

非线性 Kelvin-Helmholtz 不稳定性*

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摘 要

本文用导数展开法对液体薄层与亚音速气流接壤时的界面稳定性作非线性分析。文中考虑了液体的表面张力与体积力, 故非线性的 Rayleigh-Taylor 不稳定性可作为特例而导出; 液体与气体均不计粘性。虽然 Nayfeh^[1] 曾算过这一情况, 但其三阶方程有遗漏 (如213页的式(2.29))。同时解也不自洽 (如其一阶解 (2.31) 并不满足他的初始条件 (2.20)), 此外, 在截止波数附近, 对行波他并未考虑。本文弥补了这些, 并得出了新的结论。

一、物 理 模 型

本文所用的模型是: 未扰时, 静止液体层的深度为 h , 一端与固壁相邻, 另一端与一亚音速的气流相接壤。气体延伸至无限远, 均匀流速 U_0 平行于固壁, 液体与气体的粘性均不考虑。液体还受到一个体积力 \mathbf{g} 的作用, \mathbf{g} 与界面垂直, 并由液体指向气体, 模型中还考虑了液体的表面张力 T 。

设未扰界面在 $t=0$ 时受到一个正弦形的扰动, 波幅为 a , 波数为 k' , $ak'=\varepsilon$ 虽然小但为有限值, 并假定扰动运动是二元、无旋的。分别以 k'^{-1} 及 $(gk')^{-\frac{1}{2}}$ 作为长度与时间的特征尺度来无量纲化, 于是液体与气体的速度位分别是: $g^{\frac{1}{2}} \cdot k'^{-\frac{3}{2}} \cdot \varphi(x, y, t)$ 及 $U_0 \cdot [x + \Phi(x, y, t)] \cdot k'^{-1}$, 其中 φ 及 Φ 分别是液体与气体的无量纲扰动速度位。

二、数 学 提 法

显然, 对 $-\infty < x < \infty$,

当 $-h \leq y < \eta$ 时, $\nabla^2 \varphi = 0$ (2.1)

而当 $\eta \leq y < \infty$ 时, 则有^[2]

$$m^2 \Phi_{xx} + \Phi_{yy} = M^2 \left[\frac{1}{2} (\gamma - 1) (2\Phi_x + \Phi_x^2 + \Phi_y^2) (\Phi_{xx} + \Phi_{yy}) + (2\Phi_x + \Phi_x^2) \Phi_{xx} + 2(1 + \Phi_x) \Phi_y \Phi_{xy} + \Phi_y^2 \Phi_{yy} \right] \quad (2.2)$$

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其中 $\eta(x, t)$ 是受扰界面相对于未扰界面的垂直位移, M 是未扰气流的马赫数, $m^2 = 1 - M^2$.

在固壁处, 液体的法向速度为零, 即

$$\varphi_y(x, -h, t) = 0 \quad (2.3)$$

在远离液面处, 气体的扰动速度为零

$$\Phi_x(x, \infty, t) = \Phi_y(x, \infty, t) = 0 \quad (2.4)$$

在液-气交界面 $y = \eta(x, t)$ 上, 运动学条件^[3]分别为

$$\eta_t + \eta_x \varphi_x = \varphi_y \quad (2.5)$$

$$\eta_x(1 + \Phi_x) = \Phi_y \quad (2.6)$$

由于假定了 $U_0 \gg (gk')^{\frac{1}{2}}$ (即气流速度 \gg 波速), 故 (2.6) 式中 η_x 不出现. 动力学条件^[4-5]是

$$-\eta + \varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2) = k^2 \eta_{xx} (1 + \eta_x^2)^{-\frac{3}{2}} - \frac{1}{2} mk \chi C_p \quad (\text{在 } y = \eta \text{ 处}) \quad (2.7)$$

其中 $k = k'/k'_c$, $k'_c = (\rho g/T)^{\frac{1}{2}}$ 是气体不动时的线性截止波数, $\chi = \rho_0 U_0^2 k'_c / (m \rho g)$, ρ 是液体密度, 而 C_p 是压强扰动系数, 是由于界面上形成了波而由气体在界面上作用于液体的, 它由文献[6]给出

$$C_p = (2/\gamma M^2) \left\{ \left[1 - \frac{\gamma-1}{2} M^2 (2\Phi_x + \Phi_x^2 + \Phi_y^2) \right]^{\frac{\gamma}{\gamma-1} - 1} - 1 \right\} \quad (2.8)$$

其中 γ 是气体的比热比.

初始条件是在 $t=0$ 时,

$$\eta(x, 0) = \varepsilon \cos x \quad (2.9)$$

$$\eta_t(x, 0) = 0 \quad (\text{对行波则不应提初速为零的条件}) \quad (2.10)$$

今采用文献[7]中的导数展开法来解. 引入下列时间尺度

$$T_0 = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t \quad (2.11)$$

于是

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \dots \quad (2.12)$$

设
$$\eta(x, t; \varepsilon) = \sum_{n=1}^3 \varepsilon^n \cdot \eta_n(x, T_0, T_1, T_2) + O(\varepsilon^4) \quad (2.13)$$

$$\varphi(x, y, t; \varepsilon) = \sum_{n=1}^3 \varepsilon^n \cdot \varphi_n(x, y, T_0, T_1, T_2) + O(\varepsilon^4) \quad (2.14)$$

$$\Phi(x, y, t; \varepsilon) = \sum_{n=1}^3 \varepsilon^n \cdot \Phi_n(x, y, T_0, T_1, T_2) + O(\varepsilon^4) \quad (2.15)$$

既然 (2.1), (2.3) 及 (2.4) 是线性的, 故每一 φ_n 均应满足 (2.1) 及 (2.3); 而每一 Φ_n 均应满足 (2.4).

把上述展开式代入其余的方程, 并令 ε 的同幂次项前的系数相等, 可得关于 ε 的一至三阶方程组.

ε 阶:

$$\nabla^2 \varphi_1 = 0, \quad (2.16a)$$

$$m^2 \frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2} = 0 \quad (2.16b)$$

$$\frac{\partial \eta_1}{\partial T_0} - \frac{\partial \varphi_1}{\partial y} = 0 \quad (\text{在 } y=0 \text{ 处}) \quad (2.17)$$

$$\frac{\partial \eta_1}{\partial x} - \frac{\partial \Phi_1}{\partial y} = 0 \quad (\text{在 } y=0 \text{ 处}) \quad (2.18)$$

$$-\eta_1 + \frac{\partial \varphi_1}{\partial T_0} - k^2 \frac{\partial^2 \eta_1}{\partial x^2} - mk\chi \frac{\partial \Phi_1}{\partial x} = 0 \quad (\text{在 } y=0 \text{ 处}) \quad (2.19)$$

$$\eta_1(x, 0, 0, 0) = \cos x, \quad \frac{\partial \eta_1}{\partial T_0}(x, 0, 0, 0) = 0 \quad (\text{仅用于驻波}) \quad (2.20)$$

