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环形和圆形薄板在各种支承条件下的非对称弯曲问题(Ⅱ)

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摘 要

本文研究当作用于板周的张力为零时，弹性柔韧板在各种支承条件下的非对称弯曲问题．

一、引 言

正如钱伟长[1]在1948年所指出的，如果我们求得了弯曲问题的薄膜解，则可以应用摄动方法逐步地求出解的准确到任意阶小量的渐近近似式——半解析解．作者在文[3]中曾应用文[2]所提出的摄动方法，研究了弹性柔韧板在各种支承条件下的非对称弯曲问题，作出解的渐近近似式．但是在文[3]中，我们没有考虑作用于板周的张力取零值的情形，因为在这些取零值的点上，解的渐近近似式将出现奇性．本文应用类似研究转变（transition）点问题的方法[4]，研究了作用于板周的张力为零的情形，特别地对于周边固支的薄板，本文得到了比Срубщик[6]得到的更精确的结果．

作为一个典型的例子，我们仍考察文[3]中所考虑的环形板和圆形板的非对称弯曲问题．对于具有其它光滑周界的弹性柔韧板，或是以其它方式支承的柔韧板的弯曲问题，本方法仍然适用．

二、摄 动 方 法

采用文[3]中的记号，以r表示板中面上的点到环形（或圆形）板中心的距离，θ表示该点的极角，h表示板的厚度，E表示弹性模数，ν是泊松比．我们知道板的挠度$w(r,\theta)$和应力函数$F(r,\theta)$确定于下面著名的von Kármán方程：

$$\left.\begin{aligned}\Delta^2 w&=\frac{h}{D}L(w,F)+\frac{q}{D}\\ \Delta^2 F&=-\frac{E}{2}L(w,w)\end{aligned}\right\}\qquad(2.1)$$

其中 $D=\dfrac{Eh^3}{12(1-\nu^2)}$ 是板的弯曲刚度，和

$$\Delta\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\tag{2.2}$$

$$L(w,F)\equiv\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)+\frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$$

$$-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\tag{2.3}$$

由于尚无法求其真实解，近来多采用摄动方法[6]-[9]在薄膜解的基础上求其渐近解，进而研究薄板的各类弯曲问题．本文采用文［2］的摄动方法，研究薄板在各种支承条件下的非对称弯曲问题．

先考察环形板的弯曲问题．设板的内缘的半径为r_0，外缘的半径为r_1．如文[3]一样地无量纲化，引进无量纲量：

$$\tilde{w}=\frac{w}{r_1},\quad \tilde{r}=\frac{r}{r_1},\quad \tilde{F}=\frac{F}{Er_1^2},\quad \tilde{q}=\frac{r_1 q}{hE}$$

方程（2.1）化为（略去字母上的“～”号）

$$\left.\begin{aligned}\Pi_\varepsilon(w,F)&\equiv\varepsilon^2\Delta^2 w-L(w,F)=q\\ \Pi(w,F)&\equiv\Delta^2 F+\frac{1}{2}L(w,w)=0\end{aligned}\right\}\tag{2.4}$$

其中 $\varepsilon^2=\dfrac{h^2}{12(1-\nu^2)r_1^2}$．设给出边界条件：

$$w|_{r=b}=f_0(\theta),\qquad \left.\frac{\partial w}{\partial r}\right|_{r=b}=g_0(\theta)\tag{2.5}$$

$$w|_{r=1}=f_1(\theta),\qquad \left.\left[\frac{\partial^2 w}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\right|_{r=1}=g_1(\theta)\tag{2.6}$$

和

$$\left.\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)\right|_{r=b}=0,\qquad \left.\frac{\partial^2 F}{\partial r^2}\right|_{r=b}=S_0(\theta)\tag{2.7}$$

$$\left.\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)\right|_{r=1}=0,\qquad \left.\frac{\partial^2 F}{\partial r^2}\right|_{r=1}=S_1(\theta)\tag{2.8}$$

其中 $b=\dfrac{r_0}{r_1}$．

1．微分算子的展开式

在内缘 $r=b$ 的邻域引进两变量：

$$\xi=\frac{u(r,\theta)}{\varepsilon^p},\qquad \eta=r\tag{2.9}$$

其中 $u(r,\theta)$是待定函数，p 是待定常数．将关于 r 的偏导数借两变量 ξ 和 η 在 $r=b$的邻域展开

$$\frac{\partial^i}{\partial r^i}=\varepsilon^{-ip}(\delta_{i,0}+\varepsilon^p\delta_{i,1}+\cdots+\varepsilon^{ip}\delta_{i,i})\qquad(i=1,2,3,4)\tag{2.10}$$

其中

$$\delta_{1,0}\equiv u_r\frac{\partial}{\partial\xi},\quad \delta_{1,1}\equiv\frac{\partial}{\partial\eta}$$

$$\delta_{2,0}\equiv u_r^2\frac{\partial^2}{\partial\xi^2},\quad \delta_{2,1}\equiv 2u_r\frac{\partial^2}{\partial\xi\partial\eta}+u_{rr}\frac{\partial}{\partial\xi},\quad \delta_{2,2}\equiv\frac{\partial^2}{\partial\eta^2}$$

$$\delta_{3,0}\equiv u_r^3\frac{\partial^3}{\partial\xi^3},\quad \delta_{3,1}\equiv 3u_r^2\frac{\partial^3}{\partial\xi^2\partial\eta}+3u_r u_{rr}\frac{\partial^2}{\partial\xi^2}$$

$$\delta_{3,2}\equiv 3u_r\frac{\partial^3}{\partial\xi\partial\eta^2}+3u_{rr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{rrr}\frac{\partial}{\partial\xi},\quad \delta_{3,3}\equiv\frac{\partial^3}{\partial\eta^3}$$

