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Darcy 渗流定律的微观界定及其应用*

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摘要: 将 Boltzmann 微观方程用 Chapman_Enskog 展开, 结合 BGK 近似理论得到 Navier_Stokes 方程, 通过在多孔介质的某种表征体元上的不同的方式平均 Navier_Stokes 方程后得出 Darcy 渗流定律的一种表示方法, 并用实例证明了方法的可靠性•

关 键 词: Darcy 定律; Boltzmann 方程; Navier_Stokes 方程; Chapman_Enskog 展开
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引 言

使用微观方法研究复杂问题具有广泛而深刻的物理背景和现实意义, 在渗流力学领域, 人们已对它产生了很大的兴趣, 并用它去解释和揭示出新的渗流物理现象与本质[1,2]• 多孔介质流体运动的复杂性是公认的, 对它的动力学模型既可以通过宏观的也可以通过微观的特征推得, 而宏观的方法是比较普遍采用的, Darcy 定律本身就是描述流量与速率、渗透率、粘度以及压力梯度等宏观参数相互关系的方程• 在实践中, 由于多孔介质内部结构的复杂性, 求解 Darcy 方程不是件容易的事情• 多孔介质内部的异质结构导致难以方便的应用其特有的边界条件, 而微观的方法主要是通过理想化后可以使多孔介质内部的空隙、裂缝、内部流管的几何特征得到简化• 早些时候, 曾有过许多推测认为微观方法可以模拟真实的物理系统[3], 这种推测后来得到了证实•

本文中我们先从 Boltzmann 方程出发, 通过一系列数学方法后得出了 Navier_Stokes 方程, 再采用 Irmay[4] 的平均方法得到 Darcy 定律的表达式, 经过实例证明, 结果正确可靠, 这种研究对进一步阐明渗流机理是有意义的•

1 原理和方法

用 $f(\boldsymbol{x}, \boldsymbol{u}, t)\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{u}$ 表示在 t 时刻, 位于体积元 $\mathrm{d}^3\boldsymbol{x}$ 和速度间隔 $\mathrm{d}^3\boldsymbol{u}$ 内的分子数(这个分子数是统计平均值), 过了时间 $\mathrm{d}t$ 后, 在时刻 $t+\mathrm{d}t$, 位于同一体积元 $\mathrm{d}^3\boldsymbol{x}$ 和同一速度间隔 $\mathrm{d}^3\boldsymbol{u}$ 内

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的分子数将为$f(\boldsymbol{x}, \boldsymbol{u}, t+dt)d^3\boldsymbol{x}d^3\boldsymbol{u}$• 当 dt 很小时, 用泰勒级数展开, 只取头两项, 得:

$$f(\boldsymbol{x}, \boldsymbol{u}, t+dt)d^3\boldsymbol{x}d^3\boldsymbol{u} = \left[f(\boldsymbol{x}, \boldsymbol{u}, t) + \frac{\partial f}{\partial t}dt\right]d^3\boldsymbol{x}d^3\boldsymbol{u} \tag{1}$$

引入如下假定:

①只考虑两个碰撞;

②两个分子的统计分布是相互独立的, 略去相关性, 即分子混沌性假设;

③外力不影响局部的碰撞动力学, 取分子质量为 1•

可以推出 Boltzmann 分布函数$f(\boldsymbol{x}, \boldsymbol{u}, t)$方程

$$\partial_t f + \boldsymbol{u}\partial_{\boldsymbol{x}} f + \frac{\boldsymbol{K}}{m}\partial_{\boldsymbol{u}} f = Q(f, f), \tag{2}$$

其中 $\boldsymbol{K}$ 为外力, m 为分子质量, 这里取 1, $\partial_t f + \boldsymbol{u}\partial_{\boldsymbol{x}} f$ 为分布函数的漂移变化率, (2) 式即为玻耳兹曼微分方程, 其中

$$Q(f, f) = \int d^3\boldsymbol{u}_1 \int d\Omega\,\sigma(\Omega)\,|\boldsymbol{u} - \boldsymbol{u}_1|\,[f(\boldsymbol{u}_1)f(\boldsymbol{u}_1') - f(\boldsymbol{u})f(\boldsymbol{u}_1)] \tag{3}$$