ε^2 阶:

$$\nabla^2 \varphi_2 = 0 \quad (2.21a)$$

$$m^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = M^2 \left[(\gamma+1) \frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial^2 \Phi_1}{\partial x^2} + (\gamma-1) \frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial^2 \Phi_1}{\partial y^2} + 2 \frac{\partial \Phi_1}{\partial y} \cdot \frac{\partial^2 \Phi_1}{\partial y \partial x} \right] \quad (2.21b)$$

$$\frac{\partial \eta_2}{\partial T_0} - \frac{\partial \varphi_2}{\partial y} = -\frac{\partial \varphi_1}{\partial x} \cdot \frac{\partial \eta_1}{\partial x} + \eta_1 \frac{\partial^2 \varphi_1}{\partial y^2} - \frac{\partial \eta_1}{\partial T_1} \quad (\text{在 } y=0 \text{ 处}) \quad (2.22)$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = \eta_1 \frac{\partial^2 \Phi_1}{\partial y^2} - \frac{\partial \eta_1}{\partial x} \cdot \frac{\partial \Phi_1}{\partial x} \quad (\text{在 } y=0 \text{ 处}) \quad (2.23)$$

$$\begin{aligned} -\eta_2 + \frac{\partial \varphi_2}{\partial T_0} - k^2 \frac{\partial^2 \eta_2}{\partial x^2} - mk\chi \frac{\partial \Phi_2}{\partial x} = & -\frac{\partial \varphi_1}{\partial T_1} - \eta_1 \frac{\partial^2 \varphi_1}{\partial T_0 \partial y} - \frac{1}{2} \left[\left(\frac{\partial \varphi_1}{\partial x} \right)^2 \right. \\ & \left. + \left(\frac{\partial \varphi_1}{\partial y} \right)^2 \right] + mk\chi \left[\eta_1 \frac{\partial^2 \Phi_1}{\partial x \partial y} + \frac{m^2}{2} \left(\frac{\partial \Phi_1}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \Phi_1}{\partial y} \right)^2 \right] \end{aligned} \quad (2.24)$$

(在 $y=0$ 处)

$$\text{当 } T_0 = T_1 = T_2 = 0 \text{ 时, } \eta_2 = 0, \quad \frac{\partial \eta_2}{\partial T_0} = -\frac{\partial \eta_1}{\partial T_1} \quad (\text{仅用于驻波}) \quad (2.25)$$

ε^3 阶:

$$\nabla^2 \varphi_3 = 0 \quad (2.26a)$$

$$\begin{aligned} m^2 \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} = & M^2 \left\{ (\gamma+1) \left(\frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial \Phi_2}{\partial x} \cdot \frac{\partial^2 \Phi_1}{\partial x^2} \right) + (\gamma-1) \left(\frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial^2 \Phi_2}{\partial y^2} \right. \right. \\ & \left. \left. + \frac{\partial \Phi_2}{\partial x} \cdot \frac{\partial^2 \Phi_1}{\partial y^2} \right) + 2 \left(\frac{\partial \Phi_1}{\partial y} \cdot \frac{\partial^2 \Phi_2}{\partial x \partial y} + \frac{\partial \Phi_2}{\partial y} \cdot \frac{\partial^2 \Phi_1}{\partial x \partial y} \right) \right. \\ & \left. + \frac{\gamma-1}{2} \left[\left(\frac{\partial \Phi_1}{\partial x} \right)^2 + \left(\frac{\partial \Phi_1}{\partial y} \right)^2 \right] \left[\frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2} \right] + \left(\frac{\partial \Phi_1}{\partial x} \right)^2 \cdot \frac{\partial^2 \Phi_1}{\partial x^2} \right. \\ & \left. + 2 \frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_1}{\partial y} \frac{\partial^2 \Phi_1}{\partial x \partial y} + \left(\frac{\partial \Phi_1}{\partial y} \right)^2 \frac{\partial^2 \Phi_1}{\partial y^2} \right\} \end{aligned} \quad (2.26b)$$

$$\begin{aligned} \frac{\partial \eta_3}{\partial T_0} - \frac{\partial \varphi_3}{\partial y} = & -\frac{\partial \eta_1}{\partial T^2} - \frac{\partial \eta_2}{\partial T_1} + \eta_1 \frac{\partial^2 \varphi_2}{\partial y^2} + \eta_2 \frac{\partial^2 \varphi_1}{\partial y^2} - \frac{\partial \eta_1}{\partial x} \left(\frac{\partial \varphi_2}{\partial x} + \eta_1 \frac{\partial^2 \varphi_1}{\partial x \partial y} \right) \\ & - \frac{\partial \eta_2}{\partial x} \frac{\partial \varphi_1}{\partial x} + \frac{1}{2} \eta_1^2 \frac{\partial^3 \varphi_1}{\partial y^3} \quad (\text{在 } y=0 \text{ 处}) \end{aligned} \quad (2.27)$$

$$\begin{aligned} \frac{\partial \eta_3}{\partial x} - \frac{\partial \Phi_3}{\partial y} = & -\frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial \eta_2}{\partial x} - \left(\frac{\partial \Phi_2}{\partial x} + \eta_1 \frac{\partial^2 \Phi_1}{\partial x \partial y} \right) \frac{\partial \eta_1}{\partial x} + \eta_1 \frac{\partial^2 \Phi_2}{\partial y^2} \\ & + \frac{1}{2} \eta_1^2 \frac{\partial^3 \Phi_1}{\partial y^3} + \eta_2 \frac{\partial^2 \Phi_1}{\partial y^2} \quad (\text{在 } y=0 \text{ 处}) \end{aligned} \quad (2.28)$$

$$\begin{aligned} -\eta_3 + \frac{\partial \varphi_3}{\partial T_0} - k^2 \frac{\partial^2 \eta_3}{\partial x^2} - mk\chi \frac{\partial \Phi_3}{\partial x} = & -\frac{\partial \varphi_2}{\partial T_1} - \frac{\partial \varphi_1}{\partial T_2} - \eta_1 \frac{\partial^2 \varphi_2}{\partial T_0 \partial y} - \eta_2 \frac{\partial^2 \varphi_1}{\partial T_0 \partial y} \\ & - \eta_1 \frac{\partial^2 \varphi_1}{\partial T_1 \partial y} - \frac{\partial \varphi_1}{\partial x} \cdot \frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_1}{\partial y} \cdot \frac{\partial \varphi_2}{\partial y} + mk\chi \left[\eta_1 \frac{\partial^2 \Phi_2}{\partial x \partial y} + \eta_2 \frac{\partial^2 \Phi_1}{\partial x \partial y} \right. \\ & \left. + m^2 \frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_2}{\partial x} + \frac{\partial \Phi_1}{\partial y} \frac{\partial \Phi_2}{\partial y} \right] - \frac{1}{2} \eta_1^2 \frac{\partial^3 \varphi_1}{\partial T_0 \partial y^2} - \eta_1 \frac{\partial \varphi_1}{\partial x} \frac{\partial^2 \varphi_1}{\partial x \partial y} - \eta_1 \frac{\partial \varphi_1}{\partial y} \cdot \frac{\partial^2 \varphi_1}{\partial y^2} \\ & - \frac{3}{2} k^2 \left(\frac{\partial \eta_1}{\partial x} \right)^2 \cdot \frac{\partial^2 \eta_1}{\partial x^2} + mk\chi \left[\frac{1}{2} \eta_1^2 \frac{\partial^3 \Phi_1}{\partial x \partial y^2} + m^2 \frac{\partial \Phi_1}{\partial x} \cdot \frac{\partial^2 \Phi_1}{\partial x \partial y} \cdot \eta_1 \right. \\ & \left. + \eta_1 \frac{\partial \Phi_1}{\partial y} \frac{\partial^2 \Phi_1}{\partial y^2} - \frac{M^2}{2} \left(\frac{\partial \Phi_1}{\partial x} \right)^3 - \frac{M^2}{2} \cdot \frac{\partial \Phi_1}{\partial x} \left(\frac{\partial \Phi_1}{\partial y} \right)^2 + \frac{2-\gamma}{6} M^4 \left(\frac{\partial \Phi_1}{\partial x} \right)^3 \right] \end{aligned} \quad (2.29)$$

