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对称正交铺设矩形叠层板的非线性弯曲*

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摘 要

本文应用[1]中提出的奇异摄动方法，在[3]的基础上，研究了在各种支承条件下承受均布载荷的对称正交铺设矩形叠层板的非线性弯曲问题，导出了挠度和应力函数的一致有效的 N 阶形式渐近解．对承受均布压力，边界位移为零的简支矩形板进行了分析、计算．

一、问题的提出

随着复合材料在各个部门得到越来越广泛的应用，对复合材料结构的力学分析也就提出了更迫切的要求．复合材料的优越性使它更适用于大跨度的结构．因此，本文应用江福汝提出的新的构造边界层的方法[1,2]，在研究了正交异性矩形薄板的非线性弯曲后[3]，进一步研究了承受均布荷载的对称正交铺设矩形叠层板在各种支承条件下的非线性弯曲，简化了这类问题[4,5]．

假设所研究的矩形叠层板宽a，长b，厚度为t，承受横向均布载荷q；并设共有N层单向增强（正交异性）层．其材料主方向与层合板坐标轴交错成0°和90°角．奇数层的纤维方向是层合板的x轴向，偶数层的纤维方向是层合板的y轴向（图1）．

（1） **基本方程** 对称正交铺设叠层板的本构方程为

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \tag{1.1}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} \tag{1.2}$$

因此，中面应变[6,7]

* 江福汝推荐．本文曾在1983年“全国奇异摄动理论及应用”学术会议上宣读．

$$\left\{\begin{matrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{matrix}\right\}=\begin{bmatrix}A_{11} & A_{12} & 0\\ A_{12} & A_{22} & 0\\ 0 & 0 & A_{66}\end{bmatrix}^{-1}\left\{\begin{matrix}N_x\\ N_y\\ N_{xy}\end{matrix}\right\}=t\begin{bmatrix}A_{11}^* & A_{12}^* & 0\\ A_{12}^* & A_{22}^* & 0\\ 0 & 0 & A_{66}^*\end{bmatrix}\left\{\begin{matrix}\dfrac{\partial^2\varphi}{\partial y^2}\\ \dfrac{\partial^2\varphi}{\partial x^2}\\ -\dfrac{\partial^2\varphi}{\partial x\partial y}\end{matrix}\right\} \tag{1.3}$$

对于常用的叠层板，刚度A_{ij}和D_{ij}可用总层数N，正交铺设比M和主单层刚度比F表达[7]。对称正交铺设叠层板非线性弯曲的von Kármán方程为

$$\begin{bmatrix}L_1^* & 0\\ 0 & L_2^*\end{bmatrix}\left\{\begin{matrix}w\\ \varphi\end{matrix}\right\}=\left\{\begin{matrix}L_3(\varphi,w)+q\\ -L_3(w,w)/2\end{matrix}\right\} \tag{1.4}$$

形中，算子

$$\left.\begin{aligned}L_1^*&=D_{11}\frac{\partial^4}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4}{\partial y^4}\\ L_2^*&=A_{22}^*\frac{\partial^4}{\partial x^4}+(2A_{12}^*+A_{66}^*)\frac{\partial^4}{\partial x^2\partial y^2}+A_{11}^*\frac{\partial^4}{\partial y^4}\end{aligned}\right\} \tag{1.5}$$

$$L_3=\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}-2\frac{\partial^2}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y} \tag{1.6}$$

引入无量纲量

$$\tilde{w}=\frac{w}{a}\qquad \tilde{x}=\frac{x}{a}\qquad \tilde{y}=\frac{y}{a}\qquad \tilde{\varphi}=\frac{\varphi A_{11}^*}{a^2}\qquad \tilde{q}=aA_{11}^*q \tag{1.7}$$

方程(1.4)化为(略去字母上的"～"号)

$$\begin{bmatrix}\varepsilon^2L_1 & 0\\ 0 & L_2\end{bmatrix}\left\{\begin{matrix}w\\ \varphi\end{matrix}\right\}=\left\{\begin{matrix}L_3(\varphi,w)+q\\ -\dfrac{1}{2}L_3(w,w)\end{matrix}\right\} \tag{1.8}$$

式中

$$\varepsilon^2=\frac{[(F-1)P+1]t^2}{12\left[\dfrac{M+F}{1+M}-\dfrac{1+M}{1+MF}\gamma_{21}^2\right]a^2} \tag{1.9}$$

$$\left.\begin{aligned}L_1&=\frac{\partial^4}{\partial x^4}+2\frac{D_{12}+2D_{66}}{D_{11}}\frac{\partial^4}{\partial x^2\partial y^2}+\frac{D_{22}}{D_{11}}\frac{\partial^4}{\partial y^4}\\ L_2&=\frac{A_{22}^*}{A_{11}^*}\frac{\partial^4}{\partial x^4}+\frac{2A_{12}^*+A_{66}^*}{A_{11}^*}\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\end{aligned}\right\} \tag{1.10}$$

(2) **边界条件** 假设挠度函数w的边界条件为

$$w\Big|_{x=0}=f_1(y),\qquad \frac{\partial w}{\partial x}\Big|_{x=0}=g_1(y) \tag{1.11}$$

$$w\Big|_{x=1}=f_2(y),\qquad \frac{\partial w}{\partial x}\Big|_{x=1}=g_2(y) \tag{1.12}$$

$$w\Big|_{y=0}=f_3(x),\qquad -\frac{D_{21}}{D_{22}}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\Big|_{y=0}=g_3(x) \tag{1.13}$$

$$w\Big|_{y=\frac{b}{a}}=f_4(x),\qquad -\frac{D_{21}}{D_{22}}\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\Big|_{y=\frac{b}{a}}=g_4(x) \tag{1.14}$$

应力函数φ的边界条件为

$$\frac{\partial^2\varphi}{\partial y^2}\bigg|_{x=0}=h_1(y),\qquad \frac{\partial^2\varphi}{\partial x\partial y}\bigg|_{x=0}=I_1(y) \tag{1.15}$$

$$\frac{\partial^2\varphi}{\partial y^2}\bigg|_{x=1}=h_2(y),\qquad \frac{\partial^2\varphi}{\partial x\partial y}\bigg|_{x=1}=I_2(y) \tag{1.16}$$

$$\frac{\partial^2\varphi}{\partial x^2}\bigg|_{y=0}=h_3(x),\qquad \frac{\partial^2\varphi}{\partial x\partial y}\bigg|_{y=0}=I_3(x) \tag{1.17}$$

$$\frac{\partial^2\varphi}{\partial x^2}\bigg|_{y=\frac{b}{a}}=h_4(x),\qquad \frac{\partial^2\varphi}{\partial x\partial y}\bigg|_{y=\frac{b}{a}}=I_4(x) \tag{1.18}$$

