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小参数椭圆-抛物偏微分方程一致收敛差分格式的充要条件*

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摘　要

本文研究小参数椭圆-抛物偏微分方程一致收敛差分格式的充要条件.

一、引　言

在许多实际问题中，如流体力学（边界层问题），弹性力学（壳体的边界效应问题），量子力学（WKB问题）等，都遇到在方程的最高阶导数项含有小参数 ε 的问题. 用一般的差分方法解高阶导数项含有小参数的奇异摄动问题，往往会产生一些困难，它是由于解的边界层性态而引起的. 传统的有限差分逼近，是在边界层附近减少网格尺寸，使边界层的主要特点不会失去. 然而，这种逼近的缺点是需要相当多的计算. 事实上，往往由于计算机系统的限制，当网格取太小，所得到的离散问题在数值上可能变成不适定. 因此，我们希望构造顾及到边界层存在，以及与小参数 ε 无关的一致收敛的差分格式.

近十几年来，有不少人做了这方面的工作，例如，Il'in[1]（1969），Emelyanov[2]（1979），Doolan, Miller, Schilders[3]（1980）等人的工作，这些工作大都是对小参数的常微分方程问题进行讨论.

本文研究在正方形区域 R（$0\leqslant x\leqslant 1$，$0\leqslant y\leqslant 1$）内解椭圆-抛物偏微分方程第一边值问题：

$$\begin{cases} L_\varepsilon u_\varepsilon \equiv \varepsilon \dfrac{\partial^2 u_\varepsilon}{\partial y^2} + \dfrac{\partial^2 u_\varepsilon}{\partial x^2} - a(x,y)\dfrac{\partial u_\varepsilon}{\partial y} = f(x,y) & (1.1) \\ u_\varepsilon(x,0) = u_\varepsilon(x,1) = u_\varepsilon(0,y) = u_\varepsilon(1,y) = 0 & (1.2) \end{cases}$$

的一致收敛差分格式的充要条件，其中 $\varepsilon>0$ 是小参数，$a(x,\ y)\geqslant\alpha>0$.

当 $\varepsilon=0$ 时，方程（1.1），（1.2）退化为抛物型方程边值问题：

$$\begin{cases} L_0 W \equiv \dfrac{\partial^2 W}{\partial x^2} - a(x,y)\dfrac{\partial W}{\partial y} = f(x,y) & (1.3) \\ W(x,0) = W(0,y) = W(1,y) = 0 & (1.4) \end{cases}$$

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二、一致收敛差分格式的必要条件

对于高阶导数项含有小参数的奇异摄动问题的数值解法，我们希望：

(i) 所构造的差分格式适应边界层性质；

(ii) 当 $\varepsilon\to0$ 时，差分方程的解的渐近性态与微分方程解的渐近性态相同；

(iii) 当 $h\to0$ 时，所构造的差分格式的解关于 ε 一致收敛于微分方程的解.

文章[4]提出了满足(i)，(ii)的差分格式，并证明了所构造的差分格式是具有二阶精度的收敛的差分格式. 为了满足(iii)，我们先来讨论一致收敛差分格式的必要条件.

假设差分格式

$$\begin{cases} L_\varepsilon^h u_{ij}^h=\varepsilon\sigma_j(\tau)\dfrac{u_{i\,(j+1)}^h-2u_{i\,j}^h+u_{i\,(j-1)}^h}{h^2}+\dfrac{u_{(i+1)\,j}^h-2u_{i\,j}^h+u_{(i-1)\,j}^h}{h^2} \\ \qquad -a(x_i,\ y_j)\dfrac{u_{i\,(j+1)}^h-u_{i\,(j-1)}^h}{2h}=f(x_i,y_j) & (2.1) \\ u_{i0}^h=u_{in}^h=u_{0j}^h=u_{nj}^h=0 & (2.2)\end{cases}$$

的解关于 ε 一致收敛于奇异摄动问题 (1.1)，(1.2) 的解，其中 $\sigma_j(\tau)$ 是拟合因子，$\tau=\dfrac{h}{\varepsilon}$.