当 $T_0 = T_1 = T_2 = 0$ 时有,

$$\eta_3 = 0 \text{ 与 } \frac{\partial \eta_3}{\partial T_0} = -\frac{\partial \eta_1}{\partial T_2} - \frac{\partial \eta_2}{\partial T_1} \quad (\text{仅用于驻波}) \quad (2.30)$$

为便于与 Nayfeh 的推导逐一作比较, 本文尽量采用相同的符号, 顺序甚至编号。这里仅列出到目前为止, 与 Nayfeh 一文的不同之处:

1、Nayfeh 将 (2.29) 式的最后一项 “ $(mk\chi(2-\gamma)/6)M^4(\partial\Phi_1/\partial x)^3$ ” 遗漏了(它来自 (2.8) 式 C_3 的三阶项), 而既然一律准确到三阶, 该三阶项就没有理由略去。这一遗漏, 至少会通过 P_0 (本文 (3.56) 式, Nayfeh (A12) 式) 影响 ξ' (驻波振幅对频率的二阶修正) 的值; 通过 p_0 (本文 (3.26) 式, Nayfeh (A6) 式) 影响 β' (行波振幅对频率的二阶修正) 的值以及 σ (对截止波数的二阶修正) 的值。

2、一般说, 行波的初速度不可能处处为零,

3、本文按惯例、规定速度位的梯度等于速度(当然也可以规定速度位梯度的负值等于速度), 而 Nayfeh 文中并未明确交待他的定义。本文中 (2.5) ~ (2.7) 与 Nayfeh 相应的式子在正负号上的差别估计即来源于速度位的不同定义(但从最后效果看, 这一符号上的差别并不影响结果)。

在下一节中, 将分为远离截止波数的情况和靠近截止波数的情况来分别讨论。以下凡与 Nayfeh 文中讨论重复的部分, 倘不影响完整性, 为节省篇幅起见, 一概从简甚或省去, 而只保留为说明不同点所必需的那部分推导。

三、求解并讨论

(一) 远离线性截止波数 k_0 的情况

1、对行波的解

其一阶解为

$$\begin{cases} \eta_1 = \cos \theta & (3.1) \\ \varphi_1 = -\mu_1 [\cosh(y+h)/\sinh h] \cdot \sin \theta & (3.2) \\ \Phi_1 = m^{-1} \cdot e^{-m y} \cdot \sin \theta & (3.3) \end{cases}$$

其中 $\theta = x + \mu_1 T_0 + \beta(T_1, T_2)$, $\beta(0, 0) = 0$ (3.4)

$$\mu_1^2 = C_n^{-1} \cdot n(nk^2 - nk\chi - 1), C_n = \coth nh$$

倘取 β 为常数, 即得线性解; 当 $\mu_1^2 > 0$, 扰动是线性稳定的, $\mu_1^2 < 0$ 则线性不稳定, 使 $\mu_1^2 = 0$ 的波数称为线性截止波数 k_0 (相应于中性稳定),

$$k_0 = [\chi \pm (\chi^2 + 4)^{\frac{1}{2}}] / 2 \quad (3.5)$$

由于波数只能是正数, 上式只有取正号才有意义。

倘体积力改为由气体指向液体, (3.5) 式变为 $k_0 = [\chi \pm (\chi^2 - 4)^{\frac{1}{2}}] / 2$, 因而存在两个截止波数。只有当扰动波数位于此两截止波数之外时, 扰动才是稳定的; 反之, 则不稳定。当 $\chi < 2$ 时, 则对所有扰动波数都稳定, 此即经典的 Kelvin-Helmholtz 问题。

与上述一阶解对应的二阶方程为

$$m^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = -\frac{M^4(\gamma+1)}{2m^2} e^{-2my} \sin 2\theta, \nabla^2 \varphi_2 = 0 \quad (3.6)$$

$$\frac{\partial \eta_2}{\partial T_0} - \frac{\partial \varphi_2}{\partial y} = -C_1 \mu_1 \sin 2\theta + \frac{\partial \beta}{\partial T_1} \sin \theta \quad (\text{在 } y=0 \text{ 处}) \quad (3.7)$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = \frac{m^2+1}{2m} \sin 2\theta \quad (\text{在 } y=0 \text{ 处}) \quad (3.8)$$

$$\begin{aligned} -\eta_2 + \frac{\partial \varphi_2}{\partial T_0} - k^2 \frac{\partial^2 \eta_2}{\partial x^2} - mk\chi \frac{\partial \Phi_2}{\partial x} &= \frac{\mu_1^2}{4} (1 - C_1^2) + \frac{1}{4} [\mu_1^2 (3 - C_1^2) \\ &\quad - 2mk\chi] \cos 2\theta + \mu_1 C_1 \frac{\partial \beta}{\partial T_1} \cos \theta \quad (\text{在 } y=0 \text{ 处}) \end{aligned} \quad (3.9)$$

初始条件为 $T_0 = T_1 = T_2 = 0$ 时 $\eta_2 = 0$ 。

为了消除二阶解中的永年项, $\partial \beta / \partial T_1$ 应为零, 即 $\beta = \beta(T_2)$ 。于是满足 (3.6)~(3.9), (2.25), (2.3) 及 (2.4) 的二阶解为

$$\begin{cases} \eta_2 = a_{22} (\cos 2\theta - \cos \theta_2) & (3.10) \\ \varphi_2 = -\frac{\mu_1^2}{4} (C_1^2 - 1) T_0 + \left(\frac{a_{22} \mu_2}{2} \sin \theta_2 - \mu_1 b_{22} \sin 2\theta \right) \frac{\cosh 2(y+h)}{\sinh 2h} & (3.11) \end{cases}$$

$$\Phi_2 = \left[\frac{M^4(\gamma+1)}{8m^3} y + \frac{d_{22}}{m} \right] e^{-2my} \cdot \sin 2\theta - \frac{a_{22}}{m} e^{-2my} \cdot \sin \theta_2 \quad (3.12)$$

其中 $\theta_2 = 2x + \mu_2 T_0$ (3.13)

$$a_{22} = \left[(3 - C_1^2 - 4C_1 C_2) \mu_1^2 + \frac{2k\chi}{m} \left(1 + \frac{M^4(\gamma+1)}{4m^2} \right) \right] / [2C_2(\mu_2^2 - 4\mu_1^2)] \quad (3.14)$$

$$b_{22} = a_{22} - C_1/2 \quad (3.15)$$

$$d_{22} = a_{22} + \frac{m^2+1}{4m} + \frac{M^4(\gamma+1)}{16m^3} \quad (3.16)$$

由上述一阶及二阶解代入 (2.26) ~ (2.29) 可得

$$m^2 \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} = e^{-3my} (p_1 + qy) \sin \theta + \text{NSPT}, \nabla^2 \varphi_3 = 0 \quad (3.17)$$

$$\frac{\partial \eta_3}{\partial T_0} - \frac{\partial \varphi_3}{\partial y} = (p_2 + \beta') \sin \theta + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.18)$$

$$\frac{\partial \eta_3}{\partial x} - \frac{\partial \Phi_3}{\partial y} = p_3 \sin \theta + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.19)$$

$$-\eta_3 + \frac{\partial \varphi_3}{\partial T_0} - k^2 \frac{\partial^2 \eta_3}{\partial x^2} - mk\chi \frac{\partial \Phi_3}{\partial x} = (p_4 + k\chi p_6 + \mu_1 C_1 \beta') \cos \theta + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.20)$$