或

$$\int_0^{b/a} t\frac{\partial^2\varphi}{\partial y^2}dy=\bar{P}_x\frac{b}{a}t \tag{1.19}$$

$$\int_0^1 t\frac{\partial^2\varphi}{\partial x^2}dx=\bar{P}_y t \tag{1.20}$$

$$\int_0^1\left[\frac{\partial^2\varphi}{\partial y^2}+\frac{A_{11}^*}{A_{12}^*}\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx=\delta_x \tag{1.21}$$

$$\int_0^{b/a}\left[\frac{A_{21}^*}{A_{11}^*}\frac{\partial^2\varphi}{\partial y^2}+\frac{A_{22}^*}{A_{11}^*}\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dy=\delta_y \tag{1.22}$$

式中，$\bar{P}_x\frac{b}{a}t$和$\bar{P}_y t$是常数，为在板平面内，在x方向和在y方向的合力；δ_x和δ_y则为板在x方向和在y方向的伸长。

二、递推方程和边界条件

在边界$x=0$和$x=1$的邻域分别引入边界层函数$v_n^{(1)}, h_n^{(1)}$和$v_n^{(2)}$，$h_n^{(2)}$；在边界$y=0$和$y=b/a$的邻域分别引进边界层函数$v_n^{(3)}$，$h_n^{(3)}$和$v_n^{(4)}$，$h_n^{(4)}$。假设挠度w和应力函数φ的N阶近似式为

$$W_N(x,y,\varepsilon)=\sum_{n=0}^N\varepsilon^n w_n(x,y)+\sum_{n=0}^N\varepsilon^{n+\alpha_1}v_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^N\varepsilon^{n+\alpha_2}v_n^{(2)}(\bar{\xi},\bar{\eta},y)$$
$$+\sum_{n=0}^N\varepsilon^{n+\alpha_3}v_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^N\varepsilon^{n+\alpha_4}v_n^{(4)}(x,\bar{\alpha},\bar{\beta}) \tag{2.1}$$

$$\Phi_N(x,y,\varepsilon)=\sum_{n=0}^N\varepsilon^n\varphi_n(x,y)+\sum_{n=0}^N\varepsilon^{n+\beta_1}h_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^N\varepsilon^{n+\beta_2}h_n^{(2)}(\bar{\xi},\bar{\eta},y)$$
$$+\sum_{n=0}^N\varepsilon^{n+\beta_3}h_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^N\varepsilon^{n+\beta_4}h_n^{(4)}(x,\bar{\alpha},\bar{\beta}) \tag{2.2}$$

式中$\alpha_1,\alpha_2,\alpha_3,\alpha_4$和$\beta_1,\beta_2,\beta_3,\beta_4$为待定常数。

将(2.1)式代入边界条件(1.11)～(1.14)，我们看到应取$\alpha_1=\alpha_2=1$，$\alpha_3=\alpha_4=2$。逐次地比较等式两端ε的各次幂的系数，我们得到关于w_n，$v_n^{(1)}$，$v_n^{(2)}$，$v_n^{(3)}$ 和 $v_n^{(4)}$的边界条件[3]：

$$\left.\begin{aligned}
&w_0|_{x=0}=f_1(y) \qquad w_0|_{x=1}=f_2(y)\\
&w_0|_{y=0}=f_3(x) \qquad w_0\Big|_{y=\frac{b}{a}}=f_4(x)\\
&w_{0,x}|_{x=0}+\delta_{1,0}v_0^{(1)}|_{\eta=0}=g_1(y)\\
&w_{0,x}|_{x=1}+\tilde{\delta}_{1,0}v_0^{(2)}|_{\tilde{\eta}=1}=g_2(y)\\
&-\left\{\left[w_{0,yy}+\frac{D_{12}}{D_{22}}w_{0,xx}\right]_{y=0}+\gamma_{2,0}v_0^{(3)}\Big|_{\beta=0}\right\}=g_3(x)\\
&-\left\{\left[w_{0,yy}+\frac{D_{12}}{D_{22}}w_{0,xx}\right]_{y=\frac{b}{a}}+\tilde{\gamma}_{2,0}v_0^{(4)}\Big|_{\tilde{\beta}=\frac{b}{a}}\right\}=g_4(x)
\end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned}
&w_n|_{x=0}+v_{n-1}^{(1)}|_{\eta=0}=0 \qquad w_n|_{x=1}+v_{n-1}^{(2)}|_{\tilde{\eta}=1}=0\\
&w_n|_{y=0}+v_{n-2}^{(3)}|_{\beta=0}=0 \qquad w_n\Big|_{y=\frac{b}{a}}+v_{n-2}^{(4)}\Big|_{\tilde{\beta}=\frac{b}{a}}=0\\
&w_{n,x}|_{x=0}+(\delta_{1,0}v_n^{(1)}+\delta_{1,1}v_{n-1}^{(1)})|_{\eta=0}=0\\
&w_{n,x}|_{x=1}+(\tilde{\delta}_{1,0}v_n^{(2)}+\tilde{\delta}_{1,1}v_{n-1}^{(2)})|_{\tilde{\eta}=1}=0\\
&\left[w_{n,yy}+\frac{D_{12}}{D_{22}}w_{n,xx}\right]_{y=0}+(\gamma_{2,0}v_n^{(3)}+\gamma_{2,1}v_{n-1}^{(3)}+\gamma_{2,2}v_{n-2}^{(3)})_{\beta=0}=0\\
&\left[w_{n,yy}+\frac{D_{12}}{D_{22}}w_{n,xx}\right]_{y=\frac{b}{a}}+(\tilde{\gamma}_{2,0}v_n^{(4)}+\tilde{\gamma}_{2,1}v_{n-1}^{(4)}+\tilde{\gamma}_{2,2}v_{n-2}^{(4)})_{\tilde{\beta}=\frac{b}{a}}=0
\end{aligned}\right\} \tag{2.4}$$

$$(n=1,2,\cdots,N)$$

在上式和以后的计算中，我们都将带负下标的量取为零。将表达式(2.1)和 (2.2) 代入(1.8)式，得到w_n和φ_n的递推方程

$$L_3(w_0,\varphi_0)=-q, \qquad L_2\varphi_0+\frac{1}{2}L_3(w_0,w_0)=0 \tag{2.5}$$

$$\left.\begin{aligned}
&L_3(w_0,\varphi_n)+L_3(w_n,\varphi_0)=L_1w_{n-2}-\sum_{i=1}^{n-1}L_3(w_i,\varphi_{n-i})\\
&L_2\varphi_n+L_3(w_0,w_n)=-\frac{1}{2}\sum_{i=1}^{n-1}L_3(w_i,w_{n-i})
\end{aligned}\right\} \tag{2,6}$$

$$(n=1,2,\cdots,N)$$

并且，我们看到应取$\beta_1=\beta_2=3$，$\beta_3=\beta_4=4$。于是，边界层函数$v_n^{(i)}$和$h_n^{(i)}$($i=1,\cdots,4$)的递推方程可以导出（见[3]）。最后，将(2.1)和 (2.2) 式代入方程(1.15)～(1.22)，得到 φ_n，$h_n^{(1)}$，$h_n^{(2)}$，$h_n^{(3)}$和$h_n^{(4)}$的边界条件

