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包括激发和衰减的粘弹性Ⅱ型破裂过程的研究*

范 家 参

（云南工学院，1983年10月31日收到）

摘 要

用非线性 Rayleigh 阻尼公式描述初始破裂时有激发而加速，至一定高速时有衰减而止裂。视介质为匀质各向同性的 Voigt 线性粘弹性体，用小参数摄动法把滑开型（Ⅱ型）破裂定义的非线性偏微分方程组线性化，得出各次逼近解所定义的线性方程组，再用动坐标表示的广义 Fourier级数把问题简化为非齐次的 Mathieu 方程，用 WKBJ 法给出问题在稳定区域的渐近解。

一、前 言

地震破裂的一个最盛行的机理就是认为在水平剪应力作用下，断层（裂缝）端部因高度应力集中，超过地壳岩石的断裂韧性而失稳破裂。在文献[1]里我们已经得到此问题在弹性范围内的解析解。由于引用了非线性的 Rayleigh 阻尼，就可以反映出低速破裂时有激发而加速，但破裂速度高达一定值后，又有和破裂速度立方正比的阻尼力起主要作用而使运动衰减直到止裂。

在文献[2]里，我们根据 Taylor 的动力屈服强度理论[3]说明：在地震破裂这个动力学过程中，不可能存在塑性变形，其理由就在于很多文献中都把孕震阶段地壳应力集中而超过弹性极限的那部份当做理想塑性体，所以地壳的弹性极限就是塑性变形开始的静力屈服强度。而在地震破裂这个动力学过程中，按上述 Taylor 理论，则同一材料的塑性动力屈服强度达到它的静力屈服强度的2～3倍，因此在地震破裂中没有可能发生塑性变形。

可是地壳的力学性质中却公认存在着不可忽视的粘性，例如文献[4,5,6,7,8]。在地震过程中粘滞性表现为内摩擦的阻尼力，是使运动衰减的原因之一。故在本文中我们把地壳当做匀质各向同性的线性 Voigt 粘弹性介质[4]，即代表弹性的弹簧和代表粘性的粘壶是并联的。

二、基 本 原 理

本文考虑有限长裂缝一端产生滑开型失稳破裂，只考虑其瞬时效应，因此可以不考虑两

* 钱伟长推荐。

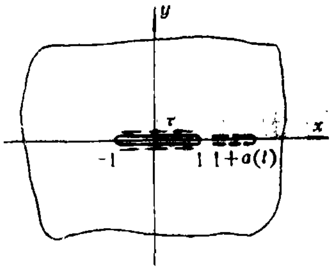

图 1

端点间动力反射的相互影响。

裂缝处于平面应变状态，其受力情况如图1所示。图1上的直角坐标已经无量纲化，x坐标为普通横坐标除以半条裂缝之长，y坐标为普通纵坐标除以半个裂缝宽度，故裂缝表面初值定义在$-1\leqslant x\leqslant 1$，$y=0\pm$。本文的时间变量t为普通时间除以纵波通过半条裂缝所需时间而无量纲化。裂缝右端位移用$a(t)$表示，故$t>0$时，裂缝右端的位置为$x=1+a(t)$，$y=0\pm$。无量纲的破裂速度为$\dot{a}(t)=da(t)/dt$，$0<\dot{a}(t)<K$，K表示横波速度和纵波速度之比，$K<1$。当$t\leqslant 0$时$\dot{a}(t)=0$。用u和v表示在无量纲坐标x和y方向的无量纲位移。考虑Voigt粘弹性介质在非线性Rayleigh阻尼作用下，问题为求解下列偏微分方程组[1,4,5]：

$$\frac{\partial^2 u}{\partial x^2}+K^2\frac{\partial^2 u}{\partial y^2}+(1-K^2)\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 u}{\partial t^2}+\left[-A+B\left(\frac{\partial u}{\partial t}\right)^2\right]\frac{\partial u}{\partial t}$$
$$+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial x\partial y}\right) \tag{2.1}$$

$$K^2\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+(1-K^2)\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 v}{\partial t^2}+\left[-A+B\left(\frac{\partial v}{\partial t}\right)^2\right]\frac{\partial v}{\partial t}$$
$$+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}\right)+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial y}\right) \tag{2.2}$$

