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Mac-Millan方程对非线性非完整力学系统的推广*

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摘 要

本文首先将Mac-Millan方程推广到最一般的非完整力学系统，得到非线性非完整系统的广义Mac-Millan方程．其次，证明广义Mac-Millan方程与广义Чаплыгин方程的等价性．最后给出一个例子．

一、引 言

1936年Mac-Millan得到线性非完整系统的一类运动微分方程[1]，这些方程称为Mac-Millan方程或自然方程．Mac-Millan方程的推导过程十分清楚地阐明了为什么第二类Lagrange方程一般不能应用于非完整系统．周培源早在五十年代初就介绍了Mac-Millan方程[2]．在苏联，六十年代、七十年代才开始重视这些方程[3][4]．苏联学者B.B. Добронравов断言，Mac-Millan方程不能推广到非线性非完整系统[3]．但是，这种说法是不对的．我们已在文献[5]中指出这点．

本文利用Jourdain原理和速度空间中虚位移的定义[6]来推导非线性非完整力学系统的广义Mac-Millan方程，证明它们和广义Чаплыгин方程的等价性，并举例说明这些新方程的应用．

二、Mac-Millan方程的推广

设力学系统的位置由n个广义坐标$q_1, q_2, \cdots, q_n$来确定，并受有g个理想非线性非完整约束

$$\dot{q}_{\varepsilon+\beta}=\dot{q}_{\varepsilon+\beta}(q_s,\ \dot{q}_\sigma,\ t)\qquad\begin{pmatrix}\beta=1,2,\cdots,g;\ \sigma=1,2,\cdots,\varepsilon;\\ \varepsilon=n-g;\ s=1,2,\cdots,n\end{pmatrix}\tag{2.1}$$

我们利用Jourdain原理和速度空间中的虚位移定义来推导系统的Mac-Millan型运动微分方程．为书写方便，采用矢量形式．由速度空间中虚位移的定义[6]，对约束(2.1)，有

$$\delta\dot{q}_{\varepsilon+\beta}=\sum_{\sigma=1}^{\varepsilon}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\delta\dot{q}_\sigma\qquad(\beta=1,2,\cdots,g)\tag{2.2}$$

Jourdain原理写成形式[7][5]

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* 樊大钧推荐．

$$\sum_{i=1}^{N}(-m_i\ddot{\bar{r}}_i+\bar{F}_i)\cdot\delta\dot{\bar{r}}_i=0 \tag{2.3}$$

其中m_i为第i个质点的质量，$\ddot{\bar{r}}_i$为此点的加速度，$\bar{F}_i$ 为作用在质点上的主动力的合力，$\delta\dot{\bar{r}}_i$ 为速度空间中的虚位移，N为系统的质点数目。

系统的动能为

$$T=\frac{1}{2}\sum_{i=1}^{N}m_i\dot{\bar{r}}_i\cdot\dot{\bar{r}}_i$$

令$(\dot{\bar{r}}_i)$为$\dot{\bar{r}}_i$中借助(2.1)消去不独立的广义速度而得表达式．由此得出的动能记作$\tilde{T}$，则有

$$\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}=\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\frac{\partial(\dot{\bar{r}}_i)}{\partial\dot{q}_\sigma},\qquad \frac{\partial\tilde{T}}{\partial q_\sigma}=\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\frac{\partial(\dot{\bar{r}}_i)}{\partial q_\sigma}$$

引入Euler 算子

$$\mathscr{E}_\sigma=\frac{d}{dt}\frac{\partial}{\partial\dot{q}_\sigma}-\frac{\partial}{\partial q_\sigma}$$

注意到 $(\dot{\bar{r}}_i)^{\cdot}=\ddot{\bar{r}}_i$，则

$$\mathscr{E}_\sigma(\tilde{T})=\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\mathscr{E}_\sigma((\dot{\bar{r}}_i))+\sum_{i=1}^{N}m_i\ddot{\bar{r}}_i\cdot\frac{\partial(\dot{\bar{r}}_i)}{\partial\dot{q}_\sigma}$$

因此有

$$\sum_{\sigma=1}^{\varepsilon}\mathscr{E}_\sigma(\tilde{T})\delta\dot{q}_\sigma=\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\sum_{\sigma=1}^{\varepsilon}\mathscr{E}_\sigma((\dot{\bar{r}}_i))\delta\dot{q}_\sigma+\sum_{i=1}^{N}m_i\ddot{\bar{r}}_i\cdot\sum_{\sigma=1}^{\varepsilon}\frac{\partial(\dot{\bar{r}}_i)}{\partial\dot{q}_\sigma}\delta\dot{q}_\sigma \tag{2.4}$$