$$\left.\begin{aligned}
&\varphi_{0,yy}|_{x=0}=h_1(y) \qquad \varphi_{0,xy}|_{x=0}=I_1(y)\\
&\varphi_{0,yy}|_{x=1}=h_2(y) \qquad \varphi_{0,xy}|_{x=1}=I_2(y)\\
&\varphi_{0,xx}|_{y=0}=h_3(x) \qquad \varphi_{0,xy}|_{y=0}=I_3(x)\\
&\varphi_{0,xx}\Big|_{y=\frac{b}{a}}=h_4(x) \qquad \varphi_{0,xy}\Big|_{y=\frac{b}{a}}=I_4(x)
\end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned}
&\varphi_{n,yy}|_{x=0}+h^{(1)}_{n-3,yy}|_{\eta=0}=0\\
&\varphi_{n,xy}|_{x=0}+[\delta_{1,0}h^{(1)}_{n-2,y}+\delta_{1,1}h^{(1)}_{n-3,y}]_{\eta=0}=0\\
&\varphi_{n,yy}|_{x=1}+h^{(2)}_{n-3,yy}|_{\bar\eta=1}=0\\
&\varphi_{n,xy}|_{x=1}+[\tilde\delta_{1,0}h^{(2)}_{n-2,y}+\tilde\delta_{1,1}h^{(2)}_{n-3,y}]_{\bar\eta=1}=0\\
&\varphi_{n,xx}|_{y=0}+h^{(3)}_{n-4,xx}|_{\beta=0}=0\\
&\varphi_{n,xy}|_{y=0}+[\gamma_{1,0}h^{(3)}_{n-3,x}+\gamma_{1,1}h^{(3)}_{n-4,x}]_{\beta=0}=0\\
&\varphi_{n,xx}\Big|_{y=\frac{b}{a}}+h^{(4)}_{n-4,xx}\Big|_{\bar\beta=\frac{b}{a}}=0\\
&\varphi_{n,xy}\Big|_{y=\frac{b}{a}}+\left[\tilde\gamma_{1,0}h^{(4)}_{n-3,x}+\tilde\gamma_{1,1}h^{(4)}_{n-4,x}\right]_{\bar\beta=\frac{b}{a}}=0\\
&\qquad\qquad (n=1,2,\cdots,N)
\end{aligned}\right\}\tag{2.8}$$

或

$$\left.\begin{aligned}
&\int_0^{b/a}t\varphi_{0,yy}dy=\bar P_x\frac{b}{a}t,\quad \int_0^1 t\varphi_{0,xx}dx=\bar P_y t\\
&\int_0^{b/a}\left[\frac{A^*_{21}}{A^*_{11}}\varphi_{0,yy}+\frac{A^*_{22}}{A^*_{11}}\varphi_{0,xx}-\frac{1}{2}(w_{0,y})^2\right]dy=\delta_y\\
&\int_0^1\left[\varphi_{0,yy}+\frac{A^*_{12}}{A^*_{11}}\varphi_{0,xx}-\frac{1}{2}(w_0,x)^2\right]dx=\delta_x
\end{aligned}\right\}\tag{2.9}$$

$$\left.\begin{aligned}
&\int_0^{b/a}t[\varphi_{n,yy}+h^{(1)}_{n-3,yy}]dy=0,\quad \int_0^{b/a}t[\varphi_{n,yy}+h^{(2)}_{n-3,yy}]dy=0\\
&\int_0^1 t[\varphi_{n,xx}+h^{(3)}_{n-4,yy}]dx=0,\quad \int_0^1 t[\varphi_{n,xx}+h^{(4)}_{n-4,yy}]dx=0\\
&\int_0^1\Big\{\varphi_{n,yy}+h^{(3)}_{n-4,yy}+\frac{A^*_{12}}{A^*_{11}}(\varphi_{n,xx}+h^{(3)}_{n-4,xx})\\
&\qquad-\frac{1}{2}\Big[w^2_{\frac{n}{2},x}+2\sum_{i=0}^{n}(w_{i,x}w_{n-i,x}+w_{i,x}v^{(3)}_{n-2-i,x})\Big]\Big\}dx=0\\
&\int_0^1\Big\{\varphi_{n,yy}+h^{(4)}_{n-4,yy}+\frac{A^*_{12}}{A^*_{11}}(\varphi_{n,xx}+h^{(4)}_{n-4,xx})\\
&\qquad-\frac{1}{2}\Big[w^2_{\frac{n}{2},x}+2\sum_{i=0}^{n}(w_{i,x}w_{n-i,x}+w_{i,x}v^{(4)}_{n-2-i,x})\Big]\Big\}dx=0\\
&\int_0^{b/a}\Big\{\frac{A^*_{21}}{A^*_{11}}(\varphi_{n,yy}+h^{(1)}_{n-3,yy})+\frac{A^*_{22}}{A^*_{11}}(\varphi_{n,xx}+h^{(1)}_{n-3,xx})\\
&\qquad-\frac{1}{2}\Big[w^2_{\frac{n}{2},x}+2\sum_{i=0}^{n}(w_{i,y}w_{n-i,y}+w_{i,y}v^{(1)}_{n-1-i,y})\Big]\Big\}dy=0\\
&\int_0^{b/a}\Big\{\frac{A^*_{21}}{A^*_{11}}(\varphi_{n,yy}+h^{(2)}_{n-3,yy})+\frac{A^*_{22}}{A^*_{11}}(\varphi_{n,xx}+h^{(2)}_{n-3,xx})\\
&\qquad-\frac{1}{2}\Big[w^2_{\frac{n}{2},y}+2\sum_{i=0}^{n}(w_{i,y}w_{n-i,y}+w_{i,y}v^{(2)}_{n-1-i,y})\Big]\Big\}dy=0\\
&\qquad\qquad (n=1,2,\cdots,N)
\end{aligned}\right\}\tag{2.10}$$

三、N阶形式渐近解的导出

方程(2.5)和边界条件(2.3)，(2.7)、(2.9)给出退化边值问题

$$L_3(w_0,\varphi_0)=-q,\qquad L_2\varphi_0+\frac{1}{2}L_3(w_0,w_0)=0 \tag{3.1}$$

$$\left.\begin{array}{ll} w_0|_{x=0}=f_1(y), & w_0|_{x=1}=f_2(y) \\ w_0|_{y=0}=f_3(x), & w_0\Big|_{y=\frac{b}{a}}=f_4(x) \end{array}\right\} \tag{3.2}$$

$$\left.\begin{array}{ll} \varphi_{0,yy}|_{x=0}=h_1(y), & \varphi_{0,yy}|_{x=1}=h_2(y) \\ \varphi_{0,xx}|_{y=0}=h_3(x), & \varphi_{0,xx}\Big|_{y=\frac{b}{a}}=h_4(x) \end{array}\right\} \tag{3.3}$$