称为碰撞积分, $\sigma(\Omega)$ 为碰撞截面• 碰撞积分具有五个基本的碰撞不变量 $\psi_k(\boldsymbol{u})$ $(k = 0, 1, 2, 3, 4)$, 而且满足[3]:

$$\int Q(f, f)\,\psi_k(\boldsymbol{u})d^3\boldsymbol{u} = 0, \tag{4}$$

这五个碰撞不变量分别为 $\psi_0 = 1$, $(\psi_1, \psi_2, \psi_3) = \boldsymbol{u}$ 和 $\psi_4 = \boldsymbol{u}^2$, 即正比于质量、动量和能量• 将玻耳兹曼方程乘以碰撞不变量, 再对 $d^3\boldsymbol{u}$ 求积分, 同时注意到

$$\int d^3\boldsymbol{u}\,\psi_k Q(f, f) = 0,$$

得到守恒定律的表达式为:

$$\int d^3\boldsymbol{u}\,\psi_k(\partial_t + \boldsymbol{u}_\alpha\partial\boldsymbol{x}_\alpha)f(\boldsymbol{x}, \boldsymbol{u}, t) = 0• \tag{5}$$

三维情况下还可以得到:

$$\partial_t\rho + \partial_{\boldsymbol{x}_\alpha}(\rho\boldsymbol{u}_\alpha) = 0, \tag{6}$$

$$\rho\partial_t\boldsymbol{u}_\alpha + \rho\boldsymbol{u}_\beta\partial_{\boldsymbol{x}_\beta}\boldsymbol{u}_\alpha = -\partial_{\boldsymbol{x}_\alpha}\boldsymbol{P}_{\alpha\beta}, \tag{7}$$

$$\rho\partial_t\theta + \rho\boldsymbol{u}_\beta\partial_{\boldsymbol{x}_\beta}\theta = -\frac{2}{3}\partial_{\boldsymbol{x}_\alpha}q_\alpha - \frac{2}{3}\boldsymbol{P}_{\alpha\beta}\Lambda_{\alpha\beta}, \tag{8}$$

而

$$n(\boldsymbol{x}, t) = \int d^3\boldsymbol{u}f(\boldsymbol{x}, \boldsymbol{u}, t), \tag{9}$$

$$\rho(\boldsymbol{x}, t) = mn(\boldsymbol{x}, t) \qquad (m \text{ 为常数}), \tag{10}$$

$$\rho\boldsymbol{u}_\alpha(\boldsymbol{x}, t) = m\int d^3\boldsymbol{u}\,\boldsymbol{u}_\alpha f(\boldsymbol{x}, \boldsymbol{u}, t), \tag{11}$$

$$Q(\boldsymbol{x}, t) = K_B T(\boldsymbol{x}, t) = \frac{m}{3n}\int d^3\boldsymbol{u}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta)^2 f(\boldsymbol{x}, \boldsymbol{u}, t), \tag{12}$$

$$\Lambda_{\alpha\beta} = \frac{m}{2}(\partial_{\boldsymbol{x}_\beta}\boldsymbol{u}_\alpha + \partial_{\boldsymbol{x}_\alpha}\boldsymbol{u}_\beta), \tag{13}$$

$$\boldsymbol{P}_{\alpha\beta} = m\int d^3\boldsymbol{u}(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)(\boldsymbol{v}_\beta - \boldsymbol{u}_\beta)f(\boldsymbol{x}, \boldsymbol{u}, t), \tag{14}$$

$$\boldsymbol{q}_\alpha(\boldsymbol{x}, t) = \frac{m^2}{2}\int d^3\boldsymbol{u}(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)(\boldsymbol{v}_\beta - \boldsymbol{u}_\beta)^2 f(\boldsymbol{x}, \boldsymbol{u}, t)• \tag{15}$$

确切地说守恒方程无多大用处, 除非我们求出 Boltzmann 方程的解f 并用f 去计算式(9)~

式(15),不过应该注意到这里张量 $\boldsymbol{P}_{\alpha\beta}$ 不存在对流项,已不同于以前文献[5]中所描述的。引入 Knudsen 数 ε, $\varepsilon \ll 1$。将分布函数进行 Chapman_Enskog 展开:

$$f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots \tag{16}$$

守恒方程(6)~(8)式可以化为:

$$\partial_t \rho + \partial_{x_\alpha}(\rho \boldsymbol{u}_\alpha) = 0, \tag{17}$$

$$\rho \partial_t \boldsymbol{u}_\alpha + \rho \boldsymbol{u}_\beta \partial_{x_\beta} \boldsymbol{u}_\alpha = -\sum_{n=0}^{\infty} \varepsilon^n \partial_{x_\alpha} \boldsymbol{P}_{\alpha\beta}^{(n)} \tag{18}$$

$$\rho \partial_t \theta + \rho \boldsymbol{u}_\beta \partial_{x_\beta} \theta = -\frac{2}{3} \sum_{n=0}^{\infty} \varepsilon^n (\partial_{x_\alpha} \boldsymbol{q}_\alpha^{(n)} + \boldsymbol{P}_{\alpha\beta}^{(n)} \Lambda_{\alpha\beta}) \cdot \tag{19}$$