今证明下述关系

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial(\dot{\bar{r}}_i)}{\partial\dot{q}_\sigma}\delta\dot{q}_\sigma=\delta\dot{\bar{r}}_i \tag{2.5}$$

实际上，点的矢径为

$$\bar{r}_i=\bar{r}_i(q_s,t)$$

点的速度为

$$\dot{\bar{r}}_i=\sum_{\sigma=1}^{\varepsilon}\frac{\partial\bar{r}_i}{\partial q_\sigma}\dot{q}_\sigma+\sum_{\beta=1}^{g}\frac{\partial\bar{r}_i}{\partial q_{\varepsilon+\beta}}\dot{q}_{\varepsilon+\beta}+\frac{\partial\bar{r}_i}{\partial t}$$

因此有

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial(\dot{\bar{r}}_i)}{\partial\dot{q}_\sigma}\delta\dot{q}_\sigma=\sum_{\sigma=1}^{\varepsilon}\left(\frac{\partial\bar{r}_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial\bar{r}_i}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\right)\delta\dot{q}_\sigma \tag{2.6}$$

以及

$$\delta\dot{\bar{r}}_i=\sum_{\sigma=1}^{\varepsilon}\frac{\partial\bar{r}_i}{\partial q_\sigma}\delta\dot{q}_\sigma+\sum_{\beta=1}^{g}\frac{\partial\bar{r}_i}{\partial q_{\varepsilon+\beta}}\delta\dot{q}_{\varepsilon+\beta}$$

注意到(2.2)，上式可写成

$$\delta\dot{\bar{r}}_i=\sum_{\sigma=1}^{\varepsilon}\left(\frac{\partial\bar{r}_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial\bar{r}_i}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\right)\delta\dot{q}_\sigma \tag{2.7}$$

比较(2.6)和(2.7)，便得(2.5)。

将(2.5)代入(2.4)，我们得到

$$\sum_{i=1}^{N} m_i\ddot{\bar{r}}_i\cdot\delta\dot{\bar{r}}_i=\sum_{\sigma=1}^{\varepsilon}\mathscr{E}_\sigma(\tilde{T})\delta\dot{q}_\sigma-\sum_{i=1}^{N} m_i(\dot{\bar{r}}_i)\cdot\sum_{\sigma=1}^{\varepsilon}\mathscr{E}_\sigma((\dot{\bar{r}}_i))\delta\dot{q}_\sigma \tag{2.8}$$

现在变换原理(2.3)中的项$\sum_{i=1}^{N}\bar{F}_i\cdot\delta\dot{\bar{r}}_i$。利用(2.7)，我们有

$$\sum_{i=1}^{N}\bar{F}_i\cdot\delta\dot{\bar{r}}_i=\sum_{\sigma=1}^{\varepsilon}\left\{\sum_{i=1}^{N}\bar{F}_i\cdot\frac{\partial\bar{r}_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\sum_{i=1}^{N}\bar{F}_i\cdot\frac{\partial\bar{r}_i}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\right\}\delta\dot{q}_\sigma$$

令广义力为

$$Q_\sigma=\sum_{i=1}^{N}\bar{F}_i\cdot\frac{\partial\bar{r}_i}{\partial q_\sigma},\qquad Q_{\varepsilon+\beta}=\sum_{i=1}^{N}\bar{F}_i\cdot\frac{\partial\bar{r}_i}{\partial q_{\varepsilon+\beta}}$$

$$\tilde{Q}_\sigma=Q_\sigma+\sum_{\beta=1}^{g}Q_{\varepsilon+\beta}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma} \tag{2.9}$$

于是有

$$\sum_{i=1}^{N}\bar{F}_i\cdot\delta\dot{\bar{r}}_i=\sum_{\sigma=1}^{\varepsilon}\tilde{Q}_\sigma\delta\dot{q}_\sigma \tag{2.10}$$

将(2.8)和(2.10)代入原理(2.3)，我们得到

$$\sum_{\sigma=1}^{\varepsilon}\left\{-\mathscr{E}_\sigma(\tilde{T})+\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\mathscr{E}_\sigma((\dot{\bar{r}}_i))+\tilde{Q}_\sigma\right\}\delta\dot{q}_\sigma=0 \tag{2.11}$$