$$\delta_{4,0}\equiv u_r\frac{\partial^4}{\partial\xi^4},\quad \delta_{4,1}\equiv 4u_r^3\frac{\partial^4}{\partial\xi^3\partial\eta}+6u_r^2 u_{rr}\frac{\partial^3}{\partial\xi^3}$$

$$\delta_{4,2}\equiv 6u_r^2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+12u_r u_{rr}\frac{\partial^3}{\partial\xi^2\partial\eta}+4u_r u_{rrr}\frac{\partial^2}{\partial\xi^2}+3u_{rr}^2\frac{\partial^2}{\partial\xi^2}$$

$$\delta_{4,3}\equiv 4u_r\frac{\partial^4}{\partial\xi\partial\eta^3}+6u_{rr}\frac{\partial^3}{\partial\xi\partial\eta^2}+4u_{rrr}\frac{\partial^2}{\partial\xi\partial\eta}+u_{rrrr}\frac{\partial}{\partial\xi}$$

$$\delta_{4,4}\equiv\frac{\partial^4}{\partial\eta^4}$$

可以得到算子 Δ^2 在 $r=b$ 近旁的展开式：

$$\Delta^2=\varepsilon^{-4p}\sum_{i=0}^{4}\varepsilon^{ip}D_i \tag{2.11}$$

以及得到$L(w(r,\theta),v(\xi,\eta,\theta))$和 $L(h(\xi,\eta,\theta),\ v(\xi,\eta,\theta))$在 $r=b$ 近旁的展开式：

$$L(w(r,\theta),v(\xi,\eta,\theta))=\varepsilon^{-2p}\sum_{i=0}^{4}\varepsilon^{ip}M_i(w,v) \tag{2.12}$$

$$L(h(\xi,\eta,\theta),v(\xi,\eta,\theta))=\varepsilon^{-3p}\sum_{i=0}^{3}\varepsilon^{ip}N_i(h,v) \tag{2.13}$$

其中

$$D_0\equiv\delta_{4,0},\quad D_1\equiv\delta_{4,1}+\frac{2}{\eta}\delta_{3,0}$$

$$D_2\equiv\delta_{4,2}+\frac{2}{\eta^2}\delta_{2,0}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\eta}\delta_{3,1}-\frac{1}{\eta^2}\delta_{2,0}$$

$$D_3\equiv\delta_{4,3}+\frac{2}{\eta^2}\delta_{2,1}\frac{\partial^2}{\partial\theta^2}-\frac{2}{\eta^3}\delta_{1,0}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\eta}\delta_{3,2}-\frac{1}{\eta^2}\delta_{2,1}+\frac{1}{\eta^3}\delta_{1,0}$$

$$D_4\equiv\delta_{4,4}+\frac{2}{\eta^2}\delta_{2,2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\eta^4}\frac{\partial^4}{\partial\theta^4}-\frac{2}{\eta^3}\delta_{1,1}\frac{\partial^2}{\partial\theta^2}+\frac{2}{\eta}\delta_{3,3}-\frac{1}{\eta^2}\delta_{2,2}+\frac{4}{\eta^4}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\eta^3}\delta_{1,1}$$

和

$$M_0(w,v)\equiv\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)\delta_{2,0}v$$

$$M_1(w,v)\equiv\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)\delta_{2,1}v+\frac{w_{rr}}{\eta}\delta_{1,0}v-2\left(\frac{-w_\theta}{\eta^2}+\frac{w_{r\theta}}{\eta}\right)\frac{\delta_{1,0}v_\theta}{\eta}$$

$$M_2(w,v)\equiv\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)\delta_{2,2}v+\left(\frac{w_{rr}}{\eta}\delta_{1,1}v+\frac{w_{rr}}{\eta^2}v_{\theta\theta}\right)$$

$$-2\left(\frac{-w_\theta}{\eta^2}+\frac{w_{r\theta}}{\eta}\right)\left(\frac{-v_\theta}{\eta^2}+\frac{\delta_{1,1}v_\theta}{\eta}\right)$$

$$N_0(h,v)\equiv\delta_{2,0}h\frac{\delta_{1,0}v}{\eta}+\delta_{2,0}v\frac{\delta_{1,0}h}{\eta}$$

$$N_1(h,v)\equiv\delta_{2,0}h\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)+\delta_{2,1}h\frac{\delta_{1,0}v}{\eta}+\delta_{2,0}v\left(\frac{\delta_{1,1}h}{\eta}+\frac{h_{\theta\theta}}{\eta^2}\right)+\delta_{2,1}v\frac{\delta_{1,0}h}{\eta}$$

$$-2\frac{\delta_{1,0}h_\theta}{\eta}\frac{\delta_{1,0}v_\theta}{\eta}$$

$$N_2(h,v)\equiv\delta_{2,1}h\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)+\delta_{2,2}h\frac{\delta_{1,0}v}{\eta}+\delta_{2,1}v\left(\frac{\delta_{1,1}h}{\eta}+\frac{h_{\theta\theta}}{\eta^2}\right)+\delta_{2,2}v\frac{\delta_{1,0}h}{\eta}$$

$$-2\left[\frac{\delta_{1,0}h_\theta}{\eta}\left(\frac{\delta_{1,1}v_\theta}{\eta}-\frac{v_\theta}{\eta^2}\right)+\frac{\delta_{1,0}v_\theta}{\eta}\left(\frac{\delta_{1,1}h_\theta}{\eta}-\frac{h_\theta}{\eta^2}\right)\right]$$

$$N_3(h,v)\equiv\delta_{2,2}h\left(\frac{\delta_{1,1}v}{\eta}+\frac{v_{\theta\theta}}{\eta^2}\right)+\delta_{2,2}v\left(\frac{\delta_{1,1}h}{\eta}+\frac{h_{\theta\theta}}{\eta^2}\right)$$

$$-2\left(\frac{\delta_{1,1}h_\theta}{\eta}-\frac{h_\theta}{\eta^2}\right)\left(\frac{\delta_{1,1}v_\theta}{\eta}-\frac{v_\theta}{\eta^2}\right)$$