利用 V-L 渐近展开，我们有

$$u_\varepsilon(x,y)=W_0(x,y)-W_0(x,1)\exp\left(-a(x,1)\frac{1-y}{\varepsilon}\right)+O(\varepsilon) \tag{2.3}$$

由渐近方法分析知，当 $\varepsilon=0$ 时，在 $y=1$ 这条边上失去一个定解条件. 此时，在 $y=1$ 附近将产生边界层现象. 所以，我们在确定拟合因子时，可以视 x 为固定情况来考虑. 于是有

$$u_\varepsilon(x,y_j)=W_0(x,jh)-W_0(x,1)\exp[-a(x,1)(n-j)\tau]+O(\varepsilon)$$

固定 $(n-j)$ 和 τ，我们有

$$\lim_{h\to0}u_\varepsilon(x,jh)=W_0(x,1)-W_0(x,1)\exp[-a(x,1)(n-j)\tau] \tag{2.4}$$

由假设知，当 $h\to0$ 时，u_{ij}^h 一致收敛于 $u_\varepsilon(x,jh)$，那么，$f(x_i,\ y_j)$ 是有界的，所以，(2.1) 乘以 h，利用 (2.4) 得到

$$\lim_{h\to0}\left\{\frac{\sigma_j(\tau)}{\tau}[u_\varepsilon(x,(j+1)h)-2u_\varepsilon(x,jh)+u_\varepsilon(x,(j-1)h)]\right.$$
$$\left.-a(x,1)\frac{u_\varepsilon(x,(j+1)h)-u_\varepsilon(x,(j-1)h)}{2}\right\}=0 \tag{2.5}$$

即

$$\lim_{h\to0}\frac{\sigma_j(\tau)}{\tau}=\frac{a(x,1)[\exp(a(x,1)\tau)-\exp(-a(x,1)\tau)]}{2\left[\exp\left(\frac{a(x,\ 1)\tau}{2}\right)-\exp\left(-\frac{a(x,\ 1)\tau}{2}\right)\right]^2}$$

$$=\frac{a(x,1)}{2}\operatorname{cth}\frac{a(x,1)\tau}{2}$$

于是，我们有

定理1 差分格式(2.1)、(2.2)的解关于ε一致收敛于微分方程(1.1)、(1.2)的解的必要条件是

$$\sigma_j(\tau)=\frac{a(x,1)\tau}{2}\,\mathrm{cth}\left(\frac{a(x,1)\tau}{2}\right)+\tau\cdot o(1) \tag{2.6}$$

如果我们取

$$\sigma_j(\tau)=\frac{a(x,y)\tau}{2}\,\mathrm{cth}\frac{a(x,y)\tau}{2}$$

这时，差分格式(2.1)，(2.2)就是文章[4]所提出的差分格式。

三、差分格式的一致收敛性

为方便起见，在(1.1)中，我们取$a(x,y)=a$为常数，并假设$0<a$，$\|f(x,y)\|_{C^2}\leqslant m$。相应的差分格式为

$$\begin{cases} L_\varepsilon^h u^h\equiv\dfrac{ah}{2}\,\mathrm{cth}\,\dfrac{ah}{2\varepsilon}\,u_{y\bar{y}}^h+u_{x\bar{x}}^h-au_{\mathring{y}}^h=f(x,y) & (3.1)\\ u_{0j}^h=u_{nj}^h=u_{i0}^h=u_{in}^h=0 & (3.2)\end{cases}$$

其中，$u_{x\bar{x}}^h$，$u_{y\bar{y}}^h$是分别关于x，y的二阶中心差商，$u_{\mathring{y}}^h$是关于y的一阶中心差商。

引理1[4] 设$V^{(h)}(x,y)$是给定在$\overline{R}_h$上的网格函数，则

$$|V^{(h)}(x,y)|\leqslant\max_{\Gamma_h}|V^{(h)}(x,y)|+\frac{M}{2}\max_{R_h}|L_\varepsilon^h V^{(h)}(x,y)|$$

其中，M是一个与x，y，ε无关的常数。

我们又注意到，在新参数$\xi=\varepsilon^{-\frac{1}{2}}x$，$\eta=\varepsilon^{-1}y$下，方程(1.1)有形式：

$$\frac{\partial^2\tilde{u}}{\partial\eta^2}+\frac{\partial^2\tilde{u}}{\partial\xi^2}-a\frac{\partial\tilde{u}}{\partial\eta}=\tilde{f}(\xi,\eta) \tag{3.3}$$

