frac{\partial \sigma'_y}{\partial a_n} - \mu_1 \left(\sigma'_x \frac{\partial \sigma'_y}{\partial a_n} + \sigma'_y \frac{\partial \sigma'_x}{\partial a_n} \right) \right] + \frac{\tau'_{xy}}{G_1} \frac{\partial \tau'_{xy}}{\partial a_n} \right\} b dx dy$$

$$+ (\alpha_1 - \alpha) (t - t_0) \frac{bl^2}{n\pi} \cos n\pi = 0 \quad (2.23)$$

将应力分量 $\sigma_x, \sigma_y, \tau_{xy}, \sigma'_x, \sigma'_y, \tau'_{xy}$ 代入 (2.23) 式, 得:

$$\frac{2}{\pi^2} \cdot \frac{a_n}{n^2} \left\{ \frac{1}{E} \left[1 + 3 \frac{kl}{\pi h} \left(1 + \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h}{h_1} \left[1 + 3 \frac{kl}{\pi h_1} \left(\frac{kl}{\pi h_1} - 1 \right) \right] \right\}$$

$$+ \frac{h^4}{l^4} \cdot \frac{\pi^2}{70} a_n n^2 \left\{ \frac{1}{E} \left[\frac{1}{3} + \frac{kl}{\pi h} \left(\frac{11}{3} + 13 \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h_1^3}{h^3} \left[\frac{1}{3} + \frac{kl}{\pi h_1} \left(13 \frac{kl}{\pi h_1} - \frac{11}{3} \right) \right] \right\}$$

$$+ \frac{h^2}{l^2} a_n \left\{ \frac{1}{10G} \left[\frac{2}{3} + \frac{kl}{\pi h} \left(1 + 6 \frac{kl}{\pi h} \right) \right] - \frac{\mu}{E} \cdot \frac{1}{5} \left[\frac{2}{3} + \frac{6kl}{\pi h} \left(1 + \frac{kl}{\pi h} \right) \right] + \frac{1}{10G_1} \cdot \frac{h_1}{h} \left[\frac{2}{3} \right. \right.$$

$$\left. + \frac{kl}{\pi h_1} \left(6 \frac{kl}{\pi h_1} - 1 \right) \right] + \frac{\mu_1}{E_1} \cdot \frac{1}{5} \frac{h_1}{h} \left[-\frac{2}{3} + \frac{6kl}{\pi h_1} \left(1 - \frac{kl}{\pi h_1} \right) \right] \right\} + \frac{4}{\pi^2} \frac{\cos n\pi}{n} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos n\pi$$

$$\cdot \left\{ \frac{1}{E} \left[1 + \frac{3kl}{\pi h} \left(1 + \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h}{h_1} \left[1 + \frac{3kl}{\pi h_1} \left(\frac{kl}{\pi h_1} - 1 \right) \right] \right\} + (\alpha_1 - \alpha) (t - t_0) \frac{h}{l} \frac{\cos n\pi}{n\pi} = 0 \quad (2.24)$$

方程 (2.24) 可用以解系数 a_n , 并可将这方程简写为:

$$\frac{a_n}{n^2} [n^4 + 2\eta n^2 + p^2] = -\frac{C}{n} \cos n\pi \quad (2.25)$$

式中的

$$p^2 = \frac{\frac{2}{\pi^2} \left\{ \frac{1}{E} \left[1 + 3 \frac{kl}{\pi h} \left(1 + \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h}{h_1} \left[1 + 3 \frac{kl}{\pi h_1} \left(\frac{kl}{\pi h_1} - 1 \right) \right] \right\}}{\frac{h^4}{l^4} \cdot \frac{\pi^2}{70} \left\{ \frac{1}{E} \left[\frac{1}{3} + \frac{kl}{\pi h} \left(\frac{11}{3} + 13 \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h_1^3}{h^3} \left[\frac{1}{3} + \frac{kl}{\pi h_1} \left(13 \frac{kl}{\pi h_1} - \frac{11}{3} \right) \right] \right\}}$$