或

$$\left.\begin{array}{l} \int_0^{b/a} t\varphi_{0,yy}dy=\bar{P}_x\,\frac{b}{a}t,\quad \int_0^1 t\varphi_{0,xx}dx=\bar{P}_y t \\ \int_0^1\left[\varphi_{0,yy}+\frac{A_{12}^*}{A_{11}^*}\varphi_{0,xx}-\frac{1}{2}w_{0,x}^2\right]dx=\delta_x \\ \int_0^{b/a}\left[\frac{A_{21}^*}{A_{11}^*}\varphi_{0,yy}+\frac{A_{22}^*}{A_{11}^*}\varphi_{0,xx}-\frac{1}{2}w_{0,y}^2\right]dy=\delta_y \end{array}\right\} \tag{3.4}$$

解此边值问题，将求得的薄膜解w_0,φ_0代入$v_0^{(i)}$及$h_0^{(i)}(i=1,\cdots,4)$之方程[3]，利用边界条件(2.3)，在$h_1(y)>0$，$h_2(y)>0$和$h_3(x)>0$，$h_4(x)>0$的条件下，得到边界层函数

$$\left.\begin{array}{l} v_0^{(1)}(\xi,\eta,y)=C_0^{(1)}(\eta,y)e^{-\xi}=C_0^{(1)}(\eta,y)\exp\left[-\frac{1}{\varepsilon}\int_0^x\sqrt{\varphi_{0,yy}(x,y)}\,dx\right] \\ v_0^{(2)}(\tilde{\xi},\tilde{\eta},y)=C_0^{(2)}(\tilde{\eta},y)e^{-\tilde{\xi}}=C_0^{(2)}(\tilde{\eta},y)\exp\left[-\frac{1}{\varepsilon}\int_x^1\sqrt{\varphi_{0,yy}(x,y)}\,dx\right] \\ v_0^{(3)}(x,\alpha,\beta)=C_0^{(3)}(x,\beta)\exp\left[-\frac{\alpha}{\sqrt{e_1}}\right]=C_0^{(3)}(x,\beta)\exp\left[-\frac{1}{\varepsilon\sqrt{e_1}}\int_0^y\sqrt{\varphi_{0,xx}(x,y)}\,dy\right] \\ v_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})=C_0^{(4)}(x,\tilde{\beta})\exp\left[-\frac{\tilde{\alpha}}{\sqrt{e_1}}\right]=C_0^{(4)}(x,\tilde{\beta})\exp\left[-\frac{1}{\varepsilon\sqrt{e_1}}\int_y^{b/a}\sqrt{\varphi_{0,xx}(x,y)}\,dy\right] \end{array}\right\} \tag{3.5}$$

$$\left.\begin{array}{l} h_0^{(1)}(\xi,\eta,y)=-A^{-1}(\eta,y)w_{0,yy}C_0^{(1)}(\eta,y)e^{-\xi} \\ h_0^{(2)}(\tilde{\xi},\tilde{\eta},y)=-A^{-1}(\tilde{\eta},\ y)w_{0,yy}\,C_0^{(2)}(\tilde{\eta},y)e^{-\tilde{\xi}} \\ h_0^{(3)}(x,\ \alpha,\ \beta)=-e_1^{-1}B^{-1}(x,\alpha)w_{0,xx}C_0^{(3)}(x,\alpha)e^{-\alpha} \\ h_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})=-e_1^{-1}B^{-1}(x,\tilde{\alpha})w_{0,xx}\,C_0^{(4)}(x,\tilde{\alpha})e^{-\tilde{\alpha}} \end{array}\right\} \tag{3.6}$$

式中$e_1=D_{22}/D_{11}$，$A(x,y)=\varphi_{0,yy}$，$B(x,y)=\varphi_{0,xx}$；$C_0^{(1)}(\eta,y)$，$\cdots$，$C_0^{(4)}(x,\tilde{\beta})$为待定函数，其边值条件为

$$\left.\begin{aligned}
&C_0^{(1)}(\eta,y)\Big|_{\eta=0}=-\frac{g_1(y)-w_{0,x}(0,y)}{\sqrt{h_1(y)}},\quad C_0^{(2)}(\tilde{\eta},y)\Big|_{\tilde{\eta}=1}=-\frac{g_2(y)-w_{0,x}(1,y)}{\sqrt{h_2(y)}}\\
&C_0^{(3)}(x,\beta)\Big|_{\beta=0}=-e_1h_3^{-1}(x)\left\{g_3(x)+\left[w_{0,yy}(x,0)+\frac{D_{12}}{D_{22}}w_{0,xx}(x,0)\right]\right\}\\
&C_0^{(4)}(x,\tilde{\beta})\Big|_{\tilde{\beta}=\frac{b}{a}}=-e_1h_4^{-1}(x)\left\{g_4(x)+\left[w_{0,yy}\left(x,\frac{b}{a}\right)+\frac{D_{12}}{D_{22}}w_{0,xx}\left(x,\frac{b}{a}\right)\right]\right\}
\end{aligned}\right\}\tag{3.7}$$

在(2.6)，(2.4)，(2.8)和(2.10)中，令$n=1$，得到w_1和φ_1的线性边值问题

$$L_3(w_0,\varphi_1)+L_3(w_1,\varphi_0)=0,\quad L_2\varphi_1+L_3(w_0,w_1)=0\tag{3.8}$$

$$\left.\begin{aligned}
&w_1|_{x=0}=-v_0^{(1)}|_{\eta=0}=-C_0^{(1)}(0,y)\\
&w_1|_{x=1}=-v_0^{(2)}|_{\tilde{\eta}=1}=-C_0^{(2)}(1,y)\\
&w_1|_{y=0}=0,\quad w_1\Big|_{y=\frac{b}{a}}=0
\end{aligned}\right\}\tag{3.9}$$

$$\left.\begin{aligned}
&\varphi_{1,yy}|_{x=0}=0,\quad \varphi_{1,yy}|_{x=1}=0\\
&\varphi_{1,xx}|_{y=0}=0,\quad \varphi_{1,xx}\Big|_{y=\frac{b}{a}}=0
\end{aligned}\right\}\tag{3.10}$$

或

$$\left.\begin{aligned}
&\int_0^{b/a}t\varphi_{1,yy}dy=0,\qquad \int_0^1 t\varphi_{1,xx}dx=0\\
&\int_0^1\left[\varphi_{1,yy}+\frac{A_{12}^*}{A_{11}^*}\varphi_{1,xx}-w_{0,x}w_{1,x}-w_{0,x}v_{0,x}^{(1)}\right]dx=0\\
&\int_0^{b/a}\left[\frac{A_{21}^*}{A_{11}^*}\varphi_{1,yy}+\frac{A_{22}^*}{A_{11}^*}\varphi_{1,xx}-w_{0,y}w_{1,x}-w_{0,y}v_{0,y}^{(1)}\right]dy=0
\end{aligned}\right\}\tag{3.11}$$