以边界Γ为区域的R变为以边界$\tilde{\Gamma}$为区域的$\tilde{R}$，且

$$\tilde{u}(\xi,\eta)=0\qquad 当(\xi,\eta)\in\tilde{\Gamma}时 \tag{3.4}$$

对于导数$\dfrac{\partial^{i+j}u_\varepsilon(x,y)}{\partial x^i\partial y^j}$的估计式由函数$\tilde{u}(\xi,\eta)$，$\tilde{u}(\varepsilon^{-\frac{1}{2}}x,\varepsilon^{-1}y)=u_\varepsilon(x,y)$的估计式推出[5]：

$$\left|\frac{\partial^{i+j}\tilde{u}(\xi,\eta)}{\partial\xi^i\partial\eta^j}\right|\leqslant M_1$$

于是有

引理2 若$u_\varepsilon(x,y)$是(1.1)，(1.2)的解，则有

$$\left|\frac{\partial^{i+j}u_\varepsilon(x,y)}{\partial x^i\partial y^j}\right|\leqslant M_1\varepsilon^{-\frac{i}{2}-j}\qquad (x,y)\in R,\quad i+j\geqslant 0 \tag{3.5}$$

其中M_1是与ε无关的常数。

由上面两个引理，我们有

定理2 假定$u_\varepsilon(x,y)$是(1.1)、(1.2)的解，且$0<a$，$\|f(x,y)\|_{C^2}\leqslant m$，则对于固定的$\varepsilon>0$，在$R_h$上差分方程问题(3.1)、(3.2)的解$u^h(x,y)$与微分方程问题(1.1)、(1.2)的解$u_\varepsilon(x,y)$之间

的差有下面的估计式：

$$|u^h(x,y)-u_\varepsilon(x,y)|\leqslant\frac{Mh^2}{\varepsilon^3} \tag{3.6}$$

其中M为与x,y,ε无关的常数.

证明：令 $e^h(x,y)=u^h(x,y)-u_\varepsilon(x,y)$

$$L_\varepsilon^h e^h(x,y)=L_\varepsilon^h u^h(x,y)-L_\varepsilon^h u_\varepsilon(x,y)=f(x,y)-L_\varepsilon^h u_\varepsilon(x,y)$$
$$=L_\varepsilon u_\varepsilon(x,y)-L_\varepsilon^h u_\varepsilon(x,y)$$
$$=\varepsilon\left(\frac{\partial^2 u_\varepsilon}{\partial y^2}-u_{\varepsilon y\bar y}\right)+\left(\frac{\partial^2 u_\varepsilon}{\partial x^2}-u_{\varepsilon x\bar x}\right)-a\left(\frac{\partial u_\varepsilon}{\partial y}-u_{\varepsilon\dot y}\right)$$
$$+\left(\varepsilon-\frac{ah}{2}\operatorname{cth}\frac{ah}{2\varepsilon}\right)u_{\varepsilon y\bar y}$$

由[3]，
$$\left|\varepsilon-\frac{ah}{2}\operatorname{cth}\frac{ah}{2\varepsilon}\right|\leqslant M_2'\varepsilon^{-1}h^2$$

故有

$$|L_\varepsilon^h e^h(x,y)|\leqslant M_2(\varepsilon q_4+p_4+q_3+\varepsilon^{-1}q_2)h^2 \tag{3.7}$$

其中 $p_i=\max\left|\frac{\partial^i u_\varepsilon}{\partial x^i}\right|$, $q_j=\max\left|\frac{\partial^j u_\varepsilon}{\partial y^j}\right|$

由(3.5)，(3.7)得到

$$|L_\varepsilon^h e^h(x,y)|\leqslant M_1M_2(\varepsilon^{-3}+\varepsilon^{-2}+\varepsilon^{-3}+\varepsilon^{-3})h^2\leqslant M\varepsilon^{-3}h^2$$

又在Γ_h上

$$e^h(x,y)=u^h(x,y)-u_\varepsilon(x,y)=0$$

由引理1得到

$$|u^h(x,y)-u_\varepsilon(x,y)|=|e^h(x,y)|\leqslant M\varepsilon^{-3}h^2$$

其中M是一个与x,y,ε无关的常数.