$$2\eta = \frac{\frac{1}{10G} \left[\frac{2}{3} + \frac{kl}{\pi h} \left(1 + 6 \frac{kl}{\pi h} \right) \right] - \frac{\mu}{E} \cdot \frac{1}{5} \left[\frac{2}{3} + \frac{6kl}{\pi h} \left(1 + \frac{kl}{\pi h} \right) \right] + \frac{1}{10G_1} \cdot \frac{h_1}{h} \left[\frac{2}{3} \right.}{\frac{h^2}{l^2} \cdot \frac{\pi^2}{70} \left\{ \frac{1}{E} \left[\frac{1}{3} + \frac{kl}{\pi h} \left(\frac{11}{3} + 13 \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h_1^3}{h^3} \left[\frac{1}{3} + \frac{kl}{\pi h_1} \left(13 \frac{kl}{\pi h_1} - \frac{11}{3} \right) \right] \right\}}$$

$$\left. + \frac{kl}{\pi h_1} \left(6 \frac{kl}{\pi h_1} - 1 \right) \right] + \frac{\mu_1}{E_1} \cdot \frac{h_1}{5h} \left[-\frac{2}{3} + \frac{6kl}{\pi h_1} \left(1 - \frac{kl}{\pi h_1} \right) \right]}{\left. \right\} \quad (2.26)$$

$$C = \frac{(\alpha_1 - \alpha) (t - t_0) \frac{h}{l} \cdot \frac{1}{\pi} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos n\pi \left\{ \frac{1}{E} \left[1 + \frac{3kl}{\pi h} \left(1 + \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h}{h_1} \left[1 + \frac{3kl}{\pi h_1} \left(\frac{kl}{\pi h_1} - 1 \right) \right] \right\}}{\frac{h^4}{l^4} \cdot \frac{\pi^2}{70} \left\{ \frac{1}{E} \left[\frac{1}{3} + \frac{kl}{\pi h} \left(\frac{11}{3} + 13 \frac{kl}{\pi h} \right) \right] + \frac{1}{E_1} \cdot \frac{h_1^3}{h^3} \left[\frac{1}{3} + \frac{kl}{\pi h_1} \left(13 \frac{kl}{\pi h_1} - \frac{11}{3} \right) \right] \right\}}$$

在解方程 (2.24) 时, 设

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \cos n\pi = S \quad (2.27)$$

由方程(2.25), 得:

$$a_n = \frac{-Cn \cos n\pi}{n^4 + 2\eta n^2 + p^2} \quad (2.28)$$

代入方程(2.27), 得:

$$-C \sum_{n=1}^{\infty} \frac{1}{n^4 + 2\eta n^2 + p^2} = S \quad (2.29)$$

由于这级数的和为已知, 故可以用C来表达S的值. 将它代入方程(2.26)的第三式, 即可从而得到C的值.

按附录中级数的极限和计算, 得:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 2\eta n^2 + p^2} = \frac{\pi}{4p\sqrt{p^2 - \eta^2}} \cdot \frac{\gamma \sinh 2\pi\beta + \beta \sin 2\pi\gamma}{\sinh^2 \beta\pi + \sin^2 \gamma\pi} - \frac{1}{2p^2} \quad (2.30)$$

式中的 $\beta = \sqrt{\frac{1}{2}(p + \eta)}$, $\gamma = \sqrt{\frac{1}{2}(p - \eta)}$

当 β 为一较大值时, 上式可简化为:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 2\eta n^2 + p^2} = \frac{\pi}{4p\beta} - \frac{1}{2p^2} \quad (2.31)$$

将方程(2.28)代入(2.1)式, 并用附录中级数之和, 得到两金属条的接触面内的剪应力 τ_0 .

$$\begin{aligned} \tau_0 &= \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = -C \sum_{n=1}^{\infty} \frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^4 + 2\eta n^2 + p^2} \\ &= 4\beta\gamma \frac{C\pi}{(\sinh^2 \beta\pi + \sin^2 \gamma\pi)} \left\{ \sinh \frac{\pi\beta}{l} x \cos \frac{\pi\gamma}{l} x \cosh \pi\beta \sin \pi\gamma \right. \\ &\quad \left. - \cosh \frac{\pi\beta}{l} x \sin \frac{\pi\gamma}{l} x \sinh \pi\beta \cos \pi\gamma \right\} \quad (2.32) \end{aligned}$$

当 $\pi\beta$ 为较大值时, $\cosh \pi\beta = \sinh \pi\beta$, 并且在金属条的端部 $x \rightarrow l$, $\sinh \frac{\pi\beta}{l} x = \cosh \frac{\pi\beta}{l} x$.