其中

$$p_1 = M^2 \left[\left(\gamma - 3 - \frac{\gamma+1}{m^2} \right) d_{22} + \frac{(3+2\gamma-\gamma^2)m^4 - 2\gamma m^2(1-\gamma) + (1-\gamma^2)}{4m^3} \right] \quad (3.21)$$

$$q = \frac{M^2(\gamma+1)}{8m^2} \left(\gamma - 3 - \frac{\gamma+1}{m^2} \right) \quad (3.22)$$

$$p_2 = -\mu_1 \left(\frac{1}{2} C_1 a_{22} + C_2 b_{22} + \frac{3}{8} \right) \quad (3.23)$$

$$p_3 = \left(2m - \frac{1}{m} \right) d_{22} + \left(\frac{1}{m} - \frac{m}{2} \right) a_{22} - \frac{1}{4} - \frac{M^4(\gamma+1)}{4m^2} - \frac{m^2}{8} \quad (3.24)$$

$$p_4 = \mu_1^2 \left[b_{22}(1 - C_1 C_2) + \frac{a_{22}}{2} - \frac{5C_1}{8} \right] + \frac{3k^2}{8} \quad (3.25)$$

$$p_6 = \frac{-(\gamma+3)M^4 + 12M^2 - 10}{16m^2} - \frac{ma_{22}}{2} \quad (3.26)$$

$$\beta' = \frac{d\beta}{dT_2} \quad (3.27)$$

“NSPT”代表不产生永年项的那些项。上述三阶方程的特解中含有永年项，它在 $T_0 \rightarrow \infty$ 时使 η_3/η_1 无界。为了确定消除这些永年项所必须满足的条件，今假定与这些在三阶方程中可能产生永年项的项相对应的特解为

$$\begin{cases} \eta_3 = 0 \\ \Phi_3 = \left[\frac{1}{8m^2} \left(p_1 + \frac{3q}{4m} \right) + \frac{q}{8m^2} y \right] e^{-3my} \cdot \sin \theta + D e^{-my} \sin \theta \\ \varphi_3 = -E \frac{\cosh(y+h)}{\sinh h} \sin \theta \end{cases}$$

它满足 (2.3)、(2.4) 及 (3.17)。将它代入 (3.18) ~ (3.20)，得

$$\begin{aligned} E = p_2 + \beta', \quad mD + 3p_1/(8m) + 5q/(32m^2) = p_3 \\ -\mu_1 C_1 E - \frac{k\chi}{8m} \left(p_1 + \frac{3q}{4m} \right) - mk\chi D = p_4 + k\chi p_6 + \mu_1 C_1 \beta' \end{aligned}$$

从中消去 E 及 D ，得

$$\begin{aligned} \beta' = \frac{\mu_1}{4C_1} \left[(C_1^2 + 4C_1 C_2 - 3) a_{22} + 3C_1 - 2C_1^2 C_2 \right] - \frac{3k^2}{16C_1 \mu_1} \\ + \frac{k\chi}{2C_1 \mu_1} \left[\frac{p_1}{4m} + \frac{q}{16m^2} - p_3 - p_6 \right] \quad (3.28) \end{aligned}$$

于是准确到二阶的解为:

$$\left\{ \begin{aligned} \eta &= \varepsilon \cos \theta + \varepsilon^2 a_{22} (\cos 2\theta - \cos \theta_2) + O(\varepsilon^3) & (3.29) \\ \varphi &= -\varepsilon \mu_1 \sin \theta \cdot \frac{\cosh(y+h)}{\sinh h} + \varepsilon^2 \left[\frac{\mu_1^2}{4} (1 - C_1^2) t + \left(\frac{1}{2} a_{22} \mu_2 \sin \theta_2 \right. \right. \\ &\quad \left. \left. - \mu_1 b_{22} \sin 2\theta \right) \frac{\cosh 2(y+h)}{\sinh 2h} \right] + O(\varepsilon^3) & (3.30) \\ \Phi &= \varepsilon m^{-1} \cdot e^{-m y} \cdot \sin \theta + \varepsilon^2 \left[\left(\frac{M^4(\gamma+1)}{8m^3} y + \frac{d_{22}}{m} \right) \sin 2\theta - \frac{a_{22}}{m} \sin \theta_2 \right] e^{-2m y} + O(\varepsilon^3) & (3.31) \end{aligned} \right.$$

$$\text{其中} \quad \theta = x + (\mu_1 + \varepsilon^2 \beta') t + O(\varepsilon^3) \quad (3.32)$$

$$\theta_2 = 2x + \mu_2 t + O(\varepsilon^2) \quad (3.33)$$

只要 μ_1 为实数, 上述解在一个很广的波数范围内是有效的, 但当 $\mu_1 \rightarrow 0$ 时, $\beta' \rightarrow \infty$, 即在线性截止波数附近该解失效, 对此情况将在三、(二) 节中讨论。

上述解表明: 初始的谐波扰动以两个波速传播, 其一与振幅的大小有关, 另一则无关。

2. 对驻波的解

满足 (2.16) ~ (2.20) 的一阶驻波解为

$$\eta_1 = \cos \alpha \cdot \cos x \quad (3.34)$$

$$\varphi_1 = -\mu_1 \cdot \sin \alpha \cdot \cos x \cdot \cosh(y+h) / \sinh h \quad (3.35)$$

$$\Phi_1 = m^{-1} \cdot e^{-m y} \cdot \cos \alpha \cdot \sin x \quad (3.36)$$

$$\text{其中} \quad \alpha = \mu_1 T_0 + \xi(T_1, T_2), \quad \xi(0, 0) = 0 \quad (3.37)$$

于是二阶方程组 (2.21) ~ (2.24) 变为:

$$m^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = -\frac{M^4(\gamma-1)}{2m^2} (1 + \cos 2\alpha) \cdot (\sin 2x) \cdot e^{-2m y}, \quad \nabla^2 \varphi_2 = 0 \quad (3.38)$$

$$\frac{\partial \eta_2}{\partial T_0} - \frac{\partial \varphi_2}{\partial y} = \frac{\partial \xi}{\partial T_1} \cos x \cdot \sin \alpha - \frac{C_1 \mu_1}{2} \cos 2x \cdot \sin 2\alpha \quad (\text{在 } y=0 \text{ 处}) \quad (3.39)$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = \frac{1}{4} \left(m + \frac{1}{m} \right) (\sin 2x) (1 + \cos 2\alpha) \quad (\text{在 } y=0 \text{ 处}) \quad (3.40)$$

$$\begin{aligned} -\eta_2 + \frac{\partial \varphi_2}{\partial T_0} - k^2 \frac{\partial^2 \eta_2}{\partial x^2} - mk \chi \frac{\partial \Phi_2}{\partial x} &= C_1 \mu_1 \frac{\partial \xi}{\partial T_1} (\cos \alpha) \cos x + \frac{\mu_1^2}{8} (1 - C_1^2) \\ &+ \frac{\mu_1^2}{8} (3 + C_1^2) \cos 2\alpha + \left\{ \frac{1}{8} [\mu_1^2 (1 + C_1^2) - 2mk\chi] \right. \\ &\left. + \frac{\cos 2\alpha}{8} [\mu_1^2 (3 - C_1^2) - 2mk\chi] \right\} \cos 2x \quad (\text{在 } y=0 \text{ 处}) \end{aligned} \quad (3.41)$$

$$\text{初始条件是 } T_0 = T_1 = T_2 = 0 \text{ 时, } \eta_2 = 0, \quad \frac{\partial \eta_2}{\partial T_0} + \frac{\partial \eta_1}{\partial T_1} = 0. \quad (3.42)$$