类似地，在外缘 $r=1$ 的邻域引进两变量：

$$\tilde{\xi}=\frac{\tilde{u}(r,\theta)}{\varepsilon^p},\qquad \tilde{\eta}=r \tag{2.14}$$

可以得到 Δ^2，$L(w(r,\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))$，$L(\tilde{h}(\tilde{\xi},\tilde{\eta},\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))$在 $r=1$ 近旁的展开式：

$$\Delta^2=\varepsilon^{-4p}\sum_{i=0}^{4}\varepsilon_{ip}\tilde{D}_i \tag{2.15}$$

$$L(w(r,\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))=\varepsilon^{-2p}\sum_{i=0}^{2}\varepsilon^{ip}\tilde{M}_i(w,\tilde{v}) \tag{2.16}$$

$$L(h(\xi,\eta,\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))=\varepsilon^{3p}\sum_{i=0}^{3}\varepsilon^{ip}\tilde{N}_i(h,\tilde{v}) \tag{2.17}$$

$$L(\tilde{h}(\tilde{\xi},\tilde{\eta},\theta),\tilde{v}(\tilde{\xi},\tilde{\eta},\theta))=\varepsilon^{-3p}\sum_{i=0}^{3}\varepsilon^{ip}\tilde{\tilde{N}}_i(\tilde{h},\tilde{v}) \tag{2.18}$$

将 $\delta_{i,j}$ 中的 u,ξ,η 分别换成 $\tilde{u}$，$\tilde{\xi}$，$\tilde{\eta}$ 就得到 $\tilde{\delta}_{i,j}$； 将 D_i 中的$\delta_{i,j}$，η 分别换成 $\tilde{\delta}_{i,j}$，$\tilde{\eta}$ 就得到 $\tilde{D}_i$，将 $M_i(w,v)$ 中的 $\delta_{i,j}$，η，v 分别换成 $\tilde{\delta}_{i,j}$，$\tilde{\eta}$，$\tilde{v}$ 就得到 $\tilde{M}_i(w,\tilde{v})$；在 $N_i(h,v)$ 中将 $\delta_{i,j}v$，$\frac{\partial^i v}{\partial\theta^i}$分别换成$\tilde{\delta}_{i,j}\tilde{v}$，$\frac{\partial^i\tilde{v}}{\partial\theta^i}$就得到$\tilde{N}_i(h,\tilde{v})$；在$N_i(h,v)$中将 $\delta_{i,j}$，η，h，v 分别换成

$\tilde{\delta}_{i,j}$，$\tilde{\eta}$，$\tilde{h}$，$\tilde{v}$就得到$\tilde{N}_i(h,\tilde{v})$。

2. 递推方程和边界条件

假设挠度和应力函数的 N 阶近似式是

$$W_N(r,\theta;\varepsilon)=\sum_{n=0}^{N}\varepsilon^{np}w_n(r,\theta)+\sum_{n=0}^{N}\varepsilon^{(n+\alpha_1)p}\ v_n(\xi,\eta,\theta)+\sum_{n=0}^{N}\varepsilon^{(n+\alpha_2)p}\ \tilde{v}_n(\tilde{\xi},\tilde{\eta},\theta) \tag{2.19}$$

$$F_N(r,\theta;\varepsilon)=\sum_{n=0}^{N}\varepsilon^{np}f_n(r,\theta)+\sum_{n=0}^{N}\varepsilon^{(n+\beta_1)p}h_n(\xi,\eta,\theta)+\sum_{n=0}^{N}\varepsilon^{(n+\beta_2)p}\ \tilde{h}_n(\tilde{\xi},\tilde{\eta},\theta) \tag{2.20}$$

其中α_1，α_2，β_1，β_2是待定常数；v_n 和 h_n 是待求的在 $r=b$ 邻域的边界层型函数（即只在边界附近有意义，当点离开边界时，其值随 ε 趋于零而指数型地趋于零）；$\tilde{v}_n$ 和 $\tilde{h}_n$ 是待求的在 $r=1$ 邻域的边界层型函数。

将（2.19）和（2.20）式代入方程（2.4）得

$$\begin{aligned}\Pi_\varepsilon(W_N,F_N)\equiv&\Big\{\varepsilon^2\Delta^2\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n\Big)-L\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n,\sum_{n=0}^{N}\varepsilon^{np}f_n\Big)\Big\}\\
&+\Big\{\varepsilon^{-4p+2+\alpha_1p}\sum_{i=0}^{4}\varepsilon^{ip}D_i\Big(\sum_{n=0}^{N}\varepsilon^{np}v_n\Big)-\varepsilon^{-2p+\beta_1p}\sum_{i=0}^{2}\varepsilon^{ip}M_i\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n,\\
&\sum_{n=0}^{N}\varepsilon^{np}h_n\Big)-\varepsilon^{-2p+\alpha_1p}\sum_{i=0}^{2}\varepsilon^{ip}M_i\Big(\sum_{n=0}^{N}\varepsilon^{np}f_n,\sum_{n=0}^{N}\varepsilon^{np}v_n\Big)\\
&-\varepsilon^{-3p+\alpha_1p+\beta_1p}\sum_{i=0}^{3}\varepsilon^{ip}N_i\Big(\sum_{n=0}^{N}\varepsilon^{np}v_n,\sum_{n=0}^{N}\varepsilon^{np}h_n\Big)\Big\}\\
&+\Big\{\varepsilon^{-4p+2+\alpha_2p}\sum_{i=0}^{4}\varepsilon^{ip}\tilde{D}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n\Big)-\varepsilon^{-2p+\beta_2p}\sum_{i=0}^{2}\varepsilon^{ip}\tilde{M}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n,\\
&\sum_{n=0}^{N}\varepsilon^{np}\tilde{h}_n\Big)-\varepsilon^{-2p+\alpha_2p}\sum_{i=0}^{2}\varepsilon^{ip}\tilde{M}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}f_n,\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n\Big)\\
&-\varepsilon^{-3p+\alpha_2p+\beta_2p}\sum_{i=0}^{3}\varepsilon^{ip}\tilde{N}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n,\sum_{n=0}^{N}\varepsilon^{np}\tilde{h}_n\Big)\Big\}\\
&-\Big\{\varepsilon^{-3p+\alpha_1p+\beta_2p}\sum_{i=0}^{3}\tilde{N}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}v_n,\sum_{n=0}^{N}\varepsilon^{np}\tilde{h}_n\Big)\end{aligned}$$