算式(2.32)可简化为:

$$\tau_0 = \frac{C\pi}{4\beta\gamma} \cdot \frac{\sinh \frac{\pi\beta}{l} x}{\sinh \pi\beta} \sin \pi\gamma \left(1 - \frac{x}{l} \right) \quad (2.33)$$

剪应力 τ_0 所组成的剪力, 由方程(2.32)得到:

$$\begin{aligned} P &= b \int_0^l \tau_0 dx = 4\beta\gamma \frac{C\pi}{(\sinh^2 \beta\pi + \sin^2 \gamma\pi)} \int_0^l \left\{ \sinh \frac{\pi\beta}{l} x \cos \frac{\pi\gamma}{l} x \cosh \pi\beta \sin \pi\gamma \right. \\ &\quad \left. - \cosh \frac{\pi\beta}{l} x \sin \frac{\pi\gamma}{l} x \sinh \pi\beta \cos \pi\gamma \right\} dx \\ &= \frac{Cbl}{4\beta p} \cdot \frac{1}{\sinh^2 \beta\pi + \sin^2 \gamma\pi} \left\{ \sinh \pi\beta \cosh \pi\beta - \sinh \pi\beta \cos \pi\gamma \right. \\ &\quad \left. + \frac{\beta}{\gamma} (\sin \gamma\pi \cos \gamma\pi - \cosh \beta\pi \sin \gamma\pi) \right\} \quad (2.34) \end{aligned}$$

当 $\beta\pi$ 是个较大的数时, 上式可简化为:

$$P = \frac{Cbl}{4\beta p} \quad (2.35)$$

由方程(2.8)及(2.32), 得两金属条的接触面内的正应力 σ_0 。一般的情况下, $\beta\pi$ 为一较大的数, 于是 $\sinh \pi\beta = \cosh \pi\beta$ 。所得的两金属条的接触面内的正应力 σ_0 为:

$$\sigma_0 = k \frac{l}{\pi} \frac{d\tau_0}{dx} = \frac{kC\pi}{4\beta\gamma \sinh \pi\beta} \left\{ (\beta \sin \pi\gamma - \gamma \cos \pi\gamma) \cosh \frac{\pi\beta}{l} x \cos \frac{\pi\gamma}{l} x - (\beta \cos \pi\gamma + \gamma \sin \pi\gamma) \sinh \frac{\pi\beta}{l} x \sin \frac{\pi\gamma}{l} x \right\} \quad (2.36)$$

在 $x \rightarrow l$ 的端部附近, 由于 $\sinh \frac{\pi\beta}{l} x = \cosh \frac{\pi\beta}{l} x$, 得 σ_0 为:

$$\sigma_0 = \frac{kC\pi}{4\beta\gamma} \cdot \frac{\sinh \frac{\pi\beta}{l} x}{\sinh \pi\beta} \left\{ \beta \sin \pi\gamma \left(1 - \frac{x}{l} \right) - \gamma \cos \pi\gamma \left(1 - \frac{x}{l} \right) \right\} \quad (2.37)$$

现计算正应力 σ_0 所组成的力偶 M 。由算式(2.8), 得到

$$\begin{aligned} M &= - \int_0^l b \sigma_0 x dx = -bk \frac{l}{\pi} \int_0^l x \frac{d\tau_0}{dx} dx \\ &= -b \frac{kl}{\pi} \left\{ x \tau_0 \Big|_0^l - \int_0^l \tau_0 dx \right\} = b \frac{kl}{\pi} \cdot \frac{Cl}{4\beta p} \end{aligned} \quad (2.38)$$

M 式中的积分式的负号, 表示当 σ_0 为张应力时所得的弯矩为负。

三、两个数例

(1) 设整温器的两金属条的 $h=h_1$, 并且 $G=G_1$, $E=E_1$, $\mu=\mu_1$ 。膨胀系数各为 α 与 α_1 , 而 $\alpha_1 > \alpha$ 。温度由 t_0 增至 t 。计算两金属条的接触面内的热应力。

由(2.7)式, $k=0$ 。故在接触面内只有剪应力 τ_0 。自方程(2.24), 得到:

$$\begin{aligned} a_n n^2 + \frac{l^2}{h^2} \cdot \frac{28}{\pi^2} a_n + \frac{l^4}{h^4} \cdot \frac{420}{\pi^4} \cdot \frac{a_n}{n^2} &= -\frac{l^4}{h^4} \cdot \frac{105}{\pi^2} \cdot \frac{\cos n\pi}{n} \\ &\cdot \left[\frac{h}{l\pi} E (\alpha_1 - \alpha) (t - t_0) + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos n\pi \right] \end{aligned}$$