为了证明差分方程(3.1)，(3.2)的解关于ε一致收敛于微分方程(1.1)，(1.2)的解，我们把微分方程(1.1)，(1.2)的渐近解表为如下形式：

$$U_\varepsilon(x,y)=\sum_{i=0}^{\infty}\varepsilon^i W_i(x,y)+\sum_{i=0}^{\infty}\varepsilon^i V_i\left(x,\frac{1-y}{\varepsilon}\right)$$
$$\overset{\text{记}}{=}W_\varepsilon+V_\varepsilon \tag{3.8}$$

由V-L渐近方法，我们有

第一分解过程：

令 $W_\varepsilon(x,y)=W_0(x,y)+\varepsilon W_1(x,y)+\cdots$

代入 $L_\varepsilon W_\varepsilon(x,y)=f(x,y)$

得到确定$W_i(x,y)$的递推过程：

$$\left.\begin{aligned}&\frac{\partial^2 W_0}{\partial x^2}-a\frac{\partial W_0}{\partial y}=f(x,y)\\&W_0(0,y)=W_0(1,y)=W_0(x,0)=0\end{aligned}\right\} \tag{3.9}$$

$$\left.\begin{aligned}&\frac{\partial^2 W_i}{\partial x^2}-a\frac{\partial W_i}{\partial y}=-\frac{\partial^2 W_{i-1}}{\partial y^2}\qquad(i=1,2,\cdots)\\&W_i(0,y)=W_i(1,y)=W_i(x,0)=0\end{aligned}\right\} \tag{3.10}$$

第二分解过程：

令 $$V_\varepsilon(x,t)=V_0(x,t)+\varepsilon V_1(x,t)+\cdots$$

代入 $$L_\varepsilon V_\varepsilon(x,t)=0$$

其中 $$t=\frac{1-y}{\varepsilon},\quad \frac{\partial^j}{\partial y^j}=\left(-\frac{1}{\varepsilon}\right)^j\frac{\partial^j}{\partial t^j}$$

得到确定V_i的递推过程：

$$\text{(i)}\quad \left.\begin{array}{l}\dfrac{\partial^2 V_0}{\partial t^2}+a\dfrac{\partial V_0}{\partial t}=0\\[2ex] V_0(x,0)=-W_0(x,1),\quad \lim\limits_{t\to\infty}V_0(x,t)=0\end{array}\right\}\tag{3.11}$$

解得 $$V_0(x,t)=-W_0(x,1)\exp(-at)$$

$$\text{(ii)}\quad \left.\begin{array}{l}\dfrac{\partial^2 V_1}{\partial t^2}+a\dfrac{\partial V_1}{\partial t}=-\dfrac{\partial^2 V_0}{\partial x^2}=\dfrac{\partial^2 W_0(x,1)}{\partial x^2}\exp(-at)\\[2ex] V_1(x,0)=-W_1(x,1),\quad \lim\limits_{t\to\infty}V_1(x,t)=0\end{array}\right\}\tag{3.12}$$

解得 $$V_1(x,t)=\left[-W_1(x,1)-\frac{1}{a}\frac{\partial^2 W_0(x,1)}{\partial x^2}t\right]\exp(-at)$$

$$\overset{\text{记}}{=}Q_1(x,t)\exp(-at)$$

$$\text{(iii)}\quad \left.\begin{array}{l}\dfrac{\partial^2 V_2}{\partial t^2}+a\dfrac{\partial V_2}{\partial t}=-\dfrac{\partial^2 V_1}{\partial x^2}=\left[\dfrac{\partial^2 W_1(x,1)}{\partial x^2}+\dfrac{1}{a}\dfrac{\partial^4 W_0(x,1)}{\partial x^4}t\right]\exp(-at)\\[2ex] V_2(x,0)=-W_2(x,1),\quad \lim\limits_{t\to\infty}V_2(x,t)=0\end{array}\right\}\tag{3.13}$$