将$w_0,\varphi_0,v_0^{(1)},\cdots,v_0^{(4)}$和$h_0^{(1)},\cdots,h_0^{(4)}$代入以上方程，求得$w_1$和$\varphi_1$后，代入$v_n^{(4)}$ 的公式[3]，并令各等式右端为零，得到确定 $C_0^{(1)}(\eta,y),\cdots,C_0^{(4)}(x,\tilde{\beta})$的一阶线性偏微分方程

$$\left.\begin{aligned}
&2A\frac{\partial C_0^{(1)}}{\partial\eta}+2\varphi_{0,xy}\frac{\partial C_0^{(1)}}{\partial y}+\left[\frac{5}{2}A_{,x}+\varphi_{1,yy}A^{\frac{1}{2}}\right]C_0^{(1)}=0\\
&2A\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}+2\varphi_{0,xy}\frac{\partial C_0^{(2)}}{dy}+\left[\frac{5}{2}A_{,x}+\varphi_{1,yy}A^{\frac{1}{2}}\right]C_0^{(2)}=0\\
&2B\frac{\partial C_0^{(3)}}{\partial\beta}+2\varphi_{0,xy}\frac{\partial C_0^{(3)}}{\partial x}+\left[\frac{5}{2}B_{,y}+\varphi_{1,xx}\left(\frac{B}{e_1}\right)^{\frac{1}{2}}\right]C_0^{(3)}=0\\
&2B\frac{\partial C_0^{(4)}}{\partial\tilde{\beta}}+2\varphi_{0,xy}\frac{\partial C_0^{(4)}}{\partial x}+\left[\frac{5}{2}B_{,y}+\varphi_{1,xx}\left(\frac{B}{e_1}\right)^{\frac{1}{2}}\right]C_0^{(4)}=0
\end{aligned}\right\}\tag{3.12}$$

在$A(x,y)>0$和$B(x,y)>0$的条件下，由Cauchy条件(3.7)可以唯一地求得$C_0^{(1)}(\eta,\ y),\ \cdots,\ C_0^{(4)}(x,\tilde{\beta})$。至此，边界层函数$v_0^{(1)},\cdots,v_0^{(4)}$；$h_0^{(1)},\cdots,h_0^{(4)}$完全确定。同样运算下去，我们可逐步地求得展开式(2.1)和(2.2)中的w_n，$v_n^{(1)},\cdots,v_n^{(4)}$和$\varphi_n,h_n^{(1)},\cdots,h_n^{(4)}(n=1,2,\cdots,N)$。

四、承受均布法向压力，边界位移为零的简支矩形板

我们以承受均布法向压力q和边缘载荷 $\bar{P}_x\frac{b}{a}t$、$\bar{P}_y t$，使边界位移为零的简支、正交铺设矩形叠层板（图1）为例进行具体的分析和讨论。

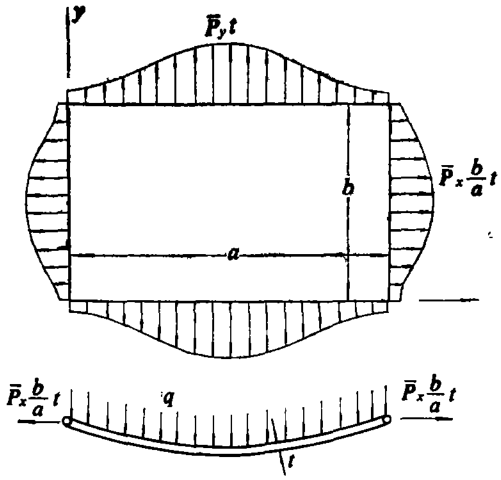


图 1

板的（无量纲）方程为

$$\left.\begin{aligned}&\varepsilon^2 L_1 w - L_3(w,\varphi) = q\\&L_2\varphi + L_3(w,w)/2 = 0\end{aligned}\right\}\tag{4.1}$$

边界条件为

$$\left.\begin{aligned}&w\Big|_{x=0}=w\Big|_{x=1}=w\Big|_{y=0}=w\Big|_{y=\frac{b}{a}}=0\\&\frac{\partial^2 w}{\partial x^2}\Big|_{x=0}=\frac{\partial^2 w}{\partial x^2}\Big|_{x=1}=\frac{\partial^2 w}{\partial y^2}\Big|_{y=0}=\frac{\partial^2 w}{\partial y^2}\Big|_{y=\frac{b}{a}}=0\end{aligned}\right\}\tag{4.2}$$

$$\left.\begin{aligned}&\int_0^{b/a} t\frac{\partial^2\varphi}{\partial y^2}dy=\bar{P}_x\frac{b}{a}t,\qquad \int_0^1 t\frac{\partial^2\varphi}{\partial x^2}dx=\bar{P}_y t\\&\int_0^1\left[\frac{\partial^2\varphi}{\partial y^2}+\frac{A^*_{12}}{A^*_{11}}\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx=0\\&\int_0^{b/a}\left[\frac{A^*_{21}}{A^*_{11}}\frac{\partial^2\varphi}{\partial y^2}+\frac{A^*_{22}}{A^*_{11}}\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dy=0\end{aligned}\right\}\tag{4.3}$$

展开式(2.1)和(2.2)中的首项w_0,φ_0确定于以下退化边值问题($\varepsilon=0$)

$$L_3(w_0,\varphi_0)=-q,\qquad L_2\varphi_0=L_3(w_0,w_0)/2\tag{4.4}$$

$$w_0|_{x=0}=w_0|_{x=1}=w_0|_{y=0}=w_0\Big|_{y=\frac{b}{a}}=0\tag{4.5}$$

$$\int_0^{b/a} t\varphi_{0,yy}dy=\bar{P}_x\frac{b}{a}t,\qquad \int_0^1 t\varphi_{0,xx}dx=\bar{P}_y t\tag{4.6}$$

$$\left.\begin{aligned}&\int_0^1\left[\varphi_{0,yy}+\frac{A_{12}^*}{A_{11}^*}\varphi_{0,xx}-\frac{1}{2}(w_{0,x})^2\right]dx=0\\&\int_0^{b/a}\left[\frac{A_{21}^*}{A_{11}^*}\varphi_{0,yy}+\frac{A_{22}^*}{A_{11}^*}\varphi_{0,xx}-\frac{1}{2}(w_{0,y})^2\right]dy=0\end{aligned}\right\}\tag{4.7}$$

已知均布法向压力可以表为

$$q=\sum_{r=1,3,5}^{\infty}\sum_{s=1,3,5}^{\infty}q_{rs}\sin r\pi x\sin\frac{as\pi}{b}y\tag{4.8}$$

式中$q_{rs}=1/rs\cdot(4/\pi)^2q_0$，我们假设挠度函数为

$$w_0=\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}w_0^{mn}\sin m\pi x\sin\frac{an\pi}{b}y\tag{4.9}$$