这里:

$$\boldsymbol{P}_{\alpha\beta}^{(n)} = m \int \mathrm{d}^3 \boldsymbol{v} f^{(n)} (\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)(\boldsymbol{v}_\beta - \boldsymbol{u}_\beta), \tag{20}$$

$$\boldsymbol{q}_\alpha^{(n)} = \frac{m^2}{2} \int \mathrm{d}^3 \boldsymbol{v} f^{(n)} (\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha) | \boldsymbol{v} - \boldsymbol{u} |^2 \cdot \tag{21}$$

由于 f 可以通过 ρ, $\boldsymbol{u}$ 和 T 表示成时间 t 的函数,所以:

$$\partial_t f = \partial_\rho f \partial_t \rho + \partial_{\boldsymbol{u}_\alpha} f \partial_t \boldsymbol{u}_\alpha + \partial_\theta f \partial_t \theta, \tag{22}$$

将式(16)分别代入到 f 对 ρ, $\boldsymbol{u}_\alpha$ 及 T 的导数后可得:

$$\partial_\rho f = \partial_\rho f^{(0)} + \varepsilon \partial_\rho f^{(1)} + \varepsilon^2 \partial_\rho f^{(2)} + \cdots, \tag{23}$$

$$\partial_{\boldsymbol{u}_\alpha} f = \partial_{\boldsymbol{u}_\alpha} f^{(0)} + \varepsilon \partial_{\boldsymbol{u}_\alpha} f^{(1)} + \varepsilon^2 \partial_{\boldsymbol{u}_\alpha} f^{(2)} + \cdots, \tag{24}$$

$$\partial_\theta f = \partial_\theta f^{(0)} + \varepsilon \partial_\theta f^{(1)} + \varepsilon^2 \partial_\theta f^{(2)} + \cdots, \tag{25}$$

式(23)、(24)、(25)中 ε 前面的内容应该与守恒方程的展开式(17)、(18)、(19)同价 ε 前面的内容是一致的,引入:

$$\partial_t = \varepsilon \partial_t^{(1)} + \varepsilon^2 \partial_t^{(2)} + \cdots \tag{26}$$

由守恒定律可以导出:

$$\partial_t^{(1)} \rho = -\partial_{x_\alpha}(\rho \boldsymbol{u}_\alpha), \tag{27}$$

$$\partial_t^{(n+1)} \rho = 0 \qquad (n > 0), \tag{28}$$

$$\partial_t^{(1)} \boldsymbol{u}_\alpha = -\boldsymbol{u}_\beta \partial_{x_\beta} \boldsymbol{u}_\alpha - \frac{1}{\rho} \partial_{x_\beta} \boldsymbol{P}_{\alpha\beta}^{(0)}, \tag{29}$$

$$\partial_t^{(n+1)} \boldsymbol{u}_\alpha = -\frac{1}{\rho} \partial_{x_\beta} \boldsymbol{P}_{\alpha\beta}^{(n)} \qquad (n > 0), \tag{30}$$

$$\partial_t^{(1)} \theta = -\boldsymbol{u}_\beta \partial_{x_\beta} \theta - \frac{2}{3\rho} (\partial_{x_\alpha} \boldsymbol{q}_\alpha^{(0)} + \boldsymbol{P}_{\alpha\beta}^{(0)} \Lambda_{\alpha\beta}), \tag{31}$$

$$\partial_t^{(n+1)} \theta = -\frac{2}{3\rho} (\partial_{x_\alpha} \boldsymbol{q}^{(n)} \boldsymbol{P}_{\alpha\beta}^{(n)} \Lambda_{\alpha\beta}) \qquad (n > 0), \tag{32}$$

将上述式子引入 $\partial_t f$ 的展开式:

$$\begin{aligned} \partial_t f = {} & (\varepsilon \partial_t^{(1)} + \varepsilon^2 \partial_t^{(2)} + \cdots)(f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots) = \\ & \varepsilon \partial_t^{(1)} f^{(0)} + \varepsilon^2 (\partial_t^{(1)} f^{(1)} + \partial_t^{(2)} f^{(0)}) + \varepsilon^3 \cdots \end{aligned} \tag{33}$$

将分布函数 f 的展开式代入 Boltzmann 方程的碰撞积分 $Q(f, f)$,同时考虑到 BGK 近似关系[6],我们得到:

$$\begin{aligned} Q(f, f) = {} & -\omega(f - f^{(0)}) = \\ & -\omega(\varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots) = \\ & J^{(0)} + \varepsilon J^{(1)} + \varepsilon^2 J^{(2)} + \cdots, \end{aligned} \tag{34}$$