解得

$$V_2(x,t)=\left\{-W_2(x,1)-\left[\frac{1}{a^3}\frac{\partial^4 W_0(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_1(x,1)}{\partial x^2}\right]t\right.$$
$$\left.-\frac{1}{2a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}t^2\right\}\exp(-at)\overset{\text{记}}{=}Q_2(x,t)\exp(-at)$$

$$\text{(iv)}\quad \left.\begin{array}{l}\dfrac{\partial^2 V_3}{\partial t^2}+a\dfrac{\partial V_3}{\partial t}=-\dfrac{\partial^2 V_2}{\partial x^2}=\left\{\dfrac{\partial^2 W_2(x,1)}{\partial x^2}+\left[\dfrac{1}{a^3}\dfrac{\partial^6 W_0(x,1)}{\partial x^6}\right.\right.\\[2ex] \qquad\left.\left.+\dfrac{1}{a}\dfrac{\partial^4 W_1(x,1)}{\partial x^4}\right]t+\dfrac{1}{2a^2}\dfrac{\partial^6 W_0(x,1)}{\partial x^6}t^2\right\}\exp(-at)\\[2ex] V_3(x,0)=-W_3(x,1),\quad \lim\limits_{t\to\infty}V_3(x,t)=0\end{array}\right\}\tag{3.14}$$

解得

$$V_3(x,t)=\left\{-W_3(x,1)-\left[\frac{2}{a^5}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^3}\frac{\partial^4 W_1(x,1)}{\partial x^4}\right.\right.$$
$$\left.+\frac{1}{a}\frac{\partial^2 W_2(x,1)}{\partial x^2}\right]t-\left[\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{2a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}\right]t^2$$
$$\left.-\frac{1}{6a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t^3\right\}\exp(-at)\overset{\text{记}}{=}Q_3(x,t)\exp(-at)$$

令 $$U_\varepsilon(x,y)=U_{\varepsilon_3}(x,y)+z_{\varepsilon_3}$$

而 $$U_{\varepsilon_3}(x,y)=W_{\varepsilon_3}(x,y)+V_{\varepsilon_3}\left(x,\frac{1-y}{\varepsilon}\right)$$

$$=\sum_{i=0}^{3}\varepsilon^i W_i(x,y)+\sum_{i=0}^{3}\varepsilon^i V_i\left(x,\frac{1-y}{\varepsilon}\right) \tag{3.15}$$

由引理1容易得到

$$|z_{\varepsilon_3}|\leqslant M_3\varepsilon^4 \tag{3.16}$$

令 $$\left.\begin{aligned}&\rho(x,y)=U_{\varepsilon_3}(x,y)-u^h(x,y),\quad \gamma=\frac{ah}{2}\operatorname{cth}\frac{ah}{2\varepsilon}\\&L_\varepsilon^h\rho(x,y)=L_\varepsilon^h U_{\varepsilon_3}(x,y)-L_\varepsilon^h u^h(x,y)\overset{\text{记}}{=}F_1+F_2\end{aligned}\right\} \tag{3.17}$$

其中

$$\begin{aligned}F_1=&\gamma(W_{0y\bar y}+\varepsilon W_{1y\bar y}+\varepsilon^2W_{2y\bar y}+\varepsilon^3W_{3y\bar y})+(W_{0x\bar x}+\varepsilon W_{1x\bar x}+\varepsilon^2W_{2x\bar x}\\&+\varepsilon^3W_{3x\bar x})-a(W_{0\hat y}+\varepsilon W_{1\hat y}+\varepsilon^2W_{2\hat y}+\varepsilon^3W_{3\hat y})-f\\F_2=&\gamma(V_{0y\bar y}+\varepsilon V_{1y\bar y}+\varepsilon^2V_{2y\bar y}+\varepsilon^3V_{3y\bar y})+(V_{0x\bar x}+\varepsilon V_{1x\bar x}+\varepsilon^2V_{2x\bar x}\\&+\varepsilon^3V_{3x\bar x})-a(V_{0\hat y}+\varepsilon V_{1\hat y}+\varepsilon^2V_{2\hat y}+\varepsilon^3V_{3\hat y})\end{aligned}$$

因为 $W_i(x,y)$ $(i=0,1,2,3)$ 满足(3.9)，(3.10)且应用不等式[2]：

$$|\gamma-\varepsilon|\leqslant Mh\operatorname{th}\frac{ah}{2\varepsilon} \tag{3.18}$$

所以，我们有

$$|F_1|\leqslant M(h+\varepsilon^4) \tag{3.19}$$

事实上，

$$\begin{aligned}F_1=&(W_{0x\bar x}-aW_{0\hat y}-f)+\varepsilon(W_{0y\bar y}+W_{1x\bar x}-aW_{1\hat y})\\&+\varepsilon^2(W_{1y\bar y}+W_{2x\bar x}-aW_{2\hat y})+\varepsilon^3(W_{2y\bar y}+W_{3x\bar x}-aW_{3\hat y})\\&+(\gamma-\varepsilon)(W_{0y\bar y}+\varepsilon W_{1y\bar y}+\varepsilon^2W_{2y\bar y}+\varepsilon^3W_{3y\bar y})+\varepsilon^4W_{3y\bar y}\end{aligned}$$