为了使驻波的两阶解中不含永年项, 应有 $\partial \xi / \partial T_1 = 0$, 即 $\xi = \xi(T_2)$. 于是满足 (3.38) ~ (3.42) 的二阶解为

$$\left\{ \begin{array}{l} \eta_2 = \left[\frac{1}{2} a_{22} (\cos 2\alpha - \cos \mu_2 T_0) + a_{20} (1 - \cos \mu_2 T_0) \right] \cos 2x \end{array} \right. \quad (3.43)$$

$$\left\{ \begin{array}{l} \varphi_2 = \frac{\mu_1^2}{8} (1 - C_1^2) T_0 + \frac{\mu_1}{16} (C_1^2 + 3) \sin 2\alpha + \left[\left(\frac{a_{22}}{4} + \frac{a_{20}}{2} \right) \mu_2 \cdot \sin \mu_2 T_0 \right. \\ \left. - \frac{\mu_1 b_{22}}{2} \sin 2\alpha \right] (\cos 2x) \frac{\cosh 2(y+h)}{\sinh 2h} \end{array} \right. \quad (3.44)$$

$$\left\{ \begin{array}{l} \Phi_2 = \left[\frac{M^4(\gamma+1)}{16m^3} (1 + \cos 2\alpha) y + \frac{d_{22}}{2m} \cos 2\alpha + \frac{d_{20}}{m} \right. \\ \left. - \frac{1}{m} \left(a_{20} + \frac{a_{22}}{2} \right) \cos \mu_2 T_0 \right] (\sin 2x) e^{-2my} \end{array} \right. \quad (3.45)$$

其中
$$a_{20} = \left[2(1 + C_1^2) \mu_1^2 + \frac{k\chi}{m} \left(4 + \frac{M^4(\gamma+1)}{m^2} \right) \right] / 8C_2\mu_2^2 \quad (3.46)$$

$$d_{20} = a_{20} + M^4(\gamma+1)/32m^3 + (m+m^{-1})/8 \quad (3.47)$$

将一阶解 (3.34) ~ (3.36) 及二阶解 (3.43) ~ (3.45) 代入 (2.26) ~ (2.29) 右端得三阶方程组

$$\nabla^2 \varphi_3 = 0 \quad (3.48a)$$

$$m^2 \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} = (P_1 + Qy) e^{-3my} \cdot (\cos \alpha) \cdot \sin x + \text{NSPT} \quad (3.48b)$$

$$\frac{\partial \eta_3}{\partial T_0} - \frac{\partial \varphi_3}{\partial y} = (P_2 + \xi') \cdot (\sin \alpha) \cdot \cos x + \text{NSPT} \quad (3.49)$$

$$\frac{\partial \eta_3}{\partial x} - \frac{\partial \Phi_3}{\partial y} = P_3 \cdot (\cos \alpha) \cdot \cos x + \text{NSPT} \quad (3.50)$$

$$- \eta_3 + \frac{\partial \varphi_3}{\partial T_0} - k^2 \frac{\partial^2 \eta_3}{\partial x^2} - mk\chi \frac{\partial \Phi_3}{\partial x} = (P_4 + k\chi P_5 + \mu_1 C_1 \xi') \cdot (\cos \alpha) \cdot \cos x + \text{NSPT} \quad (3.51)$$

其中

$$P_1 = M^2 \left(\frac{d_{22}}{4} + d_{20} \right) \left(\gamma - 3 - \frac{\gamma+1}{m^2} \right) + \frac{3M^2 [(3+2\gamma-\gamma^2)m^4 + 2m^2\gamma(\gamma-1) + (1-\gamma^2)]}{16m^3} \quad (3.52)$$

$$P_2 = \mu_1 \left[\frac{1}{2} C_1 a_{20} - \frac{1}{4} C_2 b_{22} - \frac{1}{8} C_1 a_{22} - \frac{1}{32} \right] \quad (3.53)$$

$$P_3 = \left(\frac{a_{22}}{4} + a_{20} \right) \left(\frac{1}{m} - \frac{m}{2} \right) + \left(\frac{d_{22}}{4} + d_{20} \right) \left(2m - \frac{1}{m} \right) - \frac{3}{16} - \frac{3m^2}{32} - \frac{3M^4(\gamma+1)}{16m^2} \quad (3.54)$$

$$P_4 = \mu_1^2 \left[\frac{b_{22}}{2} (1 - C_1 C_2) + \frac{1}{2} \left(\frac{a_{22}}{4} + a_{20} \right) + \frac{C_1}{32} \right] + \frac{9k^2}{32} \quad (3.55)$$

$$P_5 = \frac{3}{64m^2} [-(\gamma+3)M^4 + 12M^2 - 10] - \frac{m}{2} \left(\frac{a_{22}}{4} + a_{20} \right) \quad (3.56)$$

$$Q = \frac{3M^6(\gamma+1)}{32m^2} \left(\gamma - 3 - \frac{\gamma+1}{m^2} \right) \quad (3.57)$$

为了使三阶解中不出现永年项, 要求:

$$\xi' = \frac{d\xi}{dT_2} = \left[-P_4 - \mu_1 C_1 P_2 + k\chi \left(\frac{P_1}{4m} + \frac{Q}{16m^2} - P_3 - P_6 \right) \right] / 2C_1 \mu_1 \quad (3.58)$$

于是准确到二阶解的界面位移

$$\eta = \varepsilon \cdot [\cos(\mu_1 + \varepsilon^2 \xi')t] \cos x + \varepsilon^2 \{ a_{22} [\cos 2(\mu_1 + \varepsilon^2 \xi')t - \cos \mu_2 t] / 2 + a_{20} (1 - \cos \mu_2 t) \} \cos 2x + O(\varepsilon^3) \quad (3.59)$$

倘 μ_1 为实数, (3.59) 式代表一个具有两个频率的驻波, 一个频率为 $\mu_1 + \varepsilon^2 \xi'$, 它与振幅有关, 另一为 μ_2 , 它与振幅无关. 但当 $\mu_1 \rightarrow 0$ 时 $\xi' \rightarrow \infty$, 这种情况将在三、(二) 节中专门讨论. 当 μ_1 为虚数时, (3.59) 代表一个振幅随时间增长的波, 且随着时间的流逝, 二阶项很快就变得比一阶项更大, 从而使这种展开失效.

(二) 在 k_0 附近的展开

$$\text{设 } k = k_0 + \varepsilon^2 \sigma, \quad \sigma = O(1) \quad (3.60)$$

并只考虑到随 T_0 及 T_1 的变化. 这时, (2.16) ~ (2.30) 大致保持不变, 只是 k 变成了 k_0 , 而 (2.29) 右端则多出了两项 $\sigma \left(2k_0 \frac{\partial^2 \eta_1}{\partial x^2} + m\chi \frac{\partial \Phi_1}{\partial x} \right)$. 由于 Nayfeh^[1] 未考虑在截止波数附近的行波情况, 即使对驻波情况, 三阶方程中的遗漏项也会影响到二阶解, 故需重新推导.