其中:

$$J^{(0)}(f^{(0)}) = 0, \tag{35}$$

$$J^{(1)}(f^{(0)}, f^{(1)}) = J^{(1)}(f^{(1)}) = -\omega f^{(1)}, \tag{36}$$

$$J^{(2)}(f^{(0)}, f^{(1)}, f^{(2)}) = J^{(2)}(f^{(2)}) = -\omega f^{(2)}. \tag{37}$$

其中碰撞频率 ω 为常数. 一般情况下, $J^{(n)}$ 不会与所有 $f^{(k)}(k \leqslant n)$ 有关联, 而是只于 $f^{(n)}$ 有关. 这种简化原理使得碰撞积分经过 BGK 近似后对于分布函数 f 来说是线性的, Boltzmann 方程左边的偏导数 ∂_x 应该与时间导数的 ε 相同, 也就是说

$$\partial_{x_\alpha} = \varepsilon \partial^{(1)}_{x_\alpha}. \tag{38}$$

式(38)看起来就像一个展开式的首项. 只有一种宏观空间尺度值得考虑, 象对流、扩散这些不同的宏观过程是可以通过时间尺度区别开来的, 事实上它们是在相同空间尺度上的. 比较 Boltzmann 方程的两边, 使同阶 ε 前面的内容相等, 可以得到:

$$J^{(0)}(f^{(0)}) = 0, \tag{39}$$

$$\partial^{(1)}_t f^{(0)} + \boldsymbol{u}_\alpha \partial^{(1)}_{x_\alpha} f^{(0)} = J^{(1)}(f^{(0)}, f^{(1)}) = -\omega f^{(1)}, \tag{40}$$

$$\partial^{(1)}_t f^{(1)} + \partial^{(2)}_t f^{(0)} + \boldsymbol{u}_\alpha \partial^{(1)}_{x_\alpha} f^{(1)} = J^{(2)}(f^{(0)}, f^{(1)}, f^{(2)}) = -\omega f^{(2)}. \tag{41}$$

引入 Maxwell 分布函数[3]

$$f^{(M)} = f(\boldsymbol{x}, \boldsymbol{u}, t) = n\left(\frac{m}{2\pi K_B T}\right)^{3/2} \exp\left[-\frac{m}{2K_B T}(\boldsymbol{v} - \boldsymbol{u})^2\right], \tag{42}$$

由 BGK 近似可以得出关系式:

$$J(f) = \omega\left[f^{(M)}(\boldsymbol{x}, \boldsymbol{u}) - f(\boldsymbol{x}, \boldsymbol{u})\right]. \tag{43}$$

因此我们可以利用上述关系消去式(39)~式(41)中的 J 系列, 同时, 可以很方便的从式(40)中求得

$$f^{(1)} = -\frac{1}{\omega}\left(\partial^{(1)}_t f^{(0)} + \boldsymbol{u}_\alpha \partial^{(1)}_{x_\alpha} f^{(0)}\right), \tag{44}$$

式(44) $f^{(0)}$ 可以由 Maxwell 分布函数求得.

下一步按照式(20)计算张量 $\boldsymbol{P}^{(1)}_{\alpha\beta}$:

$$\begin{aligned}\boldsymbol{P}^{(1)}_{\alpha\beta} &= m\int \mathrm{d}^3\boldsymbol{u}(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)(\boldsymbol{v}_\beta - \boldsymbol{u}_\beta) f^{(1)} = \\ &- \frac{m}{\omega}\int \mathrm{d}^3\boldsymbol{u}(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)(\boldsymbol{v}_\beta - \boldsymbol{u}_\beta)\left(\partial^{(1)}_t f^{(0)} + \boldsymbol{v}_\gamma \partial^{(1)}_{x_\gamma} f^{(0)}\right)\end{aligned} \tag{45}$$