$$+\varepsilon^{-3p+\alpha_2 p+\beta_1 p}\sum_{i=0}^{3}\tilde{N}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n,\ \sum_{n=0}^{N}\varepsilon^{np}h_n\Big)\Big\}=q \tag{2.21}$$

$$\Pi(W_N,F_N)\equiv\Big\{\Delta^2\Big(\sum_{n=0}^{N}\varepsilon^{np}f_n\Big)+\frac{1}{2}L\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n,\ \sum_{n=0}^{N}\varepsilon^{np}w_n\Big)\Big\}$$

$$+\Big\{\varepsilon^{-4p+\beta_1 p}\sum_{i=0}^{4}\varepsilon^{ip}D_i\Big(\sum_{n=0}^{N}\varepsilon^{np}h_n\Big)+\frac{1}{2}\varepsilon^{-3p+2\alpha_1 p}\sum_{i=0}^{3}\varepsilon^{ip}N_i\Big(\sum_{n=0}^{N}\varepsilon^{np}v_n,$$

$$\sum_{n=0}^{N}\varepsilon^{np}v_n\Big)+\varepsilon^{-2p+\alpha_1 p}\sum_{i=0}^{2}\varepsilon^{ip}M_i\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n,\ \sum_{n=0}^{N}\varepsilon^{np}v_n\Big)\Big\}$$

$$+\Big\{\varepsilon^{-4p+\beta_2 p}\sum_{i=0}^{4}\varepsilon^{ip}\tilde{D}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}\tilde{h}_n\Big)+\frac{1}{2}\varepsilon^{-3p+2\alpha_2 p}\sum_{i=0}^{3}\varepsilon^{ip}\tilde{N}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n,$$

$$\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n\Big)+\varepsilon^{-2p+\alpha_2 p}\sum_{i=0}^{2}\varepsilon^{ip}\tilde{M}_i\Big(\sum_{n=0}^{N}\varepsilon^{np}w_n,\ \sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n\Big)\Big\}$$

$$-\varepsilon^{-3p}\sum_{i=0}^{3}\varepsilon^{ip}\tilde{N}_i\Big(\sum_{n=0}^{N}\varepsilon^{(n+\alpha_1)p}v_n,\ \sum_{n=0}^{N}\varepsilon^{(n+\alpha_2)p}\tilde{v}_n\Big)=0 \tag{2.22}$$

再代入边界条件(2.5)～(2.8)得

$$\sum_{n=0}^{N}\varepsilon^{np}w_n\Big|_{r=b}+\varepsilon^{\alpha_1 p}\sum_{n=0}^{N}\varepsilon^{np}v_n\Big|_{\eta=b}=f_0(\theta) \tag{2.23}$$

$$\sum_{n=0}^{N}\varepsilon^{np}w_{n,r}\Big|_{r=b}+\varepsilon^{\alpha_1 p-p}(\delta_{1,0}+\varepsilon^{p}\delta_{1,1})\sum_{n=0}^{N}\varepsilon^{np}v_n\Big|_{\eta=b}=g_0(\theta) \tag{2.24}$$

$$\sum_{n=0}^{N}\varepsilon^{np}w_n\Big|_{r=1}+\varepsilon^{\alpha_2 p}\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n\Big|_{\tilde{\eta}=1}=f_1(\theta) \tag{2.25}$$

$$\sum_{n=0}^{N}\varepsilon^{np}[w_{n,rr}+\nu(w_{n,r}+w_{n,\theta\theta})]\Big|_{r=1}+\varepsilon^{\alpha_2 p-2p}\Big[\tilde{\delta}_{2,0}+\varepsilon^{p}(\tilde{\delta}_{2,1}+\nu\tilde{\delta}_{1,0})$$

$$+\varepsilon^{2p}\Big(\tilde{\delta}_{2,2}+\nu\tilde{\delta}_{1,1}+\nu\frac{\partial^2}{\partial\theta^2}\Big)\Big]\sum_{n=0}^{N}\varepsilon^{np}\tilde{v}_n\Big|_{\tilde{\eta}=1}=g_1(\theta) \tag{2.26}$$

和

$$\sum_{n=0}^{N}\varepsilon^{np}\Big(\frac{f_{n,r}}{r}+\frac{f_{n,\theta\theta}}{r^2}\Big)\Big|_{r=b}+\varepsilon^{\beta_1 p-p}\Big[\frac{\delta_{1,0}}{\eta}+\varepsilon^{p}\Big(\frac{\delta_{1,1}}{\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\Big)\Big]\cdot$$

$$\cdot\sum_{n=0}^{N}\varepsilon^{np}h_n\Big|_{\eta=b}=0 \tag{2.27}$$

$$\sum_{n=0}^{N} \varepsilon^{np} f_{n,rr}\Big|_{r=b} + \varepsilon^{\beta_1 p-2p}[\delta_{2,0}+\varepsilon^{p}\delta_{2,1}+\varepsilon^{2p}\delta_{2,2}] \sum_{n=0}^{N} \varepsilon^{np} h_n\Big|_{\eta=b} = S_0(\theta) \tag{2.28}$$