因(2.11)中的$\delta\dot{q}_\sigma$是彼此独立的，故得

$$\mathscr{E}_\sigma(\tilde{T})=\tilde{Q}_\sigma+\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\mathscr{E}_\sigma((\dot{\bar{r}}_i))\qquad(\sigma=1,2,\cdots,\varepsilon) \tag{2.12}$$

或者写成下述形式

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}=\tilde{Q}_\sigma+\sum_{i=1}^{N}m_i(\dot{\bar{r}}_i)\cdot\left(\frac{d}{dt}\frac{\partial(\dot{\bar{r}}_i)}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\bar{r}}_i)}{\partial q_\sigma}\right)\qquad(\sigma=1,2,\cdots,\varepsilon) \tag{2.13}$$

方程(2.13)称为非线性完整系统的广义Mac-Millan方程。

现在证明，当约束是线性非完整的情形，广义Mac-Millan 方程(2.13)便是 Mac-Millan 原来的方程。设约束是线性非完整的，即有形式

$$\dot{q}_{\varepsilon+\beta}=\sum_{\sigma=1}^{\varepsilon}B_{\varepsilon+\beta,\sigma}\dot{q}_\sigma+B_{\varepsilon+\beta}\qquad(\beta=1,2,\cdots,g;\ \varepsilon=n-g) \tag{2.14}$$

我们有

$$(\dot{x}_i)=\sum_{\sigma=1}^{\varepsilon}a_{i\sigma}\dot{q}_\sigma+a_i,\quad(\dot{y}_i)=\sum_{\sigma=1}^{\varepsilon}b_{i\sigma}\dot{q}_\sigma+b_i,\quad(\dot{z}_i)=\sum_{\sigma=1}^{\varepsilon}c_{i\sigma}\dot{q}_\sigma+c_i \tag{2.15}$$

其中

$$a_{i\sigma}=\frac{\partial x_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial x_i}{\partial q_{\varepsilon+\beta}}B_{\varepsilon+\beta,\sigma},\qquad a_i=\sum_{\beta=1}^{g}\frac{\partial x_i}{\partial q_{\varepsilon+\beta}}B_{\varepsilon+\beta}+\frac{\partial x_i}{\partial t}$$

$$b_{i\sigma}=\frac{\partial y_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial y_i}{\partial q_{\varepsilon+\beta}}B_{\varepsilon+\beta,\sigma},\qquad b_i=\sum_{\beta=1}^{g}\frac{\partial y_i}{\partial q_{\varepsilon+\beta}}B_{\varepsilon+\beta}+\frac{\partial y_i}{\partial t}$$

$$c_{i\sigma}=\frac{\partial z_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial z_i}{\partial q_{\varepsilon+\beta}}B_{\varepsilon+\beta,\sigma},\qquad c_i=\sum_{\beta=1}^{g}\frac{\partial z_i}{\partial q_{\varepsilon+\beta}}B_{\varepsilon+\beta}+\frac{\partial z_i}{\partial t}$$

于是有

$$\left.\begin{aligned}
\frac{d}{dt}\frac{\partial(\dot{x}_i)}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{x}_i)}{\partial q_\sigma}&=\dot{a}_{i\sigma}-\sum_{\nu=1}^{\varepsilon}\frac{\partial a_{i\nu}}{\partial q_\sigma}\dot{q}_\nu-\frac{\partial a_i}{\partial q_\sigma}\equiv\xi_{i\sigma}\\
\frac{d}{dt}\frac{\partial(\dot{y}_i)}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{y}_i)}{\partial q_\sigma}&=\dot{b}_{i\sigma}-\sum_{\nu=1}^{\varepsilon}\frac{\partial b_{i\nu}}{\partial q_\sigma}\dot{q}_\nu-\frac{\partial b_i}{\partial q_\sigma}\equiv\eta_{i\sigma}\\
\frac{d}{dt}\frac{\partial(\dot{z}_i)}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{z}_i)}{\partial q_\sigma}&=\dot{c}_{i\sigma}-\sum_{\nu=1}^{\varepsilon}\frac{\partial c_{i\nu}}{\partial q_\sigma}\dot{q}_\nu-\frac{\partial c_i}{\partial q_\sigma}\equiv\zeta_{i\sigma}
\end{aligned}\right\} \tag{2.16}$$