故有

$$|F_1|\leqslant M_3\left(h^2+\varepsilon h^2+\varepsilon^2h^2+\varepsilon^3h^2+h\operatorname{th}\frac{ah}{2\varepsilon}+\varepsilon^4\right)\leqslant M(h+\varepsilon^4)$$

现估计 F_2，考虑到(3.11)～(3.14)中，$V_i\left(x,\frac{1-y}{\varepsilon}\right)=V_i(x,t)$ 的显式表达式，我们有：

$$\gamma V_{0y\bar y}=-\frac{2a}{h}\operatorname{ch}\frac{a\tau}{2}\operatorname{sh}\frac{a\tau}{2}W_0(x,1)\exp(-at) \tag{3.20}$$

$$-aV_{0\hat y}=\frac{2aW_0(x,1)}{h}\operatorname{sh}\frac{a\tau}{2}\operatorname{ch}\frac{a\tau}{2}\exp(-at) \tag{3.21}$$

$$V_{0x\bar x}=\left[-\frac{\partial^2W_0(x,1)}{\partial x^2}-\frac{h^2}{12}\frac{\partial^4W_0(\xi_1,1)}{\partial x^4}\right]\exp(-at) \tag{3.22}$$

$$\begin{aligned}\varepsilon\gamma V_{1y\bar y}=&\left\{\frac{2\varepsilon a}{h}\operatorname{ch}\frac{a\tau}{2}\operatorname{sh}\frac{a\tau}{2}\left[-W_1(x,1)-\frac{1}{a}\frac{\partial^2W_0(x,1)}{\partial x^2}t\right]\right.\\&\left.+\frac{2\varepsilon\tau}{h}\frac{\partial^2W_0(x,1)}{\partial x^2}\operatorname{ch}^2\frac{a\tau}{2}\right\}\exp(-at)\end{aligned} \tag{3.23}$$

$$-a\varepsilon V_{1\hat{y}}=-\frac{\varepsilon a}{h}\left\{2\left[-\frac{1}{a}\frac{\partial^2 W_0(x,1)}{\partial x^2}t-W_1(x,1)\right]\mathrm{ch}\frac{a\tau}{2}\mathrm{sh}\frac{a\tau}{2}+\frac{\tau}{a}\frac{\partial^2 W_0(x,1)}{\partial x^2}\mathrm{ch}(a\tau)\right\}\exp(-at) \tag{3.24}$$

$$\varepsilon V_{1x\bar{x}}=\left\{\varepsilon\left[-\frac{\partial^2 W_1(x,1)}{\partial x^2}-\frac{1}{a}\frac{\partial^4 W_0(x,1)}{\partial x^4}t\right]+\frac{\varepsilon h^2}{12}\left[-\frac{\partial^4 W_1(\xi_2,1)}{\partial x^4}-\frac{1}{a}\frac{\partial^6 W_0(\xi_2,1)}{\partial x^6}t\right]\right\}\exp(-at) \tag{3.25}$$

$$\begin{aligned}\varepsilon^2\gamma V_{2y\bar{y}}=\frac{\varepsilon^2 a}{2h}\Big\{&4\left[-\frac{1}{2a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}t^2-\left(\frac{1}{a^3}\frac{\partial^4 W_0(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_1(x,1)}{\partial x^2}\right)t\right.\\&\left.-W_2(x,1)\right]\mathrm{sh}\frac{a\tau}{2}\mathrm{ch}\frac{a\tau}{2}-4\tau\left[-\frac{1}{a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}t-\left(\frac{1}{a^3}\frac{\partial^4 W_0(x,1)}{\partial x^4}\right.\right.\\&\left.\left.+\frac{1}{a}\frac{\partial^2 W_1(x,1)}{\partial x^2}\right)\right]\mathrm{ch}^2\frac{a\tau}{2}+\tau^2\left(-\frac{1}{a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}\right)\mathrm{ch}(a\tau)\\&\cdot\mathrm{cth}\frac{a\tau}{2}\Big\}\exp(-at)\end{aligned} \tag{3.26}$$