则边界条件(4.5)自然满足。为满足方程(4.4)和边界条件(4.6)，选取应力函数

$$\varphi_0=\frac{\bar{P}_xy^2}{2}+\frac{\bar{P}_yx^2}{2}+\sum_{m=0,2,4}^{\infty}\sum_{n=0,2,4}^{\infty}\varphi_0^{mn}\cos m\pi x\cos\frac{an\pi}{b}y\tag{4.10}$$

边界位移为零的条件(4.7)化为

$$\left.\begin{aligned}&\bar{P}_x+\frac{A_{12}^*}{A_{11}^*}\bar{P}_y=\frac{\pi^2}{8}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}m^2(w_0^{mn})^2\\&\frac{A_{21}^*}{A_{11}^*}\bar{P}_x+\frac{A_{22}^*}{A_{11}^*}\bar{P}_y=\frac{a^2\pi^2}{8b^2}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}n^2(w_0^{mn})^2\end{aligned}\right\}\tag{4.11}$$

将(4.9)和(4.10)式代入方程(4.4)的第二式，比较方程两边同类三角项的系数，得到φ_0^{mn}和w_0^{mn}之间的关系为

$$\varphi_0^{mn}=\frac{1}{4\left(c_2m^4+d_2\dfrac{m^2n^2a^2}{b^2}+e_2\dfrac{n^4a^4}{b^4}\right)}\sum a_{rspq}w_0^{rs}\ w_0^{pq}\tag{4.12}$$

式中求和号包括了$w_0^{rs}\ w_0^{pq}$对于$m=r\pm p$，$n=s\pm q$的所有乘积，系数

$$a_{rspq}=2rspq\pm(r^2q^2+s^2p^2)\tag{4.13}$$

当$m=0$或$n=0$且$r=q$，$s=p$时，应将上式除以2；当$m=r+p$，$n=s-q$或$m=r-p$，$n=s+q$时，上式括号前的符号取正号，否则取负号。

再将(4.8)，(4.9)和(4.10)式代入方程(4.4)的第一式，比较方程两边同类三角项的系数，得到挠度、应力函数与法向载荷的系数w_0^{mn}、φ_0^{mn}和q_{mn}之间的关系式，即

$$q_{mn}=\bar{P}_x\cdot m^2\pi^2w_0^{mn}+\bar{P}_y\frac{a^2n^2\pi^2}{b^2}w_0^{mn}+\frac{a^2\pi^4}{4b^2}\sum b_{rspq}\varphi_0^{rs}\ w_0^{pq}\tag{4.14}$$

式中求和号包括了$\varphi_0^{rs}\ w_0^{pq}$对于$m=r\pm p$，$n=s\pm q$的所有乘积，系数

$$b_{rspq}=\pm(rq\pm sp)^2\qquad(\text{当}r\neq0,s\neq0)\tag{4.15}$$

当$r=0$或$s=0$时应将上式乘以2；当$r-p=m$或$s-q=n$（但非同时），则括号外的符号取正号，否则取负号。当$m=r+p$，$n=s-q$或$m=r-p$，$n=s+q$时，括号内的符号取正号，否则为负。

将关系式(4.12)代入(4.14)式，便得到关于挠度系数 w_0^{mn} 的三次方程组，解此方程组。求得w_0^{mn}后再代回(4.12)式，就可计算应力函数系数φ_0^{mn}。

在(4.8)和(4.9)式中若取两项计算，即假设

$$w_0=w_0^{11}\sin\pi x\sin\frac{a\pi}{b}y+w_0^{31}\sin3\pi x\sin\frac{a\pi}{b}y \tag{4.16}$$

$$q=\frac{16}{\pi^2}q_0\sin\pi x\sin\frac{a\pi}{b}y+\frac{16}{3\pi^2}q_0\sin3\pi x\sin\frac{a\pi}{b}y \tag{4.17}$$

选取

$$\varphi_0=\frac{\bar{P}_xy^2}{2}+\frac{\bar{P}_yx^2}{2}+\varphi_0^{20}\cos2\pi x+\varphi_0^{02}\cos\frac{2a\pi}{b}y+\varphi_0^{22}\cos2\pi x\cos\frac{2a\pi}{b}y$$
$$+\varphi_0^{40}\cos4\pi x+\varphi_0^{42}\cos4\pi x\cos\frac{2a\pi}{b}y+\varphi_0^{60}\cos6\pi x \tag{4.18}$$

则由(4.12)式，有

$$\left.\begin{aligned}
&\varphi_0^{20}=\frac{a^2}{32c_2b^2}[(w_0^{11})^2-2w_0^{11}w_0^{31}],\quad \varphi_0^{02}=\frac{b^2}{32d_2a^2}[(w_0^{11})^2+9(w_0^{31})^2]\\
&\varphi_0^{22}=\frac{w_0^{11}w_0^{31}}{4\left[c_2\dfrac{b^2}{a^2}+d_2+e_2\dfrac{a^2}{b^2}\right]},\qquad \varphi_0^{42}=-\frac{w_0^{11}w_0^{31}}{16\left[16c_2\dfrac{b^2}{a^2}+4d_2+e_2\dfrac{a^2}{b^2}\right]}\\
&\varphi_0^{40}=\frac{a^2}{64c_2b^2}w_0^{31}w_0^{11},\qquad \varphi_0^{60}=\frac{a^2}{288c_2b^2}(w_0^{31})^2
\end{aligned}\right\} \tag{4.19}$$

然后，由(4.14)式，我们得到

$$\left.\begin{aligned}
\frac{16q_0}{\pi^2}=&\pi^2\bar{P}_xw_0^{11}+\frac{a^2\pi^2}{b^2}\bar{P}_yw_0^{11}+\frac{\pi^4}{16}\left(\frac{a^4}{c_2b^4}+\frac{1}{d_2}\right)(w_0^{11})^3+\frac{3a^4\pi^4}{16c_2b^4}(w_0^{11})^2w_0^{31}\\
&-\frac{\pi^4}{16}\left[\frac{4a^4}{c_2b^4}+\frac{9}{d_2}+\frac{16}{\left(c_2\dfrac{b^4}{a^4}+d_2\dfrac{b^2}{a^2}+e_2\right)}+\frac{1}{\left(16c_2\dfrac{b^4}{a^4}+4d_2\dfrac{b^2}{a_2}+e_2\right)}\right]w_0^{11}\cdot(w_0^{31})^2\\
\frac{16q_0}{3\pi^2}=&9\pi^2\bar{P}_xw_0^{31}+\frac{a^2\pi^2}{b^2}\bar{P}_yw_0^{31}+\frac{a^4\pi^4}{16c_2b^4}(w_0^{11})^3-\frac{\pi^4}{16}\left(\frac{81}{d_2}+\frac{a^4}{c_2b^4}\right)(w_0^{31})^3\\
&-\frac{\pi^4}{16}\left[\frac{4a^4}{c_2b^4}+\frac{9}{d_2}+\frac{16}{\left(c_2\dfrac{b^4}{a^4}+d_2\dfrac{b^2}{a^2}+e_2\right)}+\frac{1}{\left(16c_2\dfrac{b^4}{a^4}+4d_2\dfrac{b^2}{a^2}+e_2\right)}\right](w_0^{11})^2\cdot w_0^{31}
\end{aligned}\right\} \tag{4.20}$$