1. 对驻波的解

满足 (2.16) ~ (2.20) 的一阶驻波解为

$$\begin{cases} \eta_1 = \eta_{11}(T_1) \cdot \cos x & (3.61) \\ \varphi_1 = 0 & (3.62) \\ \Phi_1 = m^{-1} \cdot \eta_{11} \cdot (\sin x) \cdot e^{-m y} & (3.63) \end{cases}$$

$$\text{其中 } \eta_{11}(0) = 1 \quad (3.64)$$

于是 (2.21) ~ (2.24) 就成了

$$m^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = - \frac{M^4 (\gamma + 1)}{2m^2} \cdot \eta_{11}^2 \cdot e^{-2m y} \cdot \sin 2x, \quad \nabla^2 \varphi_2 = 0 \quad (3.65)$$

$$\frac{\partial \eta_2}{\partial T_0} - \frac{\partial \varphi_2}{\partial y} = -\eta'_{11} \cdot \cos x \quad (\text{在 } y=0 \text{ 处}) \quad (3.66)$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = \frac{1+m^2}{2m} \eta_{11}^2 \cdot \sin 2x \quad (\text{在 } y=0 \text{ 处}) \quad (3.67)$$

$$-\eta_2 + \frac{\partial \varphi_2}{\partial T_0} - k_0^2 \frac{\partial^2 \eta_2}{\partial x^2} - m k_0 \chi \frac{\partial \Phi_2}{\partial x} = -\frac{1}{2} m k_0 \cdot \chi \cdot \eta_{11}^2 \cdot \cos 2x \quad (\text{在 } y=0 \text{ 处}) \quad (3.68)$$

满足 (3.65) ~ (3.68) 及初始条件 (2.25) 的二阶解为

$$\eta_2 = a_{22} (\eta_{11}^2 - \cos \mu_2 T_0) \cos 2x \quad (3.69)$$

$$\varphi_2 = \eta'_{11} (\cos x) \frac{\cosh(y+h)}{\sinh h} + \frac{\mu_2 a_{22}}{2} (\sin \mu_2 T_0) (\cos 2x) \frac{\cosh 2(y+h)}{\sinh 2h} \quad (3.70)$$

$$\Phi_2 = \eta_{11}^2 \left[\frac{M^4 (\gamma + 1)}{8m^3} y + \frac{d_{22}}{m} \right] e^{-2m y} \cdot \sin 2x - \frac{a_{22}}{m} e^{-2m y} \cdot (\sin 2x) \cdot \cos \mu_2 T_0 \quad (3.71)$$

$$\text{为满足 (2.25) 中的初速度条件, 要求 } \frac{d\eta_{11}}{dT_1} \Big|_{T_1=0} = \eta'_{11}(0) = 0 \quad (3.72)$$

而 a_{22} , d_{22} 则应按 $k=k_0$ (即 $\mu_1=0$) 代入估值。

将一阶解 (3.61) ~ (3.63) 及二阶解 (3.69) ~ (3.71) 代入 (2.26) ~ (2.28) 及修正后的 (2.29) 中, 得三阶方程组:

$$m^2 \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} = (p_1 + qy) \cdot \eta_{11}^3 \cdot e^{-3my} \cdot \sin x + \text{NSPT} \quad (3.73a)$$

$$\nabla^2 \varphi_3 = 0 \quad (3.73b)$$

$$\frac{\partial \eta_3}{\partial T_0} - \frac{\partial \varphi_3}{\partial y} = 0 + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.74)$$

$$\frac{\partial \eta_3}{\partial x} - \frac{\partial \Phi_3}{\partial y} = p_3 \cdot \eta_{11}^3 \cdot \sin x + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.75)$$

$$-\eta_3 + \frac{\partial \varphi_3}{\partial T_0} - k_c^2 \frac{\partial^2 \eta_3}{\partial x^2} - mk_c \chi \frac{\partial \Phi_3}{\partial x} = \left[\left(\frac{3}{8} k_c^2 + k_c \chi p_6 \right) \eta_{11}^3 - C_1 \cdot \eta_{11}'' \right. \\ \left. + \sigma (\chi - 2k_c) \eta_{11} \right] \cos x + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.76)$$

其中 $p_1 \sim p_6$ 应按 $\mu_1=0$ 代入估值。(3.73) ~ (3.76) 的特解中包含永年项, 当 $T_0 \rightarrow \infty$ 时, 它使 $\eta_3/\eta_1 \rightarrow \infty$, 为了消除永年项, 应有

$$C_1 \eta_{11}'' + (2k_c - \chi) \sigma \eta_{11} - \Gamma \eta_{11}^3 = 0 \quad (3.77)$$

其中
$$\Gamma = \frac{3}{8} k_c^2 + k_c \chi \left(p_6 + p_3 - \frac{p_1}{4m} - \frac{q}{16m^2} \right). \quad (3.78)$$

由 $\eta_{11}(0)=1$ 及 $\eta_{11}'(0)=0$ (参 (3.64) 及 (3.72)), 对 (3.77) 积分一次可得

$$\eta_{11}'^2 = \Gamma (\eta_{11}^2 - 1) (\eta_{11}^2 - \tilde{\Gamma}) / (2C_1) \quad (3.79)$$

其中
$$\tilde{\Gamma} = 2\sigma(2k_c - \chi) / \Gamma - 1 \quad (3.80)$$

而 (3.79) 式的左端只能是个非负的数。

当 $\Gamma > 0$ 时, 如还有 $\tilde{\Gamma} > 1$ 成立, 则因 $\eta_{11}(0)=1$, 故 η_{11}^2 不能超过 1 (否则 η_{11}' 将成为虚数); 倘 $\tilde{\Gamma} < 1$, 则 η_{11}^2 不能小于 1, 即 η_{11} 无界。于是 $\tilde{\Gamma}=1$ 是稳定与否的分界线, 因而

$$\sigma = \Gamma / (2k_c - \chi) \quad (3.81)$$

而 $k = k_c + e^2 \cdot \Gamma / (2k_c - \chi) + O(e^3)$ 则是准确到二阶的截止波数。

当 $\Gamma < 0$ 时, 倘 $\tilde{\Gamma} < 0$, 则 $0 \leq \eta_{11}^2 \leq 1$; 倘 $\tilde{\Gamma} > 0$, η_{11}^2 只在 1 与 $\tilde{\Gamma}$ 之间变化, 因而振幅 η_{11} 总是有界的。

2. 对行波的解

满足 (2.16) ~ (2.20) 的一阶解为

$$\begin{cases} \eta_1 = \eta_{11}(T_1) \cdot \cos[x + \beta(T_1)] = \eta_{11}(T_1) \cdot \cos \theta_0 & (3.82) \\ \varphi_1 = 0 & (3.83) \\ \Phi_1 = m^{-1} \eta_{11}(\sin \theta_0) \cdot e^{-my} & (3.84) \end{cases}$$