因此

$$\sum_{i=1}^{N}m_i(\dot{\vec{r}}_i)\cdot\left(\frac{d}{dt}\frac{\partial(\dot{\vec{r}}_i)}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\vec{r}}_i)}{\partial q_\sigma}\right)\sum_{i=1}^{N}m_i(\dot{x}_i\xi_{i\sigma}+\dot{y}_i\eta_{i\sigma}+\dot{z}_i\zeta_{i\sigma})=W_\sigma \tag{2.17}$$

将(2.17)代入(2.13)，我们得到

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}=\tilde{Q}_\sigma+W_\sigma\qquad(\sigma=1,2,\cdots,\varepsilon) \tag{2.18}$$

这就是Mac-Millan方程。

如果系统是完整的，则方程(2.13)成为第二类Lagrange方程。

三、广义Mac-Millan方程与广义Чаплыгин方程的等价性

现在我们由广义 Mac-Millan 方程(2.13)来推导广义 Чаплыгин 方程。我们变换方程(2.13)右边第二项。我们有

$$\frac{\partial(\dot{\vec{r}}_i)}{\partial\dot{q}_\sigma}=\frac{\partial\vec{r}_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial\vec{r}_i}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}$$

因此

$$\frac{d}{dt}\frac{\partial(\dot{\vec{r}}_i)}{\partial \dot{q}_\sigma}=\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_{\varepsilon+\beta}}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}+\sum_{\beta=1}^{g}\frac{\partial \vec{r}_i}{\partial q_{\varepsilon+\beta}}\frac{d}{dt}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}$$

以及

$$\frac{\partial(\dot{\vec{r}}_i)}{\partial q_\sigma}=\frac{\partial \dot{\vec{r}}_i}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma}$$

注意到经典Lagrange关系

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_s}=\frac{\partial \dot{\vec{r}}_i}{\partial q_s},\quad \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_s}=\frac{\partial \vec{r}_i}{\partial q_s}\cdot$$

我们有

$$\begin{aligned}\sum_{i=1}^{N}m_i(\dot{\vec{r}}_i)\cdot\left(\frac{d}{dt}\frac{\partial(\dot{\vec{r}}_i)}{\partial \dot{q}_\sigma}-\frac{\partial(\dot{\vec{r}}_i)}{\partial q_\sigma}\right)&=\sum_{i=1}^{N}m_i(\dot{\vec{r}}_i)\cdot\left\{\sum_{\beta=1}^{g}\frac{\partial \dot{\vec{r}}_i}{\partial q_{\varepsilon+\beta}}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}+\sum_{\beta=1}^{g}\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_{\varepsilon+\beta}}\right.\\&\left.\cdot\left(\frac{d}{dt}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}-\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma}\right)\right\}=\sum_{\beta=1}^{g}\frac{\partial T}{\partial q_{\varepsilon+\beta}}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}\\&+\sum_{\beta=1}^{g}\frac{\partial T}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{d}{dt}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}-\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma}\right)\end{aligned}\tag{3.1}$$

将(3.1)代入方程(2.13)，得到

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\sigma}-\frac{\partial \tilde{T}}{\partial q_\sigma}=\tilde{Q}_\sigma+\sum_{\beta=1}^{g}\frac{\partial T}{\partial q_{\varepsilon+\beta}}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}+\sum_{\beta=1}^{g}\frac{\partial T}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{d}{dt}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}-\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma}\right)\tag{3.2}$$

$$(\sigma=1,2,\cdots,\varepsilon)$$

方程(3.2)是牛青萍1964年得到的[6]。

注意到

$$\frac{\partial \tilde{T}}{\partial q_{\varepsilon+\beta}}=\frac{\partial T}{\partial q_{\varepsilon+\beta}}+\sum_{\gamma=1}^{g}\frac{\partial T}{\partial \dot{q}_{\varepsilon+\gamma}}\frac{\partial \dot{q}_{\varepsilon+\gamma}}{\partial q_{\varepsilon+\beta}}$$

方程(3.2)可写成形式

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\sigma}-\frac{\partial \tilde{T}}{\partial q_\sigma}=\tilde{Q}_\sigma+\sum_{\beta=1}^{g}\frac{\partial \tilde{T}}{\partial q_{\varepsilon+\beta}}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}+\sum_{\beta=1}^{g}\frac{\partial T}{\partial \dot{q}_{\varepsilon+\beta}}T_\sigma^{\varepsilon+\beta}\quad(\sigma=1,2,\cdots,\varepsilon)\tag{3.3}$$