边界条件(4.11)为

$$\left.\begin{aligned}
&\bar{P}_x+\frac{A_{12}^*}{A_{11}^*}\bar{P}_y=\frac{\pi^2}{8}[(w_0^{11})^2+9(w_0^{31})^2]\\
&\frac{A_{21}^*}{A_{11}^*}\bar{P}_x+\frac{A_{22}^*}{A_{11}^*}\bar{P}_y=\frac{a^2\pi^2}{8b^2}[(w_0^{11})^2+(w_0^{31})^2]
\end{aligned}\right\} \tag{4.21}$$

联解以上方程，求得$\bar{P}_x$、$\bar{P}_y$与w_0^{11}、w_0^{31} 的关系后，代入方程(4.20)，联解该方程组，即得w_0^{11}和w_0^{31}；再代入(4.19)式，便计算得φ_0^{20}，φ_0^{02}，…，φ_0^{60}。

由(3.6)和(3.7)式，可知边界层型函数

$$v_0^{(1)}=v_0^{(2)}=v_0^{(3)}=v_0^{(4)}=0,\qquad h_0^{(1)}=h_0^{(2)}=h_0^{(3)}=h_0^{(4)}=0 \tag{4.22}$$

w_1和φ_1确定于边值问题

$$L_3(w_1,\varphi_0)+L_3(w_0,\varphi_1)=0, \quad L_2\varphi_1+L_3(w_0,w_1)=0 \tag{4.23}$$

$$\left.\begin{aligned}
&\int_0^{b/a} t\varphi_{1,yy}dy=0 && w_1|_{x=0}=w_1|_{x=1}=0\\
&\int_0^1 t\varphi_{1,xx}dx=0 && w_1|_{y=0}=w_1\Big|_{y=\frac{b}{a}}=0\\
&\int_0^1\left[\varphi_{1,yy}+\frac{A_{12}^*}{A_{11}^*}\varphi_{1,xx}-w_{0,x}w_{1,x}-w_{0,x}v_{0,x}^{(1)}\right]dx=0\\
&\int_0^{b/a}\left[\frac{A_{21}^*}{A_{11}^*}\varphi_{1,yy}+\frac{A_{22}^*}{A_{11}^*}\varphi_{1,xx}-w_{0,y}w_{1,y}-w_{0,y}v_{0,y}^{(1)}\right]dy=0
\end{aligned}\right\} \tag{4.24}$$

显然，$w_1\equiv0$，$\varphi_1\equiv0$，$v_1^{(i)}=h_1^{(i)}=0 \quad (i=1,\cdots,4)$。

w_2和φ_2确定于边值问题

$$\left.\begin{aligned}
&L_3(w_0,\varphi_2)+L_3(w_2,\varphi_0)=L_1w_0\\
&L_2\varphi_2+L_3(w_0,w_2)=0
\end{aligned}\right\} \tag{4.25}$$

$$\left.\begin{aligned}
&w_2|_{x=0}=w_2|_{x=1}=0, \quad w_2|_{y=0}=w_2\Big|_{y=\frac{b}{a}}=0, \quad \int_0^{b/a} t\varphi_{2,yy}dy=0, \quad \int_0^1 t\varphi_{2,xx}dx=0\\
&\int_0^1\left\{\left[\varphi_{1,yy}+\frac{A_{12}^*}{A_{11}^*}\varphi_{2,xx}\right]-\frac{1}{2}\left[w_{1,x}^2+2\left(w_{1,x}v_{0,x}^{(1)}+w_{0,x}v_{1,x}^{(1)}+w_{0,x}v_{0,x}^{(2)}\right)\right]\right\}dx=0\\
&\int_0^{b/a}\left\{\left[\frac{A_{21}^*}{A_{11}^*}\varphi_{2,yy}+\frac{A_{22}^*}{A_{11}^*}\varphi_{2,xx}\right]-\frac{1}{2}\left[w_{1,y}^2+2\left(w_{1,y}v_{0,y}^{(1)}+w_{0,y}v_{1,y}^{(1)}+w_{0,y}v_{0,y}^{(2)}\right)\right]\right\}dy=0
\end{aligned}\right\} \tag{4.26}$$

假设挠度和应力函数分别为

$$w_2=w_2^{11}\sin\pi x\sin\frac{a\pi}{b}y+w_2^{31}\sin3\pi x\sin\frac{a\pi}{b}y \tag{4.27}$$

$$\varphi_2=\varphi_2^{20}\cos2\pi x+\varphi_2^{02}\cos\frac{a\pi}{b}y+\varphi_2^{22}\cos2\pi x\cos\frac{2a\pi}{b}y+\varphi_2^{40}\cos4\pi x$$
$$+\varphi_2^{42}\cos4\pi x\cos\frac{2a\pi}{b}y+\varphi_2^{60}\cos6\pi x \tag{4.28}$$

由(4.12)式求得

$$\left.\begin{aligned}
&\varphi_2^{20}=\frac{a^2}{16c_2b^2}\left(w_0^{11}w_2^{11}-w_0^{11}w_2^{31}-w_0^{31}w_2^{11}\right)\\
&\varphi_0^{02}=\frac{b^2}{16d_2a^2}\left(w_0^{11}w_2^{11}+9w_0^{31}w_2^{31}\right)\\
&\varphi_2^{22}=\frac{w_0^{11}w_2^{31}+w_0^{31}w_2^{11}}{4\left(c_2\frac{b^2}{a^2}+d_2+e_2\frac{a^2}{b^2}\right)}, \qquad \varphi_2^{40}=\frac{a^2}{64c_2b^2}\left(w_0^{11}w_2^{31}+w_0^{31}w_2^{11}\right)\\
&\varphi_2^{42}=-\frac{w_0^{11}w_2^{31}+w_0^{31}w_2^{11}}{16\left(16c_2\frac{b^2}{a^2}+4d_2+e_2\frac{a^2}{b^2}\right)}, \quad \varphi_2^{60}=\frac{a^2}{144c_2b^2}w_0^{31}w_2^{31}
\end{aligned}\right\} \tag{4.29}$$