其中 $\theta_0 = x + \beta(T_1)$ 。由初位移条件 (2.20), 要求

$$\eta_{11}(0) = 1 \text{ 及 } \beta(0) = 0 \quad (3.85)$$

于是 (2.21) ~ (2.24) 成了

$$\nabla^2 \varphi_2 = 0 \quad (3.86a)$$

$$m^2 \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = -\frac{M^4(\gamma+1)}{2m^2} \eta_{11}^2 e^{-2my} \sin 2\theta_0 \quad (3.86b)$$

$$\frac{\partial \eta_2}{\partial T_0} - \frac{\partial \varphi_2}{\partial y} = -\eta'_{11} \cos \theta_0 + \eta_{11} \beta' \sin \theta_0 \quad (\text{在 } y=0 \text{ 处}) \quad (3.87)$$

$$\frac{\partial \eta_2}{\partial x} - \frac{\partial \Phi_2}{\partial y} = \frac{m^2+1}{2m} \cdot \eta^2_{11} \cdot \sin 2\theta_0 \quad (\text{在 } y=0 \text{ 处}) \quad (3.88)$$

$$-\eta_2 + \frac{\partial \varphi_2}{\partial T_0} - k_c^2 \frac{\partial^2 \eta_2}{\partial x^2} - mk_o \chi \frac{\partial \Phi_2}{\partial x} = -\frac{1}{2} mk_o \chi \cdot \eta^2_{11} \cdot \cos 2\theta_0 \quad (\text{在 } y=0 \text{ 处}) \quad (3.89)$$

满足 (2.25) 及 (3.86) ~ (3.89) 的二阶解为

$$\eta_2 = a_{22} (\eta^2_{11} \cos 2\theta_0 - \cos \theta_2) \quad (3.90)$$

$$\varphi_2 = \eta'_{11} (\cos \theta_0) \frac{\cosh(y+h)}{\sinh h} - \eta_{11} \beta' (\sin \theta_0) \frac{\cosh(y+h)}{\sinh h} + \frac{\mu_2 a_{22}}{2} (\sin \theta_2) \frac{\cosh 2(y+h)}{\sinh 2h} \quad (3.91)$$

$$\Phi_2 = \eta^2_{11} \left[\frac{M^4(\gamma+1)}{8m^3} y + \frac{d_{22}}{m} \right] e^{-2m y} \cdot \sin 2\theta_0 - \frac{a_{22}}{m} e^{-2m y} \cdot \sin \theta_2 \quad (3.92)$$

其中 a_{22} 及 d_{22} 应按 $k=k_c$ (即 $\mu_1=0$) 代入估值。

将 (3.82) ~ (3.84) 及 (3.90) ~ (3.92) 代入 (2.26) ~ (2.28) 及修正后的 (2.29) 中, 得三阶方程组:

$$\nabla^2 \varphi_3 = 0 \quad (3.93a)$$

$$m^2 \frac{\partial^2 \Phi_3}{\partial x^2} + \frac{\partial^2 \Phi_3}{\partial y^2} = (p_1 + qy) \eta^3_{11} e^{-3m y} \cdot \sin \theta_0 + \text{NPST} \quad (3.93b)$$

$$\frac{\partial \eta_3}{\partial T_0} - \frac{\partial \varphi_3}{\partial y} = 0 + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.94)$$

$$\frac{\partial \eta_3}{\partial x} - \frac{\partial \Phi_3}{\partial y} = p_3 \eta^3_{11} \cdot \sin \theta_0 + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.95)$$

$$-\eta_3 + \frac{\partial \varphi_3}{\partial T_0} - k_c^2 \frac{\partial^2 \eta_3}{\partial x^2} - mk_o \chi \frac{\partial \Phi_3}{\partial x} = \left[\sigma(\chi - 2k_o) \eta_{11} - C_1 \eta''_{11} + C_1 \eta_{11} \beta'^2 \right. \\ \left. + \left(\frac{3}{8} k_c^2 + k_o \chi p_6 \right) \eta^3_{11} \right] \cos \theta_0 + (2\eta'_{11} \beta' + \eta_{11} \beta'') \sin \theta_0 + \text{NSPT} \quad (\text{在 } y=0 \text{ 处}) \quad (3.96)$$

其中 $p_1 \sim p_6$ 应按 $\mu_1=0$ 代入估值。(3.93) ~ (3.96) 的特解中含有永年项, 为了消除它, 应有下述方程组成立:

$$\begin{cases} 2\eta'_{11} \cdot \beta' + \eta_{11} \cdot \beta'' = 0 \end{cases} \quad (3.97)$$

$$C_1 \eta'_{11} + \sigma(2k_o - \chi) \eta_{11} - C_1 \eta_{11} \beta'^2 - \Gamma \eta^3_{11} = 0 \quad (3.98)$$

$$\text{由 (3.97) 要求 } \eta^2_{11} \cdot \beta' = G \quad (3.99)$$

其中 G 为任意常数。当 $G=0$, 得 $\eta^2_{11} \equiv 0$ 或 $\beta' \equiv 0$ 。其中 $\eta^2_{11} \equiv 0$ 的解不满足初位移条件, 无意义; $\beta' \equiv 0$ 的解使 (3.98) 式蜕化为 (3.77) 式, 这就是三、(二)、1 节中讨论过的驻波情况。当 $G \neq 0$ 时, η^2_{11} 与 β' 的依存关系见图 1, 这时 η_{11}^2 不能通过零点。

将 (3.99) 代入 (3.98), 得

$$C_1 \eta''_{11} + (2k_o - \chi) \sigma \eta_{11} - \frac{C_1 G^2}{\eta_{11}^3} - \Gamma \eta^3_{11} = 0 \quad (3.100)$$

倘再规定 $\eta''_{11}(0)=0$, 则对 (3.100) 积分一次, 得

$$\eta_{i1}^2 + \frac{(2k_0 - \chi)\sigma}{C_1}(\eta_{i1}^2 - 1) - \frac{\Gamma}{2C_1}(\eta_{i1}^2 + 1)(\eta_{i1}^2 - 1) + G^2\left(\frac{1}{\eta_{i1}^2} - 1\right) = 0 \quad (3.101)$$

经过整理, 可得

$$\eta_{i1}^2 \cdot \eta_{i1}'^2 = \frac{\Gamma}{2C_1}(\eta_{i1}^2 - 1) \left[\eta_{i1}^2 - \frac{1}{2}(\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 - \frac{8C_1G^2}{\Gamma}}) \right] \left[\eta_{i1}^2 - \frac{1}{2}(\tilde{\Gamma} - \sqrt{\tilde{\Gamma}^2 - \frac{8C_1G^2}{\Gamma}}) \right]$$

令 $z = \sqrt{\frac{8C_1G^2}{|\Gamma|}}$, 最后得

$$\eta_{i1}^2 \cdot \eta_{i1}'^2 = \frac{\Gamma}{2C_1}(\eta_{i1}^2 - 1) \left[\eta_{i1}^2 - \frac{1}{2}(\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 + z^2}) \right] \left[\eta_{i1}^2 - \frac{1}{2}(\tilde{\Gamma} - \sqrt{\tilde{\Gamma}^2 + z^2}) \right] \quad (\text{当 } \Gamma < 0) \quad (3.102)$$

$$\eta_{i1}^2 \cdot \eta_{i1}'^2 = \frac{\Gamma}{2C_1}(\eta_{i1}^2 - 1)(\eta_{i1}^2 - a)(\eta_{i1}^2 - c) \quad (\text{当 } \Gamma > 0) \quad (3.103)$$

其中 $a(\tilde{\Gamma}, z) = (\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 - z^2})/2 \quad (3.104)$

$c(\tilde{\Gamma}, z) = (\tilde{\Gamma} - \sqrt{\tilde{\Gamma}^2 - z^2})/2 \quad (3.105)$