其中

$$T_\sigma^{\varepsilon+\beta}=\frac{d}{dt}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_\sigma}-\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_\sigma}-\sum_{\gamma=1}^{g}\frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial q_{\varepsilon+\gamma}}\frac{\partial \dot{q}_{\varepsilon+\gamma}}{\partial \dot{q}_\sigma}$$

方程(3.3)称为非线性非完整系统的广义 Чаплыгин方程[8][9]，是 B.C. Новоселов 1957年得到的。这就证明了广义Mac-Millan方程(2.13)与广义Чаплыгин方程(3.3)的等价性。

四、例 子

一质量为m的质点，受有速度大小为常数的非完整约束，在Newton中心引力场中运动[10]。

我们利用广义Mac-Millan方程(2.13)来组成质点的运动微分方程。

以引力中心为坐标原点，取球坐标r,φ,θ为广义坐标，则点的直角坐标为

$$x=r\cos\theta\cos\varphi,\quad y=r\cos\theta\sin\varphi,\quad z=r\sin\theta \tag{4.1}$$

点的速度投影为

$$\left.\begin{aligned}\dot{x}&=\dot{r}\cos\theta\cos\varphi-r\dot{\theta}\sin\theta\cos\varphi-r\dot{\varphi}\cos\theta\sin\varphi\\ \dot{y}&=\dot{r}\cos\theta\sin\varphi-r\dot{\theta}\sin\theta\sin\varphi+r\dot{\varphi}\cos\theta\cos\varphi\\ \dot{z}&=\dot{r}\sin\theta+r\dot{\theta}\cos\theta\end{aligned}\right\} \tag{4.2}$$

约束方程为

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\varphi}^2\cos^2\theta=c^2=\text{const} \tag{4.3}$$

取$\dot{r}$，$\dot{\theta}$为独立的广义速度，则方程(4.3)可写成形式

$$\dot{\varphi}^2=\frac{c^2-\dot{r}^2-r^2\dot{\theta}^2}{r^2\cos^2\theta} \tag{4.4}$$

质点的动能，在不计非完整约束时为

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\varphi}^2\cos^2\theta)$$

考虑到非完整约束(4.3)时为

$$\tilde{T}=\frac{1}{2}mc^2=\text{const}$$

于是方程(2.13)的左边变为零。

因广义力为

$$Q_\theta=Q_\varphi=0,\quad Q_r=-\frac{mM\gamma}{r^2}$$

其中$M\gamma$为引力常数，因此有

$$\tilde{Q}_\theta=Q_\theta+Q_\varphi\frac{\partial\dot{\varphi}}{\partial\dot{\theta}}=0,\qquad \tilde{Q}_r=Q_r+Q_\varphi\frac{\partial\dot{\varphi}}{\partial\dot{r}}=-\frac{mM\gamma}{r^2} \tag{4.5}$$

现在计算方程(2.13)右边第二项。计算时需将$\dot{\varphi}$按(4.4)考虑作为$\dot{r}$，$\dot{\theta}$，r，θ的函数。由(4.2)有

$$\frac{\partial(\dot{x})}{\partial\dot{r}}=\cos\theta\cos\varphi-r\cos\theta\sin\varphi\frac{\partial\dot{\varphi}}{\partial\dot{r}}$$

$$\frac{\partial(\dot{x})}{\partial r}=-\dot{\theta}\sin\theta\cos\varphi-\dot{\varphi}\cos\theta\sin\varphi-r\cos\theta\sin\varphi\frac{\partial\dot{\varphi}}{\partial r}$$

因此

$$\mathscr{E}_r((\dot{x}))=\frac{d}{dt}\frac{\partial(\dot{x})}{\partial\dot{r}}-\frac{\partial(\dot{x})}{\partial r}=-r\cos\theta\sin\varphi\left(\frac{d}{dt}\frac{\partial\dot{\varphi}}{\partial\dot{r}}-\frac{\partial\dot{\varphi}}{\partial r}\right)-\frac{\partial\dot{\varphi}}{\partial\dot{r}}\frac{d}{dt}(r\cos\theta\sin\varphi) \tag{4.6}$$