或简写为:

$$\frac{a_n}{n^2} [n^4 + 2\eta n^2 + \rho^2] = -\frac{\cos n\pi}{n} C$$

式中的

$$2\eta = \frac{l^2}{h^2} \cdot \frac{28}{\pi^2}, \quad \rho^2 = \frac{l^4}{h^4} \cdot \frac{420}{\pi^4};$$

$$C = \frac{l^4}{h^4} \cdot \frac{105}{\pi^2} \left[\frac{E}{\pi} \frac{h}{l} (\alpha_1 - \alpha) (t - t_0) + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos n\pi \right]$$

设金属条的 $\frac{l}{h} = 10$, 于是

$$p = \sqrt{\frac{10^4}{\pi^4}} \times 420 = 207.65; \eta = \frac{1}{2} \times 100 \times \frac{28}{\pi^2} = 141.85$$

$$\beta = \sqrt{\frac{1}{2}(207.65 + 141.85)} = 13.219$$

$$\gamma = \sqrt{\frac{1}{2}(207.65 - 141.85)} = 5.7358$$

由方程(2.31), 得:

$$\sum_{n=1}^{\infty} \frac{1}{\pi^4 + 2 \times 141.85\pi^2 + 207.65^2} = \frac{\pi}{4 \times 207.65 \times 13.219} - \frac{1}{2 \times 207.65^2} = 0.00027453$$

将(2.26)式的第三式的C代入(2.29)式, 并用(2.27)式, 得到:

$$-0.00027453 \frac{(\alpha_1 - \alpha) \frac{1}{10\pi} (t - t_0) + \frac{8}{\pi^2} S \frac{1}{E}}{\frac{1}{10^4} \cdot \frac{\pi^2}{70} \cdot \frac{2}{3} \cdot \frac{1}{E}} = S$$

$$S = -0.037678E(\alpha_1 - \alpha)(t - t_0)$$

由方程(2.29)得:

$$-C \times 0.00027453 = -0.037678E(\alpha_1 - \alpha)(t - t_0)$$

$$C = 137.25E(\alpha_1 - \alpha)(t - t_0)$$

由方程(2.32), 并由于 $\sinh \pi\beta = \cosh \pi\beta$, 得剪应力 τ_0 为:

$$\tau_0 = \frac{137.25 \times 3.1416E(\alpha_1 - \alpha)(t - t_0)}{4 \times 13.219 \times 5.7358} \left\{ \frac{\sinh \frac{13.219\pi}{l} x}{\sinh 13.219\pi} \cos \frac{5.7358\pi}{l} x \sin 5.7358\pi \right. \\ \left. - \frac{\cosh \frac{13.219\pi}{l} x}{\sinh 13.219\pi} \sin \frac{5.7358\pi}{l} x \cos 5.7358\pi \right\}$$

在表1中列出了热应力 τ_0 值。

表 1

x	$0.85l(8.5h)$	$0.9l(9h)$	$0.92l$	0.94	0.95	0.96	0.97	$0.98(9.8h)$	0.99	$\frac{l}{10}(10h)$
τ_0	$\frac{0.0012}{E(\alpha_1 - \alpha)(t - t_0)}$	0.0218	0.052	0.104	0.141	0.181	0.212	0.22	0.171	0
τ_0	1.92kg/cm^2	34.4	83.5	167	224	289	348	350	272	0

表内的第三行 τ_0 为当 $E = 2 \times 10^6 \text{kg/cm}^2$, $(\alpha_1 - \alpha) = 4 \times 10^{-6}$, $(t - t_0) = 200^\circ\text{C}$ 时所得的热应力 τ_0 。从以上的计算可以看出, 分布在两金属条的接触面内的剪应力, 系集中在端部长度 $\frac{l}{10} = h$ 内, 即集中分布的长度约为金属条厚度 h 。 τ_0 的分布表示在图5中。