再代入方程(4.14)

$$\left.\begin{aligned}&-2(w_0^{11}\varphi_2^{02}+w_2^{11}\varphi_0^{02})-2(w_0^{11}\varphi_0^{20}+w_2^{11}\varphi_0^{20})-4(w_0^{31}\varphi_2^{22}+w_2^{31}\varphi_0^{22})\\&+2(w_0^{31}\varphi_2^{20}+w_2^{31}\varphi_0^{20})+(w_0^{31}\varphi_2^{42}+w_2^{31}\varphi_0^{42})-8(w_0^{31}\varphi_2^{40}+w_2^{31}\varphi_0^{40})\\&=w_0^{11}\left(c_1\frac{b^2}{a^2}+d_1+e_1\frac{a^2}{b^2}\right)\\&2(w_0^{11}\varphi_2^{20}+w_2^{11}\varphi_0^{20})-18(w_0^{31}\varphi_0^{02}+w_2^{31}\varphi_2^{02})-4(w_0^{11}\varphi_2^{22}+w_2^{11}\varphi_0^{22})\\&-8(w_0^{11}\varphi_2^{40}+w_2^{11}\varphi_0^{40})+(w_0^{11}\varphi_2^{42}+w_2^{11}\varphi_0^{42})-18(w_0^{31}\varphi_2^{60}+w_2^{31}\varphi_0^{60})\\&=w_0^{31}\left(81c_1\frac{b^2}{a^2}+9d_1+e_1\frac{a^2}{b^2}\right)\end{aligned}\right\}\tag{4.30}$$

即可解得 w_2^{11}, w_2^{31}，进一步可求得 $\varphi_2^{20},\cdots,\varphi_2^{60}$。

于是，挠度和应力函数的二阶近似式分别为

$$W=(w_0^{11}+\varepsilon^2w_2^{11})\sin\pi x\sin\frac{a\pi}{b}y+(w_0^{31}+\varepsilon^2w_2^{31})\sin3\pi x\sin\frac{a\pi}{b}y+O(\varepsilon^2)\tag{4.31}$$

$$\begin{aligned}\Phi=&\frac{\bar{P}_xy^2}{2}+\frac{\bar{P}_yx^2}{2}+(\varphi_0^{20}+\varepsilon^2\varphi_2^{20})\cos2\pi x+(\varphi_0^{02}+\varepsilon^2\varphi_2^{02})\cos\frac{2a\pi}{b}y\\&+(\varphi_0^{22}+\varepsilon^2\varphi_2^{22})\cos2\pi x\cos\frac{2a\pi}{b}y+(\varphi_0^{40}+\varepsilon^2\varphi_2^{40})\cos4\pi x\\&+(\varphi_0^{42}+\varepsilon^2\varphi_2^{42})\cos4\pi x\cos\frac{2a\pi}{b}y+(\varphi_0^{60}+\varepsilon^2\varphi_2^{60})\cos6\pi x+O(\varepsilon^2)\end{aligned}\tag{4.32}$$

下面，我们以高模量石墨/环氧复合材料为例作具体数值计算。

高模量石墨/环氧性能为

$E_1=137.9\times10^6$kpa$(20.0\times10^6$psi$)$

$E_2=E_3=14.48\times10^6$kpa$(2.1\times10^6$psi$)$

$G_{12}=G_{23}=G_{31}=4.98\times10^6kpa(0.85\times10^6psi)$

$\nu_{12}=\nu_{23}=\nu_{31}=0.21$

对于$N=1$（单层），有

$\varepsilon=0.289346$,

$$c_1=1,\quad e_1=\frac{D_{22}}{D_{11}}=0.105,\quad d_1=\frac{2D_3}{D_{11}}=0.213313$$

$$c_2=\frac{E_1}{E_2}=9.523809,\quad e_2=1,\quad d_2=\frac{E_1}{G_{12}}-2\nu_{12}=23.109412$$

对于$N=3$，有

$$M=2,\quad F=\frac{E_2}{E_1}=0.105,\quad P=\frac{1}{27},\quad \varepsilon=0.339155$$

$$c_1=1,\quad e_1=\frac{D_{22}}{D_{11}}=0.142885,\quad d_1=\frac{2(D_{12}+2D_{66})}{D_{11}}=0.220626$$

$$c_2=\frac{A^*_{22}}{A^*_{11}}=1.739669,\quad e_2=1,\quad d_2=\frac{2A^*_{12}+A^*_{66}}{A^*_{11}}=16.448774$$

我们将$N=1$（单层板）和$N=3$的叠层板的（无量纲）载荷与中心挠度的关系曲线图对长宽比取几种不同比值的情况分别作于图2中。与各向同性板相同[4]，在给定的压力下，挠度随长宽比值的减少而增加；同时，我们看到叠层板的刚度较之单层板大为提高。在叠层方板中心和角点处的最外层纤维的弯曲应力则示于图3中。

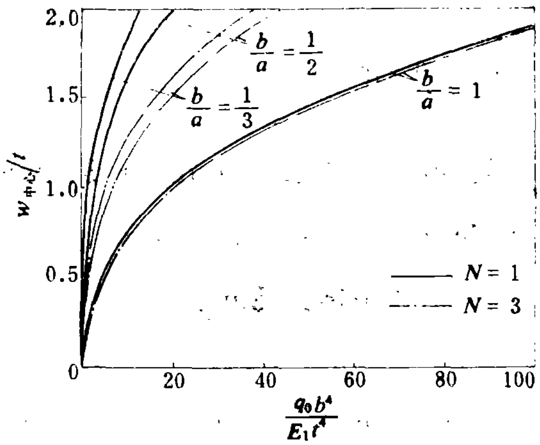
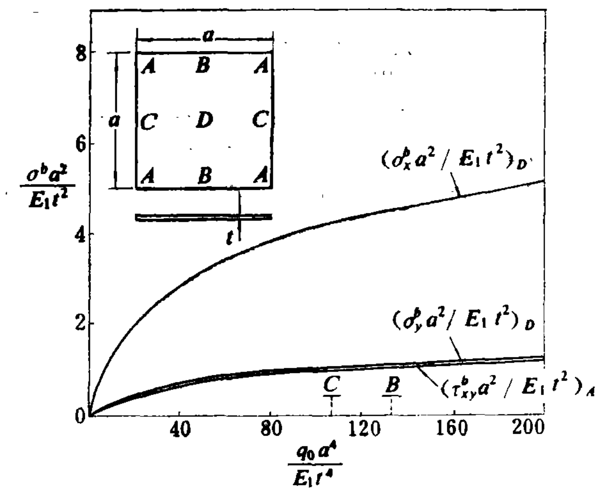


图 2



$\underline{B}$, $(\sigma^b a^2/E_1t^2)_B$　　$\underline{C}$, $(\sigma^b a^2/E_1t^2)_C$

图 3

因为参数ε是小的，所以挠度w和应力函数φ的N阶渐近表达式很快收敛。虽然我们在(4.9)式中只取两项计算，且只作到二阶近似，结果是令人满意的。其中心挠度的近似结果仅比板的von Kármán大挠度方程的解低0.126%。

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Nonlinear Bendings of Rectangular Symmetrically Laminated Cross-Ply Plates under Various Supports

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Abstract

This paper studies the nonlinear bendings of rectangular symmetrically laminated cross-ply plates subjected to uniform pressure under various supports on the bassis of [3] by the singular perturbation method offered in [1]. The uniformly valid N-order asymptotic solutions of the deflection and stress function are derived. Analyses and numerical solutions are given for simply supported rectangular laminates, whose edge displacement is vanished.