$$\sum_{n=0}^{N} \varepsilon^{np}\left(\frac{f_{n,r}}{r}+\frac{f_{n,\theta\theta}}{r^2}\right)\Big|_{r=1} + \varepsilon^{\beta_2 p-p}\left[\frac{\tilde{\delta}_{1,0}}{\tilde{\eta}}+\varepsilon^{p}\left(\frac{\tilde{\delta}_{1,1}}{\tilde{\eta}} + \frac{1}{\tilde{\eta}^2}\frac{\partial^2}{\partial\theta^2}\right)\right]\sum_{n=0}^{N} \varepsilon^{np}\tilde{h}_n\Big|_{\tilde{\eta}=1} = 0 \tag{2.29}$$

$$\sum_{n=0}^{N} \varepsilon^{np} f_{n,rr}\Big|_{r=1} + \varepsilon^{\beta_2 p-2p}[\tilde{\delta}_{2,0}+\varepsilon^{p}\tilde{\delta}_{2,1}+\varepsilon^{2p}\tilde{\delta}_{2,2}] \sum_{n=0}^{N} \varepsilon^{np}\tilde{h}_n\Big|_{\tilde{\eta}=1} = S_1(\theta) \tag{2.30}$$

在(2.21)～(2.30)式中考察ε的较低次幂项，从（2.24）和（2.26）式知应取，

$$a_1=1, \qquad a_2=2$$

从（2.21）式知应取

$$p=\frac{2}{3}$$

又从（2.21）和（2.22）式的第二和第三大括号知应取

$$\beta_1=3, \qquad \beta_2=4$$

因为v_n和h_n是$r=b$邻域的边界层型函数，$\tilde{v}_n$和$\tilde{h}$是$r=1$邻域的边界层型函数，所以（2.21）和（2.22）左端的最后一项渐近地为零．在(2.21)～(2.30)式中，逐次地比较ε的同次幂的系数，从（2.21）和（2.22）的第一大括号得到w_n和f_n（$n=0, 1, \cdots, N$）的递推方程：

$$L(w_0,f_0)=-q, \qquad \Delta^2 f_0+\frac{1}{2}L(w_0,w_0)=0 \tag{2.31}$$

$$\left.\begin{aligned} &L(w_0,f_n)+L(w_n,f_0)=\Delta^2 w_{n-3}-\sum_{i=1}^{n-1}L(w_i,f_{n-i}) \\ &\Delta^2 f_n+L(w_0,w_n)=-\frac{1}{2}\sum_{i=1}^{n-1}L(w_i,w_{n-i}) \end{aligned}\right\} \tag{2.32}$$

$$(n=1,2,\cdots, N)$$

在上式以及以后各式中，都将负下标的量取作零．再从（2.21）和（2.22）的第二大括号，考虑到$M_0(f_0,v_0)=\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)u_1^2\frac{\partial^2 v_0}{\partial\xi^2}$在$r=b$的邻域是一小量，又得到关于$v_n$和$h_n$，($n=0,1,\cdots,N$)的递推方程：

$$D_0 v_0-\varepsilon^{-p}M_0(f_0,v_0)=0 \tag{2.33}$$

$$D_0 v_n-\varepsilon^{-p}M_0(f_0,v_n)=-\sum_{i=1}^{4}D_i v_{n-i}+\sum_{i=0}^{2}\sum_{j+k=(n-1)-i}M_i(w_j,h_k)$$

$$+\varepsilon^{-p}\Big[\sum_{\substack{j+k=n\\(j\neq 0)}} M_0(f_j,v_k)+\sum_{i=1}^{2}\sum_{j+k=n-i} M_i(f_j,v_k)\Big]$$

$$+\sum_{i=0}^{3}\sum_{j+k=(n-1)-i} N_i(v_j,h_k)\qquad (n=1,2,\cdots,N) \tag{2.34}$$

$$D_0h_0=-\frac{1}{2}N_0(v_0,v_0)-M_0(w_0,v_0) \tag{2.35}$$

$$D_0h_n=-\sum_{i=1}^{4}D_ih_{n-i}-\frac{1}{2}\sum_{i=0}^{3}\sum_{j+k=n-i}N_i(v_j,v_k)-\sum_{i=0}^{2}\sum_{j+k=n-i}M_i(w_j,v_k) \tag{2.36}$$

$$(n=1,2,\cdots,N)$$

类似地，在(2.21)和 (2.22) 的第三大括号中，考虑到$\tilde{M}_0(f_0,\tilde{v}_0)=\left(\frac{f_{0,r}}{\tilde{\eta}}+\frac{f_{0,\theta\theta}}{\tilde{\eta}^2}\right)u_i^2\frac{\partial^2\tilde{v}_0}{\partial\xi^2}$在$r=1$的邻域是小量，逐次地令$\varepsilon$的各次幂的系数为零，又得到关于$\tilde{v}_n$和$\tilde{h}_n$，$(n=0,1,\cdots,N)$的递推方程：

$$\tilde{D}_0\tilde{v}_0-\varepsilon^{-p}\tilde{M}_0(f_0,\tilde{v}_0)=0 \tag{2.37}$$

$$\tilde{D}_0\tilde{v}_n-\varepsilon^{-p}\tilde{M}_0(f_0,\tilde{v}_n)=-\sum_{i=1}^{4}\tilde{D}_i\tilde{v}_{n-i}+\sum_{i=0}^{2}\sum_{j+k=(n-1)-i}\tilde{M}_i(w_j,\tilde{h}_k)$$

$$+\varepsilon^{-p}\Big[\sum_{\substack{j+k=n\\(j\neq 0)}}\tilde{M}_0(f_j,\tilde{v}_k)+\sum_{i=1}^{2}\sum_{j+k=n-i}\tilde{M}_i(f_j,\tilde{v}_k)\Big]$$

$$+\sum_{i=0}^{3}\sum_{j+k=(n-2)-i}\tilde{\tilde{N}}_i(\tilde{v}_j,h_k)\qquad (n=1,2,\cdots,N) \tag{2.38}$$