将(45)式代入式(27)~式(30)后得到(略去上标(1)以上的项):

$$\begin{aligned}\partial_t f^{(0)}(\rho, \boldsymbol{u}) = \; & \frac{\partial f^{(0)}}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial f^{(0)}}{\partial \boldsymbol{u}_\gamma}\frac{\partial \boldsymbol{u}_\gamma}{\partial t} = -\frac{f^{(0)}}{m}\frac{\partial(\rho \boldsymbol{u}_\gamma)}{\partial \boldsymbol{x}_\gamma} + \\ & \frac{m}{K_B T}(\boldsymbol{v}_\gamma - \boldsymbol{u}_\gamma) f^{(0)}\left(\boldsymbol{u}\frac{\partial \boldsymbol{u}_\gamma}{\delta \partial \boldsymbol{x}_\delta} + \frac{1}{\rho}\frac{\partial \boldsymbol{P}^{(0)}_{\gamma\delta}}{\partial \boldsymbol{x}_\delta}\right) = \\ & -\frac{\rho f^{(0)}}{m}\frac{\partial \boldsymbol{u}_\alpha}{\partial \boldsymbol{x}_\gamma} - \frac{f^{(0)}}{m}\boldsymbol{u}_\gamma \frac{\partial \rho}{\partial \boldsymbol{x}_\gamma} + \\ & \frac{m}{K_B T}(\boldsymbol{v}_\gamma - \boldsymbol{u}_\gamma) f^{(0)} \boldsymbol{u}_\delta \frac{\partial \boldsymbol{u}_\gamma}{\partial \boldsymbol{x}_\delta} + \\ & \frac{m}{K_B T}(\boldsymbol{v}_\gamma - \boldsymbol{u}_\gamma) f^{(0)} \frac{1}{\rho}\delta_{\gamma\delta}\frac{\partial \rho}{\partial \boldsymbol{x}_\delta},\end{aligned} \tag{46}$$

$$\boldsymbol{v}_\gamma \partial_{\boldsymbol{x}_\gamma} f^{(0)} = \boldsymbol{v}_\gamma \frac{\partial f^{(0)}}{\partial \rho}\frac{\partial \rho}{\partial \boldsymbol{x}_\gamma} + \boldsymbol{v}_\gamma \frac{\partial f^{(0)}}{\partial \boldsymbol{u}_\delta}\frac{\partial \boldsymbol{u}_\delta}{\partial \boldsymbol{v}_\gamma} =$$

$$v_\gamma \frac{f^{(0)}}{m}\frac{\partial \rho}{\partial \boldsymbol{x}_\gamma} + v_\gamma \frac{m}{K_B T}(v_\delta - u_\delta) f^{(0)} \frac{\partial \boldsymbol{u}_\delta}{\partial \boldsymbol{x}_\gamma}• \tag{47}$$

可以比较方便的求出各类积分值:

$$-\frac{1}{m}\frac{\partial \boldsymbol{u}_\gamma}{\partial \boldsymbol{x}_\gamma}\int \mathrm{d}^3 \boldsymbol{u}(v_\alpha - u_\alpha)(v_\beta - u_\beta) f^{(0)} = -\delta_{\alpha\beta} n \frac{K_B T}{m}\frac{\partial \boldsymbol{u}_\gamma}{\boldsymbol{x}_\gamma}, \tag{48}$$

$$\frac{1}{m}\frac{\partial \rho}{\partial \boldsymbol{x}_\gamma}\int \mathrm{d}^3 \boldsymbol{u}(v_\alpha - u_\alpha)(v_\beta - u_\beta)(v_\gamma - u_\gamma) f^{(0)} = 0, \tag{49}$$

$$\frac{m}{K_B T} f^{(0)}\left(u_\delta \frac{\partial \boldsymbol{u}_\alpha}{\partial \boldsymbol{x}_\delta} + \frac{1}{\rho}\delta_{\gamma\delta}\frac{\partial \rho}{\partial \boldsymbol{x}_\delta}\right) • \int \mathrm{d}^3 \boldsymbol{u}(v_\alpha - u_\alpha)(v_\beta - u_\beta)(v_\gamma - u_\gamma) f^{(0)} = 0, \tag{50}$$

$$\frac{m}{K_B T}\frac{\partial \boldsymbol{u}_\delta}{\partial \boldsymbol{x}_\gamma}\int \mathrm{d}^3 \boldsymbol{u}(v_\alpha - u_\alpha)(v_\beta - u_\beta) v_\gamma (v_\gamma - u_\gamma) f^{(0)} =$$

$$(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) n \frac{K_B T}{m}\frac{\partial \boldsymbol{u}_\delta}{\partial \boldsymbol{x}_\gamma}, \tag{51}$$

进而有

$$\boldsymbol{P}^{(1)}_{\alpha\beta} = -n\frac{K_\beta T}{\omega}\left[(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\frac{\partial \boldsymbol{u}_\delta}{\boldsymbol{x}_\gamma} - \delta_{\alpha\beta}\frac{\partial \boldsymbol{u}_\gamma}{\partial \boldsymbol{x}_\gamma}\right] =$$

$$n\frac{K_B T}{\omega}\begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix}• \tag{52}$$