$$\begin{aligned}-a\varepsilon^2 V_{2\hat{y}}=-\frac{a\varepsilon^2}{2h}\Big\{&4\left[-\frac{1}{2a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}t^2-\left(\frac{1}{a^3}\frac{\partial^4 W_0(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_1(x,1)}{\partial x^2}\right)t\right.\\&\left.-W_2(x,1)\right]\mathrm{ch}\frac{a\tau}{2}\mathrm{sh}\frac{a\tau}{2}-2\tau\left[-\frac{1}{a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}t-\left(\frac{1}{a^3}\frac{\partial^4 W_0(x,1)}{\partial x^4}\right.\right.\\&\left.\left.+\frac{1}{a}\frac{\partial^2 W_1(x,1)}{\partial x^2}\right)\right]\mathrm{ch}(a\tau)+2\tau^2\left(-\frac{1}{a^2}\frac{\partial^4 W_0(x,1)}{\partial x^4}\right)\mathrm{sh}\frac{a\tau}{2}\mathrm{ch}\frac{a\tau}{2}\Big\}\exp(-at)\end{aligned} \tag{3.27}$$

$$\begin{aligned}\varepsilon^2 V_{2x\bar{x}}=\varepsilon^2\Big\{&-\frac{1}{2a^2}\frac{\partial^6 W_0(x,1)}{\partial x^6}t^2-\left[\frac{1}{a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a}\frac{\partial^4 W_1(x,1)}{\partial x^4}\right]t\\&-\frac{\partial^2 W_2(x,1)}{\partial x^2}+\frac{h^2}{12}\left[-\frac{1}{2a}\frac{\partial^8 W_0(\xi_3,1)}{\partial x^8}t^2-\left(\frac{1}{a^3}\frac{\partial^8 W_0(\xi_3,1)}{\partial x^8}\right.\right.\\&\left.\left.+\frac{1}{a}\frac{\partial^6 W_1(\xi_3,1)}{\partial x^6}\right)t-\frac{\partial^4 W_2(\xi_3,1)}{\partial x^4}\right]\Big\}\exp(-at)\end{aligned} \tag{3.28}$$

$$\begin{aligned}\varepsilon^3\gamma V_{3y\bar{y}}=\frac{\varepsilon^3 a}{2h}\Big\{&4\left[-W_3(x,1)-\left(\frac{2}{a^5}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^3}\frac{\partial^4 W_1(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_2(x,1)}{\partial x^2}\right)t\right.\\&\left.-\left(\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{2a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}\right)t^2-\frac{1}{6a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t^3\right]\mathrm{sh}\frac{a\tau}{2}\mathrm{ch}\frac{a\tau}{2}\\&+4\tau\left[\left(\frac{2}{a^5}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^3}\frac{\partial^4 W_1(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_2(x,1)}{\partial x^2}\right)\right.\\&\left.+\left(\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}+\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}\right)t+\frac{1}{2a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t^2\right]\end{aligned}$$

$$\cdot \mathrm{ch}^2\frac{a\tau}{2}-\tau^2\Big[2\Big(\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{2a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}\Big)+\frac{1}{a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t\Big]$$

$$\cdot \mathrm{ch}(a\tau)\,\mathrm{cth}\frac{a\tau}{2}+\frac{2\tau^3}{3a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}\mathrm{ch}^2\frac{a\tau}{2}\Big\}\exp(-at) \tag{3.29}$$

$$-a\varepsilon^3 V_{3\hat{y}}=-\frac{a\varepsilon^3}{2h}\Big\{4\Big[-W_3(x,1)-\Big(\frac{2}{a^5}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^3}\frac{\partial^4 W_1(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_2(x,1)}{\partial x^2}\Big)t$$

$$-\Big(\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{2a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}\Big)t^2-\frac{1}{6a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t^3\Big]\mathrm{ch}\frac{a\tau}{2}\mathrm{sh}\frac{a\tau}{2}$$

$$+2\tau\Big[\Big(\frac{2}{a^5}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^3}\frac{\partial^4 W_1(x,1)}{\partial x^4}+\frac{1}{a}\frac{\partial^2 W_2(x,1)}{\partial x^2}\Big)$$