$$\tilde{D}_0\tilde{h}_0=-\tilde{M}_0(w_0,\tilde{v}_0) \tag{2.39}$$

$$\tilde{D}_0\tilde{h}_n=-\sum_{i=1}^{4}\tilde{D}_i\tilde{h}_{n-i}-\frac{1}{2}\sum_{i=0}^{3}\sum_{j+k=(n-1)-i}\tilde{\tilde{N}}_i(\tilde{v}_j,\tilde{v}_k)-\sum_{i=0}^{2}\sum_{j+k=n-i}\tilde{M}_i(w_j,\tilde{v}_k) \tag{2.40}$$

$$(n=1,2,\cdots,N)$$

又在(2.23)～(2.30)式中，逐次地比较ε的同次幂的系数，得到关于w_n，f_n，v_n，$\tilde{v}_n$，h_n和$\tilde{h}_n$ $(n=0,1,\cdots,N)$的边界条件：

$$w_0|_{r=b}=f_0(\theta),\quad w_n|_{r=b}+v_{n-1}|_{\eta=b}=0\qquad (n=1,2,\cdots,N) \tag{2.41}$$

$$\left.\begin{aligned}&w_{0,r}|_{r=b}+\delta_{1,0}v_0|_{\eta=b}=g_0(\theta)\\&w_{n,r}|_{r=b}+(\delta_{1,0}v_n+\delta_{1,1}v_{n-1})|_{\eta=b}=0\qquad (n=1,2,\cdots,N)\end{aligned}\right\} \tag{2.42}$$

$$w_0|_{r=1}=f_1(\theta),\quad w_n|_{r=1}+v_{n-2}|_{\tilde{\eta}=1}=0\qquad (n=1,2,\cdots,N) \tag{2.43}$$

$$\left.\begin{aligned}&[w_{0,rr}+\nu(w_{0,r}+w_{0,\theta\theta})]|_{r=1}+\tilde{\delta}_{2,0}\tilde{v}_0|_{\tilde{\eta}=1}=g_1(\theta)\\&[w_{n,rr}+\nu(w_{n,r}+w_{n,\theta\theta})]|_{r=1}+\Big[\tilde{\delta}_{2,0}\tilde{v}_n+(\tilde{\delta}_{2,1}+\nu\tilde{\delta}_{1,0})\tilde{v}_{n-1}\\&\quad+\left(\tilde{\delta}_{2,2}+\nu\tilde{\delta}_{1,1}+\nu\frac{\partial^2}{\partial\theta^2}\right)\tilde{v}_{n-2}\Big]\Big|_{\tilde{\eta}=1}=0\quad (n=1,2,\cdots,N)\end{aligned}\right\} \tag{2.44}$$

$$\left.\begin{aligned}&\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=b}=0\\&\left(\frac{f_{n,r}}{r}+\frac{f_{n,\theta\theta}}{r^2}\right)\Big|_{r=b}+\left[\frac{\delta_{1,0}h_{n-2}}{\eta}+\left(\frac{\delta_{1,1}}{\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)h_{n-3}\right]\Big|_{\eta=b}=0\\&\qquad(n=1,2,\cdots,N)\end{aligned}\right\}\tag{2.45}$$

$$\left.\begin{aligned}&f_{0,rr}|_{r=b}=S_0(\theta)\\&f_{n,rr}|_{r=b}+[\delta_{2,0}h_{n-1}+\delta_{2,1}h_{n-2}+\delta_{2,2}h_{n-3}]|_{\eta=b}=0\qquad(n=1,2,\cdots,N)\end{aligned}\right\}\tag{2.46}$$

$$\left.\begin{aligned}&\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=1}=0\\&\left(\frac{f_{n,r}}{r}+\frac{f_{n,\theta\theta}}{r^2}\right)\Big|_{r=1}+\left[\frac{\tilde\delta_{1,0}\tilde h_{n-3}}{\tilde\eta}+\left(\frac{\tilde\delta_{1,1}}{\tilde\eta}+\frac{1}{\tilde\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\tilde h_{n-4}\right]\Big|_{\tilde\eta=1}=0\\&\qquad(n=1,2,\cdots,N)\end{aligned}\right\}\tag{2.47}$$

$$\left.\begin{aligned}&f_{0,rr}|_{r=1}=S_1(\theta)\\&f_{n,rr}|_{r=1}+[\tilde\delta_{2,0}\tilde h_{n-2}+\tilde\delta_{2,1}\tilde h_{n-3}+\tilde\delta_{2,2}\tilde h_{n-4}]|_{\tilde\eta=1}=0\\&\qquad(n=1,2,\cdots,N)\end{aligned}\right\}\tag{2.48}$$

3. 形式渐近解

从(2.31)，(2.41)，(2.43)，(2.45)～(2.48)式我们知道 w_0 和 f_0 应是薄膜理论的解：

$$\left.\begin{aligned}&L(w_0,f_0)=-q,\quad \Delta^2f_0+\frac{1}{2}L(w_0,w_0)=0\\&w_0|_{r=b}=f_0(\theta),\quad\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=b}=0,\quad f_{0,rr}|_{r=b}=S_0(\theta)\\&w_0|_{r=1}=f_1(\theta),\quad\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=1}=0,\quad f_{0,rr}|_{r=1}=S_1(\theta)\end{aligned}\right\}\tag{2.49}$$

求得 w_0 和 f_0 后再代入（2.33）式，则得到关于 v_0 的微分方程：

$$\frac{\partial^4v_0}{\partial\xi^4}-\frac{\varepsilon^{-\gamma}}{u_r^2}\left(\frac{f_{0,r}}{\eta}+\frac{f_{0,\theta\theta}}{\eta^2}\right)\frac{\partial^2v_0}{\partial\xi^2}=0\tag{2.50}$$

若取 u 是下面微分方程

$$uu_r^2=\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\tag{2.51}$$

在边界 $r=b$ 取零值的解[1)]：

$$u(r,\theta)=\left(\frac{3}{2}\int_b^r\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\right)^{2/3}\tag{2.52}$$