类似地，计算得

$$\left.\begin{aligned}&\mathscr{E}_r((\dot y))=r\cos\theta\cos\varphi\left(\frac{d}{dt}\frac{\partial\dot\varphi}{\partial\dot r}-\frac{\partial\dot\varphi}{\partial r}\right)+\frac{\partial\dot\varphi}{\partial\dot r}\frac{d}{dt}(r\cos\theta\cos\varphi)\\&\mathscr{E}_r((\dot z))=0\end{aligned}\right\}\tag{4.7}$$

以及

$$\left.\begin{aligned}&\mathscr{E}_\theta((\dot x))=-r\cos\theta\sin\varphi\left(\frac{d}{dt}\frac{\partial\dot\varphi}{\partial\dot\theta}-\frac{\partial\dot\varphi}{\partial\theta}\right)-\frac{\partial\dot\varphi}{\partial\dot\theta}\frac{d}{dt}(r\cos\theta\sin\varphi)\\&\mathscr{E}_\theta((\dot y))=r\cos\theta\cos\varphi\left(\frac{d}{dt}\frac{\partial\dot\varphi}{\partial\dot\theta}-\frac{\partial\dot\varphi}{\partial\theta}\right)+\frac{\partial\dot\varphi}{\partial\dot\theta}\frac{d}{dt}(r\cos\theta\cos\varphi)\\&\mathscr{E}_\theta((\dot z))=0\end{aligned}\right\}\tag{4.8}$$

广义Mac-Millan 方程写成下述形式

$$\left.\begin{aligned}&m[(\dot x)\mathscr{E}_r((\dot x))+(\dot y)\mathscr{E}_r((\dot y))+(\dot z)\mathscr{E}_r((\dot z))]+\tilde Q_r=0\\&m[(\dot x)\mathscr{E}_\theta((\dot x))+(\dot y)\mathscr{E}_\theta((\dot y))+(\dot z)\mathscr{E}_\theta((\dot z))]+\tilde Q_\theta=0\end{aligned}\right\}\tag{4.9}$$

将(4.2)、(4.5)、(4.6)、(4.7)及(4.8)代入(4.9)并化简，得

$$\left.\begin{aligned}&mr^2\dot\varphi\cos^2\theta\left(\frac{d}{dt}\frac{\partial\dot\varphi}{\partial\dot r}-\frac{\partial\dot\varphi}{\partial r}\right)=-\frac{mM\gamma}{r^2}\\&mr^2\dot\varphi\cos^2\theta\left(\frac{d}{dt}\frac{\partial\dot\varphi}{\partial\dot\theta}-\frac{\partial\dot\varphi}{\partial\theta}\right)=0\end{aligned}\right\}\tag{4.10}$$

由(4.4)，得

$$\left.\begin{aligned}&\frac{\partial\dot\varphi}{\partial\dot r}=-\frac{\dot r}{\dot\varphi r^2\cos^2\theta},\quad\frac{\partial\dot\varphi}{\partial r}=-\frac{\dot\theta^2+\dot\varphi^2\cos^2\theta}{\dot\varphi r\cos^2\theta}\\&\frac{\partial\dot\varphi}{\partial\dot\theta}=-\frac{\dot\theta}{\dot\varphi\cos^2\theta},\quad\frac{\partial\dot\varphi}{\partial\theta}=\dot\varphi\,\mathrm{tg}\theta\end{aligned}\right\}\tag{4.11}$$

将(4.11)代入(4.10)并化简，我们最终得到

$$\left.\begin{aligned}&\ddot r-\frac{\dot r}{\dot\varphi}\ddot\varphi-r\dot\theta^2-\frac{2\dot r^2}{r}+2\dot r\dot\theta\,\mathrm{tg}\theta-r\dot\varphi^2\cos^2\theta=-\frac{M\gamma}{r^2}\\&r^2\ddot\theta-r^2\frac{\dot\theta}{\dot\varphi}\ddot\varphi+2r^2\dot\theta^2\mathrm{tg}\theta+r^2\dot\varphi^2\sin\theta\cos\theta=0\end{aligned}\right\}\tag{4.12}$$

方程(4.12)与文献[10]和[11]中用其它方法所得结果一致。

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Extension of the Mac-Millan's Equations to Nonlinear Nonholonomic Mechanical Systems

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Abstract

In this article, the Mac-Millan's equations are extended to the most general nonholonomic mechanical systems and the generalized Mac-Millan's equations for nonlinear nonholonomic systems are obtained. And then the equivalence between the generalized Mac-Millan's equations and the generalized Chaplygin's equations is demonstrated. Finally an example is given.