以下分别讨论 $\Gamma < 0$ 及 $\Gamma > 0$ 的情况。

$\Gamma < 0$ 的情况

当 $\tilde{\Gamma} < 0$ 时, (3.102) 右端的 $(\tilde{\Gamma} - \sqrt{\tilde{\Gamma}^2 + z^2})/2 < 0$, $(\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 + z^2})/2 \geq 0$, 故 η_{i1}^2 只有在 1 与非负的数 $(\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 + z^2})/2$ 之间变化时, $\eta_{i1} \cdot \eta_{i1}'$ 才不致于是虚数, 即 η_{i1} 有界。当 $\tilde{\Gamma} > 0$ 时, $(\tilde{\Gamma} - \sqrt{\tilde{\Gamma}^2 + z^2})/2 \leq 0$, $(\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 + z^2})/2 > 0$, η_{i1}^2 只在 1 与正数 $(\tilde{\Gamma} + \sqrt{\tilde{\Gamma}^2 + z^2})/2$ 间变化, η_{i1} 仍有界。简言之, 对 $\Gamma < 0$ 的情况, η_{i1} 总是有界的。

$\Gamma > 0$ 的情况

A. 倘若 $|\tilde{\Gamma}| \leq z$, 则 $(\eta_{i1}^2 - a)$ 与 $(\eta_{i1}^2 - c)$ 为共轭复数 (见 (3.103) 式), 故 $(\eta_{i1}^2 - a) \cdot (\eta_{i1}^2 - c) \geq 0$, 即只有 $\eta_{i1}^2 \geq 1$ 时, $\eta_{i1} \cdot \eta_{i1}'$ 才不致于是虚数。换言之, η_{i1} 无界 (见图 2 中的 1 区)。

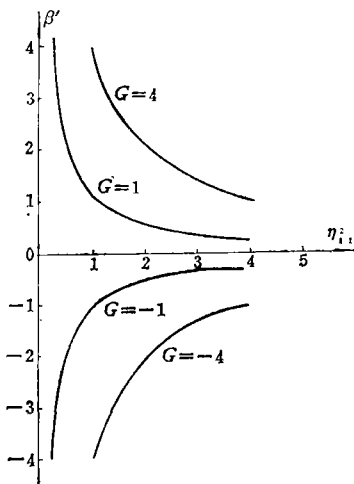


图 1

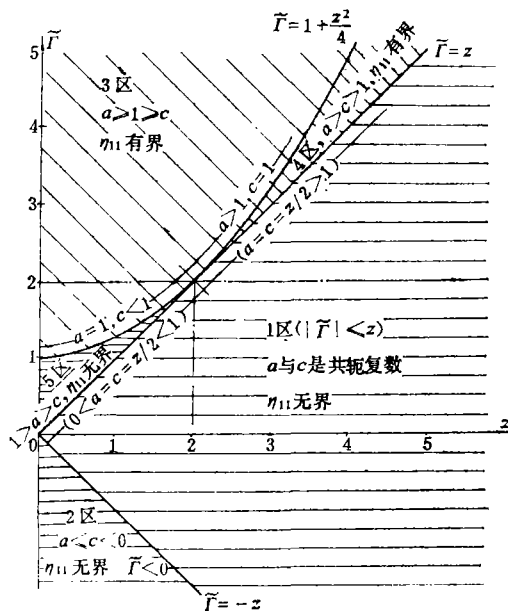


图 2

B. 倘 $\bar{\Gamma} < 0$ 而又有 $|\bar{\Gamma}| > z$ 时 (见图 2 中的 2 区), 因 $\sqrt{\bar{\Gamma}^2 - z^2} \leq |\bar{\Gamma}|$, 故 a 是非正的数, c 是负数, 因而 $(\eta_{i1}^2 - a)$ 及 $(\eta_{i1}^2 - c)$ 均是正数. 同理, $\eta_{i1}^2 \geq 1$, 即 η_{11} 无界.

C. 对 $\bar{\Gamma} > z > 0$ 的情况 (见图 2 中的 3、4、5 区)

我们先考察 a, c 相对于 1 的位置如何随 $\bar{\Gamma}$ 及 z 而变化. 不难证明, 在 $\bar{\Gamma} > z > 0$ 时, a 是 $\bar{\Gamma}$ 的递增函数, c 是 $\bar{\Gamma}$ 的递减函数; 而 $a=1$ 或 $c=1$ 又只能发生在 $\bar{\Gamma}=1+z^2/4$ 的抛物线上. 考虑到 $\sqrt{\bar{\Gamma}^2 - z^2}$ 只应取正值, 故在抛物线 $\bar{\Gamma}=1+z^2/4$ 上, 当 $z \leq 2$ 时, $\sqrt{\bar{\Gamma}^2 - z^2} = 1 - z^2/4$; 而当 $z \geq 2$ 时, $\sqrt{\bar{\Gamma}^2 - z^2} = z^2/4 - 1$. 于是图 2 中从点 $(0, 1)$ 到点 $(2, 2)$ 的这段抛物线上有 $a=1$ 及 $c < 1$ 成立; 而在 $(2, 2)$ 到 (∞, ∞) 的这段抛物线上则有 $a > 0$ 及 $c=1$ 成立.

再把图 2 中 $\bar{\Gamma}-z$ 平面内剩余的部分细分如下: 抛物线以上的区域称为 3 区, 抛物线与直线 $\bar{\Gamma}=z$ 之间的区域, 对 $z > 2$ 的那部分称为 4 区; 对 $z < 2$ 的那部分称为 5 区, 然后逐个讨论.

(a) 考虑到 a 及 c 的单调性, 只要 $\bar{\Gamma} \geq 1 + z^2/4$ (图 2 中 3 区), 就有 $a \geq 1$ 及 $c \leq 1$ 成立. 按 (3.103) 式, 遂有 $c \leq \eta_{i1}^2 \leq 1$, 即 η_{11} 有界.

(b) 对图 2 中的 4 区, 有 $a(\bar{\Gamma}, z) < c(\bar{\Gamma}, z) > 1$, 所以 $1 \leq \eta_{i1}^2 \leq c$, η_{11} 有界.

(c) 对图 2 中的 5 区, 有 $1 > a(\bar{\Gamma}, z) > c(\bar{\Gamma}, z) \geq 0$, 所以 $\eta_{i1}^2 \geq 1$, η_{11} 无界.

由图 2 可见, $\bar{\Gamma}=1$ 只在 $z=0$ (即 β 与 T_1 无关) 时, 才是稳定与否的分界, 这正是三、(二)、1 节中讨论过的情况. 而当 $z > 0$ (即 β 是 T_1 的函数) 时, 作为 η_{11} 有界与否的分界点 $\bar{\Gamma}$ 的值都将比 1 大, 即使得不稳定的波数范围比 $z=0$ 情况还要扩大. 这意味着行波不稳定的波数范围不可能比驻波小, 即行波型扰动更危险些 (“危险”的含义是具有较大的发生不稳定的波数范围).

在完成本文的过程中, 始终得到徐复同志的热情鼓励与帮助, 并与戴世强、李家春同志作过富有启发性的讨论, 谨致诚挚的谢意.

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Nonlinear Kelvin-Helmholtz Instability

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Abstract

A non-linear analysis is presented with derivative expansion method for the interfacial stability of a liquid film adjacent to a subsonic gas flow under the influence of body force and surface tension. The non-linear Rayleigh-Taylor instability is included as a special case. The gas and liquid are considered to be inviscid. Though Nayfeh (1971) gave consideration into this case, there is something omitted in his third-order equation (e. g. p. 213 expression (2.29)) and inconsistent with his solutions (e. g. the firstorder solution (2.31) does not satisfy his initial conditions (2.20)). Besides, in this paper, our solution near the cut-off wave number is extended to include the case of travelling waves and a new conclusion is drawn.