则方程（2.50）化为

$$\frac{\partial^4v_0}{\partial\xi^4}-\xi\frac{\partial^2v_0}{\partial\xi^2}=0\tag{2.53}$$

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1) 因 w_0 和 f_0 是薄膜理论的解，在薄膜的内部只存在着张力，所以在板的内部区域成立

$$\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}>0.$$

又从（2.42）得到v_0的边界条件：

$$\left.\frac{\partial v_0}{\partial \xi}\right|_{\xi=0}=\frac{g_0(\theta)-w_{0,r}(b,\theta)}{u_r(b,\theta)} \tag{2.54}$$

其中

$$u_r(b,\theta)=\lim_{r\to b}\left(\frac{3}{2}\int_b^r\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\right)^{-1/3}\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}$$

$$=\left[\frac{\partial}{\partial r}\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=b}\right]^{1/3}$$

从（2.53）和（2.54）可以求得边界层项v_0：

$$v_0=-\frac{g_0(\theta)-w_{0,r}(b,\theta)}{u_r(b,\theta)\int_0^\infty \mathrm{Ai}(t)dt}\int_\infty^\xi\int_\infty^{\xi_1}\mathrm{Ai}(t)dtd\xi_1$$

$$\equiv-\frac{g_0(\theta)-w_{0,r}(b,\theta)}{u_r(b,\theta)\int_0^\infty \mathrm{Ai}(t)dt}\int_\infty^{\frac{u(r,\theta)}{\varepsilon}}\int_\infty^{\xi_1}\mathrm{Ai}(t)dtd\xi_1 \tag{2.55}$$

其中 Ai(*t*)表示第一类 Airy 函数[9]：

$$\mathrm{Ai}(t)=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{\tau^3}{3}+t\tau\right)d\tau$$

当$t\to+\infty$时具有渐近展开式：

$$\mathrm{Ai}(t)=\frac{1}{2}\pi^{-1/2}t^{-1/4}\exp\left[-\frac{2}{3}t^{3/2}\right][1+O(t^{-3/2})]$$

将w_0和v_0再代入方程(2.35)，又求得

$$h_0=\int_\infty^\xi\int_\infty^{\xi_4}\int_\infty^{\xi_3}\int_\infty^{\xi_2}\left[\frac{1}{u_r\eta}\ \frac{\partial^2 v_0}{\partial\xi^2}\ \frac{\partial v_0}{\partial\xi}+\frac{1}{u_r^2}\left(\frac{w_{0,r}}{\eta}+\frac{w_{0,\theta\theta}}{\eta^2}\right)\frac{\partial^2 v_0}{\partial\xi^2}\right]d\xi_1d\xi_2d\xi_3d\xi_4 \tag{2.56}$$

类似地，可取（2.14）式中的$\tilde{u}$是微分方程（2.51）在边界$r=1$取零值的解：

$$\tilde{u}(r,\theta)=\left(\frac{3}{2}\int_r^1\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\right)^{2/3} \tag{2.57}$$

从微分方程（2.37）和边界条件（2.44）可以求得边界层项$\tilde{v}_0$：

$$\tilde{v}_0=\frac{g_1(\theta)-[w_{0,rr}(1,\theta)+\nu w_{0,r}(1,\theta)+\nu w_{0,\theta\theta}]}{\tilde{u}_r(1,\theta)\int_0^\infty\mathrm{Ai}(t)dt}\int_\infty^\xi\int_\infty^{\xi_1}\mathrm{Ai}(t)dtd\xi_1 \tag{2.58}$$

其中

$$\tilde{u}_r(1,\theta)=-\lim_{r\to1}\left(\frac{3}{2}\int_r^1\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\right)^{-1/3}\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}$$

$$=\left[\frac{\partial}{\partial r}\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=1}\right]^{1/3}$$

从微分方程（2.39）可以求得边界层项$\tilde{h}_0$为

$$h_0=-\int_\infty^\xi\int_\infty^{\xi_4}\int_\infty^{\xi_3}\int_\infty^{\xi_2}\frac{1}{\tilde{u}_r^2}\left(\frac{w_{0,r}}{\tilde{\eta}}+\frac{w_{0,\theta\theta}}{\tilde{\eta}^2}\right)\frac{\partial^2\tilde{v}_0}{\partial\xi^2}d\xi_1d\xi_2d\xi_3d\xi_4 \tag{2.59}$$

将求得的 w_0, f_0, v_0, $\tilde{v}_0$, h_0 和 $\tilde{h}_0$ 再代入方程（2.32）（取 $n=1$）和边界条件（2.41），（2,43），（2.45）～（2.48），又得到关于 w_1 和 f_1 的边值问题：

$$\left.\begin{aligned}&L(w_0,f_1)+L(w_1,f_0)=0,\quad \Delta^2 f_1+L(w_0,\ w_1)=0\\&w_1\Big|_{r=b}=-v_0\Big|_{\eta=b},\quad \left(\frac{f_{1,r}}{r}+\frac{f_{1,\theta\theta}}{r^2}\right)\Big|_{r=b}=0\\&f_{1,rr}\Big|_{r=b}=-u_r^2\frac{\partial^2 h_0}{\partial\xi^2}\Big|_{\eta=b}\\&w_1\Big|_{r=1}=0,\quad \left(\frac{f_{1,r}}{r}+\frac{f_{1,\theta\theta}}{r^2}\right)\Big|_{r=1}=0,\quad f_{1,rr}\Big|_{r=1}=0\end{aligned}\right\}\tag{2.60}$$

重复前面的步骤可以求得 w_n, f_n, v_n, $\tilde{v}_n$, h_n 和 $\tilde{h}_n$（$n=1$, 2, …, N）。

三、圆 形 薄 板

对于圆形薄板，只需将内缘 $r=b$ 上的边界条件代替以圆心 $r=0$ 上的条件：

$$w,\ \frac{\partial w}{\partial r},\ \frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2},\ \frac{\partial^2 F}{\partial r^2}\ \text{在}\ r=0\ \text{取有限值.}$$