Timoshenko 教授在“Analysis of Bi-Metal Thermostats”一文中, 用普通材料力学的方法, 计算了金属条的离两端较远的横截面内的最大正应力, 所得的最大值为:

$$\sigma_{\max} = \frac{1}{2}E(\alpha_1 - \alpha)(t - t_0)$$

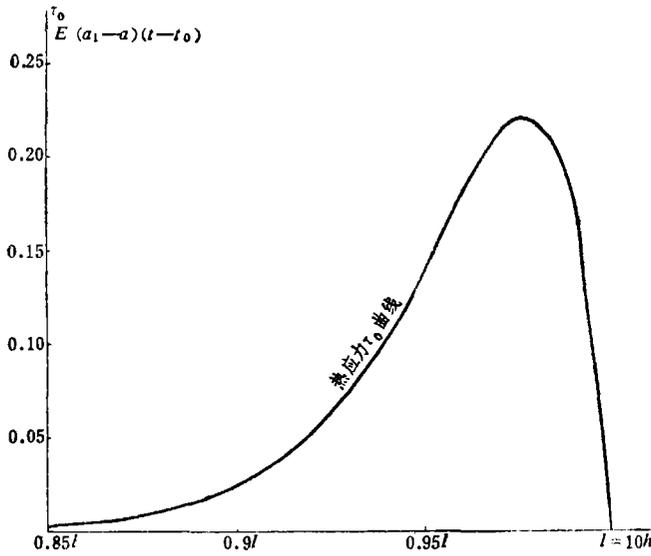


图 5 双金属条整温器的剪应力 τ_0
 $h = h_1, E = E_1, G = G_1, l = 10h$

现按照本文，先计算接触面内的应力，然后进而计算这应力值，以检验以上的分析是否正确。按照方程(2.35)，剪应力 τ_0 的合力 P 为：

$$P = \frac{Cbl}{4\beta p}$$

对于离端点较远的横截面内的弯矩为：

$$M = \frac{h}{2} P = \frac{hCb l}{8\beta p}$$

金属条的离两端较远的横截面内的最大正应力为：

$$\sigma_{\max} = \frac{P}{bh} + \frac{6M}{bh^2} = \frac{Cl}{4\beta ph} + \frac{6Cl}{8\beta ph} = \frac{Cl}{4\beta ph} (1+3) = \frac{Cl}{\beta ph}$$

将以上所得的 $C = 137.25E(\alpha_1 - \alpha)(t - t_0)$ ， $\beta = 13.219$ ， $p = 207.65$ ， $\frac{l}{h} = 10$ 代入上式得：

$$\sigma_{\max} = \frac{137.25E(\alpha_1 - \alpha)(t - t_0) \times 10}{13.219 \times 207.65} = 0.50001(\alpha_1 - \alpha)(t - t_0)E$$

这数值与 Timoshenko 教授的相同。这表明：以上所作的接触热应力的分析是正确的。

(2) 作为不仅有剪应力 τ_0 且有正应力 σ_0 的例，设 $h = h_1$ ， $l = 10h$ ， $E = 2 \times 10^8 \text{ kg/cm}^2$ ， $\mu = 0.3$ ， $G = 8 \times 10^8 \text{ kg/cm}^2$ ， $E_1 = 1.05 \times 10^8 \text{ kg/cm}^2$ ， $\mu_1 = 0.3$ ， $G_1 = 4 \times 10^5 \text{ kg/cm}^2$ ， $\alpha_1 - \alpha = 4 \times 10^{-6}$ ， $t - t_0 = 200^\circ\text{C}$ 。

由方程(2.7)，

$$k = \frac{\pi h}{2l} \frac{1}{E_1} \left(1 - \frac{E_1}{E} \right) = \frac{\pi h}{2l} \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{\pi h}{6l}$$

$$\frac{kl}{\pi h} = \frac{kl}{\pi h_1} = \frac{1}{6}$$

由方程(2.26),

$$p^2 = \frac{\frac{2}{9.8696} \left\{ 1 + \frac{3}{6} \left(1 + \frac{1}{6} \right) + 2 \left[1 + \frac{3}{6} \left(\frac{1}{6} - 1 \right) \right] \right\}}{\left(\frac{1}{10} \right)^4 \frac{9.8696}{70} \left\{ \frac{1}{3} + \frac{1}{6} \left(\frac{11}{3} + \frac{13}{6} \right) + 2 \left[\frac{1}{3} + \frac{1}{6} \left(\frac{13}{6} - \frac{11}{3} \right) \right] \right\}} = 26847$$