忽略密度与温度变分, $\boldsymbol{P}^{(1)}_{\alpha\beta}$ 的散度为:

$$\frac{\partial \boldsymbol{P}^{(1)}_{\alpha\beta}}{\partial \boldsymbol{x}_\alpha} = \mu\left[2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial z}\right] e_x + \cdots =$$

$$\mu\left[\frac{\partial}{\partial \boldsymbol{x}_\beta}\left(\frac{\partial \boldsymbol{u}_\alpha}{\boldsymbol{x}_\beta}\right) + \frac{\partial}{\partial \boldsymbol{x}_\alpha}\left(\frac{\partial \boldsymbol{u}_\beta}{\boldsymbol{x}_\beta}\right)\right] =$$

$$\mu[\nabla^2 \boldsymbol{u} + \nabla(\nabla • \boldsymbol{u})], \tag{53}$$

其中: $\mu = n\frac{K_B T}{\omega}$, (54)

为动力切变粘度系数, 最后我们得到 Navier_Stokes 方程:

$$\partial_t \boldsymbol{u}_\alpha + \boldsymbol{u}_\beta \partial_{x_\beta} \boldsymbol{u}_\alpha =$$

$$-\frac{1}{\rho}\partial_{x_\alpha}\boldsymbol{P} + \upsilon \partial_{x_\beta}\partial_{x_\beta}\boldsymbol{u}_\alpha + \xi \partial_{x_\alpha}\partial_{x_\beta}\boldsymbol{u}_\beta, \tag{55}$$

其中体积模量 ξ 与切变模量 υ 这两项粘度系数为:

$$\upsilon = \xi = \frac{K_B T}{\omega m}• \tag{56}$$

2 Darcy 运动方程

许多研究者通过在多孔介质的某种表征体元上以不同的方式平均 Navier_Stokes 方程得到 Darcy 方程, 我们采用 Irmay[4] 的方法, 引进一种由直径为 d 的圆球所组成的、代表孔隙率为 $n = L/(l+d)$ 的均质各向同性介质的模型, 计算方程(55) 所示的 Navier_Stokes 方程的空间平均, 用带上横线符号表示• 由于均质各向同性的缘故, $\overline{\boldsymbol{u}_\alpha} = 0$• 如果设速度分量 $\boldsymbol{u}_\alpha$ 互不相关, 则有:

$$\overline{\boldsymbol{u}_\alpha \boldsymbol{u}_\beta} = 0 \qquad (\alpha = 1, 2, 3,\ \beta = 1, 2, 3,\ \alpha \neq \beta, \text{以下均同}), \tag{57}$$

$$\partial_{x_\alpha} \overline{\boldsymbol{u}_\alpha \boldsymbol{u}_\beta} = 0. \tag{58}$$

对于不可压缩流体来说, 有连续方程:

$$\sum_{\alpha=1}^{3} \partial_{x_\alpha} \boldsymbol{u}_\alpha = 0. \tag{59}$$

这样, 我们可以得到

$$\partial_{x_\alpha}^2 \boldsymbol{u}_\alpha = - \sum_{\beta=1,\ \beta\neq\alpha}^{\beta=3} \partial_{x_\beta} \partial_{x_\beta} \boldsymbol{u}_\beta. \tag{60}$$

根据上述考虑, 我们可以得到:

$$\overline{\partial_{x_\alpha}^2 \boldsymbol{u}_\alpha} = - \sum_{\alpha=1,\ \beta\neq\alpha}^{\beta=3} \overline{\partial_{x_\beta} \partial_{x_\beta} \boldsymbol{u}_\beta} = 0. \tag{61}$$

将比流量及其时间导数表示为 $\overline{\boldsymbol{u}_\alpha} = q/n$ 和 $\partial \overline{\boldsymbol{u}_\alpha}/\partial t = (\partial q/\partial n)/n$. 在圆球上, 因流体粘附于壁面, 故 $\boldsymbol{u}_\alpha = 0$. 在颗粒之间 $\boldsymbol{u}_\alpha$ 按一定规律分布, $\boldsymbol{u}_\alpha$ 的最大值在相邻两颗粒的中点. 对于这种分布, 存在:

$$\frac{\partial^2 \boldsymbol{u}_\alpha}{\partial \boldsymbol{x}_\beta^2} < 0 \qquad (\alpha \neq \beta) \tag{62}$$

经过平均后可得到:

$$\sum_{\beta=1,\ \beta\neq\alpha}^{3} \frac{\partial^2 \boldsymbol{u}_\alpha}{\partial \boldsymbol{x}_\beta^2} = - [\beta (1-n)^2 / n^3 d^3]\, q. \tag{63}$$