作为一个例子，我们考察周边固支薄板的非对称弯曲问题，Срубщик 在文[5]中曾作出其准确到 $O(\varepsilon^{2/3})$ 的渐近近似式。所对应的边值问题是：

$$\left.\begin{aligned}&\varepsilon^2\Delta^2 w-L(w,F)=q\\&\Delta^2 F+\frac{1}{2}L(w,\ w)=0\end{aligned}\right\}\tag{2.4}$$

$$w\Big|_{r=1}=0,\quad \frac{\partial w}{\partial r}\Big|_{r=1}=0\tag{3.1}$$

$$\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)\Big|_{r=1}=0,\quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)\Big|_{r=1}=S(\theta)\tag{3.2}$$

$$w,\ \frac{\partial w}{\partial r},\ F,\ \frac{\partial F}{\partial r},\ \text{在}\ r=0\ \text{取有限值}\tag{3.3}$$

假设解的准确到 $O(\varepsilon^{2/3})$ 的渐近近似式是：

$$W=w_0+\varepsilon^{2/3}w_1+\varepsilon^{2/3}v_0+O(\varepsilon^{4/3})\tag{3.4}$$

$$F=f_0+\varepsilon^{2/3}f_1+O(\varepsilon^{4/3})\tag{3.5}$$

将上式代入方程（2.4）和边界条件(3.2)～(3.3)，比较 ε 的同次幂的系数，则得到关于 w_0, f_0, w_1, f_1, 和 v_0 的边值问题：

$$\left.\begin{aligned}&L(w_0,f_0)=-q,\qquad \Delta^2 f_0+\frac{1}{2}L(w_0,\ w_0)=0\\&w_0\Big|_{r=1}=0,\quad \left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=1}=0,\quad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)\Big|_{r=1}=S(\theta)\\&w_0,\ \frac{\partial w_0}{\partial r},\ f_0,\ \frac{\partial f_0}{\partial r},\ \text{在}\ r=0\ \text{取有限值}\end{aligned}\right\}\tag{3.6}$$

$$\left.\begin{array}{l}L(w_0,f_1)+L(w_1,f_0)=0,\ \Delta^2 f_1+L(w_0,w_1)=0\\ w_1\Big|_{r=1}=-v_0\Big|_{\eta=1},\ \left(\dfrac{f_{1,r}}{r}+\dfrac{f_{1,\theta\theta}}{r^2}\right)\Big|_{r=1}=0,\quad \dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\ \dfrac{\partial f_1}{\partial\theta}\right)\Big|_{r=1}=0\\ w_1,\ \dfrac{\partial w_1}{\partial r},\ f_1,\ \dfrac{\partial f_1}{\partial r},\ \text{在 } r=0 \text{ 取有限值}\end{array}\right\} \tag{3.7}$$

和

$$\left.\begin{array}{l}\dfrac{\partial^4 v_0}{\partial\xi^4}-\xi\dfrac{\partial^2 v_0}{\partial\xi^2}=0\\ u_r\dfrac{\partial v_0}{\partial\xi}\Big|_{\xi=0}=-w_{0,r}\Big|_{r=1}\end{array}\right\} \tag{3.8}$$

其中

$$\xi=\frac{u(r,\theta)}{\varepsilon^{2/3}},\quad \eta=r$$

$u(r,\theta)$ 是微分方程

$$uu_r^2=\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2} \tag{3.9}$$

在边界 $r=1$ 取零值的解:

$$u(r,\theta)=\left(\frac{3}{2}\int_r^1\sqrt{\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}}dr\right)^{2/3} \tag{3.10}$$

从 (3.8) 可以求得边界层项 v_0 为

$$v_0=\psi(r)\frac{w_{0,r}(1,\theta)}{u_r(1,\theta)\int_0^\infty \mathrm{Ai}(t)dt}\int_\infty^\xi\int_\infty^{\xi_1}\mathrm{Ai}(t)dtd\xi_1$$

$$\equiv\psi(r)\frac{w_{0,r}(1,\theta)}{u_r(1,\theta)\int_0^\infty \mathrm{Ai}(t)dt}\int_\infty^{\frac{u(r,\theta)}{\varepsilon^{2/3}}}\int_\infty^{\xi_1}\mathrm{Ai}(t)dtd\xi_1 \tag{3.11}$$

其中 $\psi(r)$ 是在$\frac{2}{3}\leqslant r<1$ 取值 1，在 $0\leqslant r\leqslant\frac{1}{3}$ 取值零的无限次可微的截断函数，用以 截断边界层项在板的内部区域的影响。

若在 (3.11) 中，将$u(r,\theta)$ 在 $r=1$ 的邻域展开

$$u(r,\theta)=u_r(1,\theta)(r-1)+u_{rr}(1,\theta)\frac{(r-1)^2}{2!}+\cdots$$

近似地取

$$u(r,\theta)\approx u_r(1,\theta)(r-1)=\left[\frac{\partial}{\partial r}\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)\Big|_{r=1}\right](r-1)$$

则 (3.11) 式给出和文[5]一致的结果。从本例我们再一次看到应用两变量展开法构造 边 界层项可以得到更精确的表达式，以及避免展开系数成幂级数的冗长计算。

附注：从上面的讨论可以看出，本方法的核心在于找出偏导数借两变量ξ和η表示后，它们的关于ε的展开式．若这些展开式预先已经知道，则我们可以容易地求出微分算子的展开式，和得到关于w_n，f_n，v_n，h_n，$\cdots$，$(n=0,1,2\cdots)$的递推的边值问题．此外，在求边界层项时，我们也只需要解常微分方程．

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Unsymmetrical Bending of Annular and Circular Thin Plates under Various Supports (Ⅱ)

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Abstract

In this paper we study the unsymmetrical bending of elastic flexible plates under various supports in case the tensile force acting on its boundary is zero.