$$p = 163.85$$

$$2\eta = \frac{\frac{1.3}{5} \left(\frac{2}{3} + \frac{1}{3} \right) - \frac{0.3}{5} \left(\frac{2}{3} + \frac{7}{6} \right) + 2 \times \frac{2}{3} \times \frac{1.3}{5} + \frac{2}{5} \times 0.3 \left(-\frac{2}{3} + \frac{5}{6} \right)}{\left(\frac{1}{10} \right)^2 \frac{9.8696}{70} \left\{ \frac{1}{3} + \frac{35}{36} + 2 \left[\frac{1}{3} + \frac{1}{6} \left(\frac{13}{6} - \frac{11}{3} \right) \right] \right\}} = 248.9$$

$$\eta = 124.45$$

$$\beta = \sqrt{\frac{1}{2} (163.85 + 124.45)} = \sqrt{144.15} = 12.007$$

$$\gamma = \sqrt{\frac{1}{2} (163.85 - 124.45)} = \sqrt{19.70} = 4.4385$$

由方程(2.32), 得:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 2 \times 124.45n^2 + 163.85^2} = \frac{3.1416}{4 \times 163.85 \times 12.007} - \frac{1}{2 \times 163.85^2} = 0.0003806$$

由方程(2.26)的第三式, 得:

$$C = \frac{E(\alpha_1 - \alpha)(t - t_0) 0.03183 + 0.40528 \left\{ 1 + \frac{1}{2} \left(1 + \frac{1}{6} \right) + 2 \left[1 + \frac{1}{2} \left(-\frac{5}{6} \right) \right] \right\}}{\left(\frac{1}{10} \right)^4 \frac{9.8696}{70} \left\{ \frac{1}{3} + \frac{35}{36} + 2 \left[\frac{1}{3} + \frac{1}{6} \left(\frac{13}{6} - \frac{11}{3} \right) \right] \right\}}$$

$$= \{E(\alpha_1 - \alpha)(t - t_0) 0.03183 + 1.1145S\} 10^4 \times 4.8176$$

将C代入方程(2.29), 并用这级数的和, 得到:

$$-0.0003806 \times 4.8176 \times 10^4 \{E(\alpha_1 - \alpha)(t - t_0) 0.03183 + 1.1145S\} = S$$

$$S = -0.02723E(\alpha_1 - \alpha)(t - t_0)$$

因而

$$C = E(\alpha_1 - \alpha)(t - t_0) 10^4 \times 4.8176 [0.03183 - 1.1145 \times 0.02723]$$

$$= 71.405E(\alpha_1 - \alpha)(t - t_0)$$

于是由方程(2.32)得剪应力 τ_0 为:

$$\tau_0 = \frac{71.405 \times 3.1416}{4 \times 12.007 \times 4.4385} E(\alpha_1 - \alpha)(t - t_0) \left\{ \begin{aligned} & \frac{\sinh \frac{12.007\pi}{l} x}{\sinh 12.007\pi} - \cos \frac{4.4385\pi}{l} x \sin 4.4385\pi \\ & - \frac{\cosh \frac{12.007}{l} x}{\sinh 12.007\pi} \sin \frac{4.4385\pi}{l} x \cos 4.4385\pi \end{aligned} \right\}$$

在 $x \rightarrow l$ 的端部, 得:

$$\tau_0 = 1.0523E(\alpha_1 - \alpha)(t - t_0) \frac{\sinh \frac{12.007\pi}{l} x}{\sinh 12.007\pi} - \sin 4.4385\pi \left(1 - \frac{x}{l} \right)$$

由方程(2.37), 得 $x \rightarrow l$ 的端部的正应力 σ_0 为:

$$\sigma_0 = \frac{\pi}{6 \times 10^{-4}} \cdot \frac{71.405\pi}{4 \times 12.007 \times 4.4385} E (\alpha_1 - \alpha) (t - t_0) \frac{\sinh \frac{12.007\pi}{l} x}{\sinh 12.007\pi} \cdot \left\{ 12.007 \sin 4.4385\pi \left(1 - \frac{x}{l}\right) - 4.4385 \cos 4.4385\pi \left(1 - \frac{x}{l}\right) \right\}$$