式中 β 是一个取决于颗粒形状, 但与孔隙率或直径无关的数值系数, 因为 $\mathrm{div}\,\boldsymbol{u} = 0$, 所以

$$\frac{1}{2} \partial_{x_\alpha} (\boldsymbol{u}_\alpha^2) = \boldsymbol{u}_{x_\alpha} \partial_{x_\alpha} \boldsymbol{u}_\alpha = - \sum_{\beta=1,\ \beta\neq\alpha}^{3} \boldsymbol{u}_\alpha \partial_x \boldsymbol{u}_\beta, \tag{64}$$

或 $$\overline{\frac{1}{2} \partial_{x_\alpha} (\boldsymbol{u}_\alpha^2)} = 0. \tag{65}$$

在颗粒之间的收缩断面上, $\boldsymbol{u}_\beta = 0\ (\beta = 1, 2, 3)$. 因此, 在收敛区, $\partial_{x_\alpha} \left(\sum_{\beta=1,\ \beta\neq\alpha}^{3} \boldsymbol{u}_\alpha^2 \right) < 0$. 由于我们考虑的是大速度的情形, 所以在发散区内我们有一种中等的发散. 这样就使发散流动的有效二次项减少. 因此, 我们可以估计:

$$\partial_{x_\alpha} \left(\sum_{\beta=1,\ \beta\neq\alpha}^{3} \overline{\boldsymbol{u}_\beta^2} \right) = - c\, \overline{\boldsymbol{u}_\alpha^2} / L = - [c(1-n)/n^3 d]\, q^2. \tag{66}$$

(66) 式中 c 是数字形状因数. 再设每单位重量流体的总能量为:

$$E = \varphi + u^2/2g = z + p/\rho g + u^2/2g. \tag{67}$$

当不考虑 z 的作用时, 则平均水力梯度[7]为:

$$J_\alpha = - \frac{\partial E}{\partial \boldsymbol{x}_\alpha} = - \frac{1}{\rho g} \frac{\partial p}{\partial \boldsymbol{x}_\alpha} - \frac{1}{\rho} \overline{\boldsymbol{u}_\alpha} \frac{\partial \boldsymbol{u}_\alpha}{\boldsymbol{x}_\alpha}. \tag{68}$$

联合 Navier_Stokes 方程(55)式, 并考虑到将平均效果为零项消去, 最后得到:

$$J_\alpha = W q_\alpha + c \frac{\partial q_\alpha}{\partial t}, \tag{69}$$

式中

$$W = [\upsilon\beta (1-n^2)\, n^3] / (g d^2), \quad c = \frac{1}{ng}. \tag{70}$$

(69) 式中在非稳定流动情况下应保留 $c\partial q_\alpha/\partial t$ 一项, 然而正如 Polubarinova_Kochina[8] 指出, 除

去运动开始的那一瞬间(大约是零点几秒)外，这个附加项将迅速趋于零。所以(69)式可以化为:

$$J_\alpha = Wq_\alpha \text{ 或 } q_\alpha = K\overline{J}_\alpha,\quad K = (gd^2/\beta\nu)[n^3/(1-n)^2] \tag{71}$$

上式即为 Darcy 定律的推广形式以及渗透率 K 的 Kozeny 公式[7]，W 为水力阻力系数。

3 应用举例

利用我们在文献[1]中介绍的格子 Boltzmann 方法来验证方程(71)所示 Darcy 定律的可靠性以及实用性。选择 3D_15 速格子 Boltzmann 模型，经过类同于文献[1]的一系列步骤后我们得到:

$$\text{格点速度:}\begin{cases} \boldsymbol{C}_0 = (0, 0, 0),\quad \boldsymbol{C}_{1,2} = (\pm 2, 0, 0); \\ \boldsymbol{C}_{3,4} = (0, \pm 2, 0),\quad \boldsymbol{C}_{5,6} = (0, 0, \pm 2), \\ \boldsymbol{C}_7 = \cdots = \boldsymbol{C}_{14} = (\pm 1, \pm 1, \pm 1), \end{cases} \tag{72}$$

几个分布函数初始值:

$$\begin{cases} F_0 = \dfrac{7}{18}\rho,\ F_2 = \dfrac{\rho}{16}\left(\dfrac{K_bT}{m}\right)^2 = \dfrac{\rho}{36}, \\ F_3 = \dfrac{\rho}{8}\left(\dfrac{K_bT}{m}\right)^2 = \dfrac{\rho}{18};\ \dfrac{K_bT}{m} = \dfrac{2}{3}, \end{cases} \tag{73}$$