$$+\Big(\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}+\frac{1}{a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}+\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}\Big)t$$

$$+\frac{1}{2a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t^2\Big]\mathrm{ch}(a\tau)-2\tau^2\Big[2\Big(\frac{1}{a^4}\frac{\partial^6 W_0(x,1)}{\partial x^6}$$

$$+\frac{1}{2a^2}\frac{\partial^4 W_1(x,1)}{\partial x^4}\Big)+\frac{1}{a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}t\Big]\mathrm{sh}\frac{a\tau}{2}\mathrm{ch}\frac{a\tau}{2}$$

$$+\frac{\tau^3}{3a^3}\frac{\partial^6 W_0(x,1)}{\partial x^6}\mathrm{ch}(a\tau)\Big\}\exp(-at) \tag{3.30}$$

其中 $|\xi_i-x|\leqslant h \quad (i=1,2,3)$

注意到:

(i) $1-2\mathrm{ch}^2\frac{a\tau}{2}+\mathrm{ch}(a\tau)=0$

(ii) $\varepsilon\Big\{2\mathrm{ch}^2\frac{a\tau}{2}-\frac{a\tau}{2}\mathrm{ch}(a\tau)\mathrm{cth}\frac{a\tau}{2}-\mathrm{ch}(a\tau)+a\tau\,\mathrm{ch}\frac{a\tau}{2}\mathrm{sh}\frac{a\tau}{2}\Big\}=\varepsilon-\gamma$

(iii) $\mathrm{ch}^2\Big(\frac{a\tau}{2}\Big)-\frac{\mathrm{ch}(a\tau)}{2}=\frac{1}{2}$

(iv) $\Big|\frac{\partial^i W_j(x,1)}{\partial x^i}\Big|\leqslant M \quad (i=0,1,\cdots,8;\ j=0,1,2,3)$

(v) $|b_0+b_1t+\cdots+b_nt^n|\exp(-at/2)=|p_n(t)|\exp(-at/2)\leqslant M_p.$

于是，由上式及(3.20)～(3.30)得到

$$|F_2|\leqslant M_4\Big(h^2+\varepsilon h^2+\varepsilon^2h^2+h\,\mathrm{th}\frac{a\tau}{2}+\varepsilon h\,\mathrm{th}\frac{a\tau}{2}+\varepsilon^3\Big)\exp\Big(-\frac{at}{2}\Big)$$
$$\leqslant M_6(h+\varepsilon^3)$$

故有

$$|F_1+F_2|\leqslant|F_1|+|F_2|\leqslant M_6(h+\varepsilon^3)$$

所以,

$$L_\varepsilon^h|U_{\varepsilon_3}-u^h|=|L_\varepsilon^h\rho|\leqslant M_6(h+\varepsilon^3)$$

由引理 1 得到

$$|U_{\varepsilon_3}(x,y)-u^h(x,y)|=|\rho(x,y)|\leqslant M_6(h+\varepsilon^3) \tag{3.31}$$

又

$$|u_\varepsilon(x,y)-U_{\varepsilon_3}(x,y)|\leqslant M_3\varepsilon^3$$

所以有

$$\begin{aligned}|u_\varepsilon(x,y)-u^h(x,y)|&\leqslant|u_\varepsilon(x,y)-U_{\varepsilon_3}(x,y)|+|U_{\varepsilon_3}(x,y)-u^h(x,y)|\\&\leqslant M_3\varepsilon^3+M_6(\varepsilon^3+h)\leqslant M(\varepsilon^3+h)\end{aligned} \tag{3.32}$$

由(3.6)和(3.32)，我们得到：

定理 3 当 $h\to 0$ 时，差分格式(3.1)、(3.2) 的解 $u^h(x,y)$ 关于小参数 ε 是一致地收敛于微分问题(1.1)、(1.2)的解 $u_\varepsilon(x, y)$，且满足估计式：

$$|u_\varepsilon(x,y)-u^h(x,y)|\leqslant Mh$$

其中 M 是与 x,y,ε 无关的常数。

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The Necessary and Sufficient Condition of Uniformly Convergent Difference Schemes for the Elliptic——Parabolic Partial Differential Equation with a Small Parameter

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Abstract

This paper studies the necessary and sufficient condition of uniformly convergent difference scheme for the elliptic-parabolic partial differential equation with a small parameter.