在表 2 内列出了 τ_0 与 σ_0 的分布

表 2

x	$\frac{0.8l}{(8h)}$	0.85l	0.9	0.92	0.94	0.95	0.96	0.97	0.98	0.99	$l=10h$
τ_0	$\frac{0.0002}{E(\alpha_1 - \alpha)(t - t_0)}$	0.0032	0.024	0.046	0.0815	0.103	0.125	0.138	0.128	0.100	0
	0.32kg/cm ²	5.12	38.4	73.6	130.4	164.8	200	220.8	205	160	0
σ_0	$\frac{0.00024}{E(\alpha_1 - \alpha)(t - t_0)}$	0.0024	0.014	0.0237	0.034	0.036	0.0315	0.0142	-0.024	-0.104	-0.244
	0.38kg/cm ²	3.8	22.4	38	54.4	57.6	50.4	22.7	-38.4	-166.4	-390.4

从以上的计算可知, 在接触面内的正应力与剪应力, 均集中分布在端部, 其长度约为金属条的厚度 h 。在第三与第五行中的数值, 各为将 $E(\alpha_1 - \alpha) \cdot (t - t_0)$ 的值代入时所得的 τ_0 与 σ_0 。在图 6 中绘出 τ_0 与 σ_0 曲线。

金属条的横截面内的正应力由两部分组成。一是轴力 P 所产生的均布应力。由 (2.35) 式, 对于金属条 A , 得到:

$$\sigma_1 = \frac{P}{bh} = \frac{Cl}{4\beta ph}$$

另一部分为由弯矩所产生的弯曲应力。弯矩 M 由两部分组成。一是由剪应力 τ_0 , 即

$$M_1 = \frac{h}{2} P = \frac{h}{2} \cdot \frac{Clb}{4\beta p}$$

σ_0 所组成的力偶, 由 (2.38) 式得:

$$M_2 = \frac{kl}{\pi} \cdot \frac{Clb}{4\beta p}$$

最大的弯曲应力为:

$$\sigma_2 = \left(\frac{h}{2} + \frac{kl}{\pi} \right) \frac{Clb}{4\beta p} \frac{6}{bh^2}$$

在离端部较远的横截面内, 金属条 A 的最大正应力为:

$$\sigma_{max} = \frac{Cl}{4\beta ph} \left\{ (1+3) + \frac{6kl}{\pi h} \right\}$$

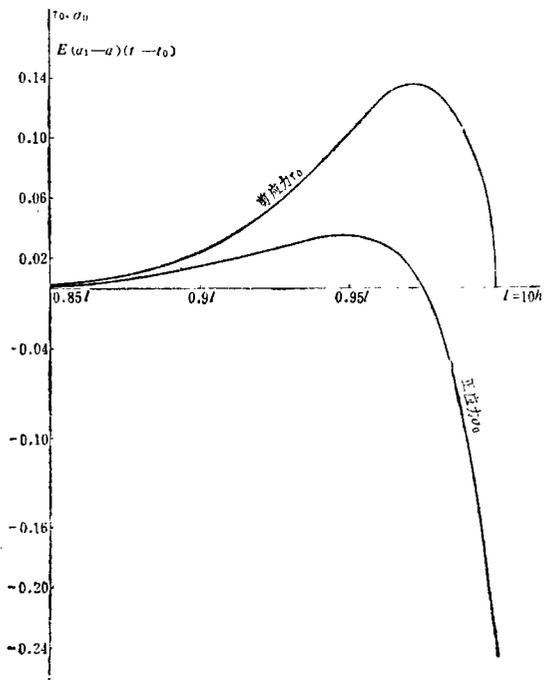


图 6 剪应力 τ_0 与正应力 σ_0 曲线

$h = h_1, l = 10h, E = 2.1 \times 10^6 \text{kg/cm}^2,$
 $G = 8 \times 10^5 \text{kg/cm}^2, \mu = 0.3, E_1 = 1.05 \times 10^6 \text{kg/cm}^2,$
 $G_1 = 4 \times 10^5 \text{kg/cm}^2, \mu = 0.3$

对于这问题 $\frac{kl}{\pi h} = \frac{1}{6}$, 于是得到:

$$\begin{aligned} \sigma_{\max} &= \frac{71.405 \times 10}{4 \times 12.007 \times 163.85} \times 5E(\alpha_1 - \alpha)(t - t_0) = 0.09079 \times 5 \\ &= 0.45369E(\alpha_1 - \alpha)(t - t_0) \end{aligned}$$