其中 m、K_b、T 与上文相同。平衡分布函数我们选用文献[9]介绍的:

$$\begin{cases} F_0^{eq} = d^{(0)} + \delta^{(0)}\boldsymbol{u}^2, \\ F_i^{eq} = d^{(1)} + \beta^{(1)}\boldsymbol{c}_i\cdot\boldsymbol{u} + \\ \qquad \gamma^{(1)}(\boldsymbol{c}_i\cdot\boldsymbol{u})^2 + \delta^{(1)}\boldsymbol{u}^2 \qquad (i = 1, \cdots, 6), \\ F_i^{eq} = d^{(2)} + \beta^{(2)}\boldsymbol{c}_i\cdot\boldsymbol{u} + \\ \qquad \gamma^{(2)}(\boldsymbol{c}_i\cdot\boldsymbol{u})^2 + \delta^{(2)}\boldsymbol{u}^2 \qquad (i = 7, \cdots, 14), \end{cases} \tag{74}$$

这里

$$\begin{cases} d^{(0)} = d^{(1)} = \dfrac{\rho}{11},\ d^{(2)} = \dfrac{\rho}{22}, \\ \alpha^{(1)} = \dfrac{\rho}{24},\ \alpha^{(2)} = \dfrac{\rho}{12}, \\ \gamma^{(1)} = \dfrac{\rho}{32},\ \gamma^{(2)} = \dfrac{\rho}{16}, \\ \delta^{(0)} = -\dfrac{7}{24}\rho,\ \delta^{(1)} = -\dfrac{\rho}{48},\ \delta^{(2)} = -\dfrac{\rho}{24}. \end{cases} \tag{75}$$

最后我们得到格子 Boltzmann 方程为:

$$\begin{cases} F_i(t + \Delta t) - F_i(t) = -\dfrac{1}{\tau}(F_i - F_i^{eq}), \\ \rho = \displaystyle\sum_{i=0}^{14} F_i = \sum_{i=0}^{14} F_i, \\ \rho\boldsymbol{u} = \displaystyle\sum_{i=0}^{14}\boldsymbol{c}_iF_i^{eg} = \sum_{i=0}^{14}\boldsymbol{c}_iF_i^{eq}, \\ \rho\varepsilon = \dfrac{1}{2}\displaystyle\sum_{i=0}^{14}(\boldsymbol{c}_i - \boldsymbol{u})^2F_i = \dfrac{1}{2}\sum_{i=0}^{14}(\boldsymbol{c} - \boldsymbol{u})^2F_i^{eq}, \end{cases} \tag{76}$$

其中 $i = 1, \cdots, 14$, τ为驰豫时间, ε为内能, 对方程(72)～(76)通过 Chapman_Enskog 展开再采用上文中的 Irmayg 平均法即可回推得(71)式所示的 Darcy 方程, 也就是说方程(72)～(76)构成了求解方程(71)式的格子 Boltzmann 迭代关系式•

选取适当的边界条件, 本例中我们选三个 3cm × 3cm × 3cm 的多孔介质样本, 分别是沙子、硅土、玻璃小珠制作, 这样:

$$F_i(x_\alpha = 0) = F_i \quad (x_\alpha = h), \tag{77}$$

其中 $h = 3$cm, 介质固相("无移动"边界)采用文献[10]介绍的"反射"边界条件, 计算这三个样本的平均渗透率, 并同时与用渗透仪测的值进行比较, 表 1 显示数值结果与实验测量结果吻合很好•

表 1 由实验测量及数值模拟得出的样本渗透率 κ 值

内容	实验测量数据			数值结果		
样本	沙子	硅土	玻璃小珠	沙子	硅土	玻璃小珠
渗透率 κ/cm^2	0.88E-7	1.22E-6	2.21E-6	1.12E-6	1.24E-6	2.28E-6

4 结　语

本文中我们用 Boltzmann 微观方程导出了 Naiver_Stokes 方程, 再采用 Irmay 平均法导出了 Darcy 定律的一种表述方式, 整个过程避免了对流项等非物理产品的产出, 结果可靠, 为对复杂结构与边界条件下方便使用 Darcy 定律提供了一定的参考方法•

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Microcosmic Bound Theorem of Daycy' s Law and Its Application

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Abstract: By combining Chapman_Enskog expansion with the BGK approximation to Baltzmann equation and Navier_Stokes equation was obtained. And an expression of Darcy' s law was obtained through taking variable average over Navier_Stokes equation on some representative space in porous media, and finally an example was taken to prove its reliability.

Key words: Darcy' s law; Boltzmann equation; Navier_Stokes equation; Chapman_Enskog expansion