附 录

两个级数的和

(1) 用以下这无穷乘积

$$\frac{e^{\pi z} - e^{-\pi z}}{2} = \pi z(1+z^2)\left(1+\frac{z^2}{4}\right)\left(1+\frac{z^2}{9}\right)\dots \quad (1)$$

等式两边取对数并计算导数, 得:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+z^2} = \frac{\pi}{2z} \coth \pi z - \frac{1}{2z^2} \quad (2)$$

当 $\eta < p$ 时, 由(2)式可计算以下这级数之和.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^4+2n^2\eta+p^2} &= \frac{1}{2i\sqrt{p^2-\eta^2}} \sum_{n=1}^{\infty} \left[\frac{1}{n^2+(\eta-i\sqrt{p^2-\eta^2})} - \frac{1}{n^2+(\eta+i\sqrt{p^2-\eta^2})} \right] \\ &= \frac{1}{2i\sqrt{p^2-\eta^2}} \cdot \frac{1}{2} \left\{ \frac{\pi \coth \pi \sqrt{\eta-i\sqrt{p^2-\eta^2}}}{\sqrt{\eta-i\sqrt{p^2-\eta^2}}} - \frac{1}{\eta-i\sqrt{p^2-\eta^2}} \right. \\ &\quad \left. - \frac{\pi \coth \pi \sqrt{\eta+i\sqrt{p^2-\eta^2}}}{\sqrt{\eta+i\sqrt{p^2-\eta^2}}} + \frac{1}{\eta+i\sqrt{p^2-\eta^2}} \right\} \end{aligned}$$

由于

$$\sqrt{\eta \pm i\sqrt{p^2-\eta^2}} = \sqrt{\frac{1}{2}(p+\eta)} \pm i\sqrt{\frac{1}{2}(p-\eta)}$$

代入上式, 经简化以后使等式两边的实部相等, 得:

$$\sum_{n=1}^{\infty} \frac{1}{n^4+2n^2\eta+p^2} = \frac{\pi}{8p\beta\gamma} \cdot \frac{\gamma \sinh 2\pi\beta + \beta \sin 2\pi\gamma}{\sinh^2\beta\pi + \sin^2\gamma\pi} - \frac{1}{2p^2} \quad (3)$$

式中的

$$\beta = \sqrt{\frac{1}{2}(p+\eta)}, \quad \gamma = \sqrt{\frac{1}{2}(p-\eta)}$$

(2) 由回路积分, 得:

$$\sum_{n=1}^{\infty} \frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^2+z^2} = (-1) \frac{\pi}{2} \cdot \frac{\sinh \frac{\pi z}{l} x}{\sinh \pi z} \quad (4)$$

对于 $\eta < p$ 这情形, 由(4)式以下这级数可写作

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^4+2n^2\eta+p^2} &= \frac{1}{2i\sqrt{p^2-\eta^2}} \sum_{n=1}^{\infty} \left[\frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^2+(\eta-i\sqrt{p^2-\eta^2})} - \frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^2+(\eta+i\sqrt{p^2-\eta^2})} \right] \\ &= \frac{\pi}{2i\sqrt{p^2-\eta^2}} \left\{ -\frac{\sinh \frac{\pi x}{l}(\beta-i\gamma)}{\sinh \pi(\beta-i\gamma)} + \frac{\sinh \frac{\pi x}{l}(\beta+i\gamma)}{\sinh \pi(\beta+i\gamma)} \right\} \end{aligned}$$

将上式展开, 并使等式两边的实部相等, 得:

$$\sum_{n=1} \frac{n \cos n\pi \sin \frac{n\pi x}{l}}{n^4 + 2n^2\eta + \rho^2} = \frac{\pi}{2\sqrt{\rho^2 - \eta^2} (\sinh^2 \pi\beta + \sin^2 \pi\gamma)} \cdot \left\{ \cosh \frac{\pi\beta}{l} x \sin \frac{\pi\gamma}{l} x \sinh \pi\beta \cos \pi\gamma - \sinh \frac{\pi\beta}{l} x \cos \frac{\pi\gamma}{l} x \cosh \pi\beta \sin \pi\gamma \right\} \quad (5)$$

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Thermal Contact Stresses of Bi-Metal Strip Thermostat

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Abstract

The distribution of shearing and normal stresses on the contact surface of the two strips composing a thermostat is found in closed form. They are of local type and concentrated at the ends of the strip along a length almost equal to the thickness of the strip.