在(2.1)和(2.2)式里，A和B是无量纲的正阻尼常数，对各种介质的A和B值，须经实验求出，目前已知一些介质的A值，但对B值则知之甚少，本文先进行理论探索。η_1和η_2是介质中分别与体积应变和剪切应变有关的两个无量纲化了的粘滞系数[4]。地壳介质有用Voigt粘弹体[4]，也有用Maxwell体的[8]。本文用前者，因它代表固体，而后者代表流体。此时本构方程为：

$$\left.\begin{aligned}
\sigma_x&=\frac{1}{K^2}\left(\frac{\partial u}{\partial x}+\eta_1\frac{\partial \dot{u}}{\partial x}\right)+\left(\frac{1}{K^2}-2\right)\left(\frac{\partial v}{\partial y}+\eta_1\frac{\partial \dot{v}}{\partial y}\right)\\
\sigma_y&=\left(\frac{1}{K^2}-2\right)\left(\frac{\partial u}{\partial x}+\eta_1\frac{\partial \dot{u}}{\partial x}\right)+\frac{1}{K^2}\left(\frac{\partial v}{\partial y}+\eta_1\frac{\partial \dot{v}}{\partial y}\right)\\
\tau_{xy}&=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\eta_2\left(\frac{\partial \dot{v}}{\partial x}+\frac{\partial \dot{u}}{\partial y}\right)
\end{aligned}\right\} \tag{2.3}$$

在(2.3)式中，位移分量上面的一点，表示对t求导数，即为：$\dot{u}=\partial u/\partial t$，$\dot{v}=\partial v/\partial t$。

裂缝表面的边界条件是：

$$\left.\begin{aligned}
&\sigma_y(x,0\pm,t)=0\\
&\tau_{xy}(x,0\pm,t)=\tau(x,t)\ \text{已知}
\end{aligned}\right\}\qquad x\leqslant 1+a(t) \tag{2.4}$$

由于对于$y=0$的反对称性而有[12]：

$$\left.\begin{aligned}
&u(x,0\pm,t)=0\\
&\sigma_y(x,0\pm,t)=0
\end{aligned}\right\}\qquad x>1+a(t),\ \text{或者}\ x<-1 \tag{2.5}$$

初始条件是：

$$u(x,y,0)=v(x,y,0)=0 \tag{2.6}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\left.\frac{\partial v}{\partial t}\right|_{t=0}=0 \tag{2.7}$$

三、用摄动法导出逐次逼近的线性化方程组

在文献[1]中我们已经证明(2.1)和(2.2)式非线性项的系数 $B<1$, 因此有下列收敛级数的展开:

$$u(x,y,t)=u_0(x,y,t)+Bu_1(x,y,t)+B^2u_2(x,y,t)+\cdots+B^nu_n(x,y,t)+\cdots \tag{3.1}$$

$$v(x,y,t)=v_0(x,y,t)+Bv_1(x,y,t)+B^2v_2(x,y,t)+\cdots+B^nv_n(x,y,t)+\cdots \tag{3.2}$$

代(3.1)和(3.2)入(2.1)和(2.2)可得:

零级近似:

$$\frac{\partial^2 u_0}{\partial x^2}+K^2\frac{\partial^2 u_0}{\partial y^2}+(1-K^2)\frac{\partial^2 v_0}{\partial x\partial y}=\frac{\partial^2 u_0}{\partial t^2}-A\frac{\partial u_0}{\partial t}+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial x\partial y}\right)+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v_0}{\partial x\partial y}\right) \tag{3.3}$$

$$K^2\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}+(1-K^2)\frac{\partial^2 u_0}{\partial x\partial y}=\frac{\partial^2 v_0}{\partial t^2}-A\frac{\partial v_0}{\partial t}+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial^2 u_0}{\partial x\partial y}\right)+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u_0}{\partial x\partial y}\right) \tag{3.4}$$

一级近似:

$$\frac{\partial^2 u_1}{\partial x^2}+K^2\frac{\partial^2 u_1}{\partial y^2}+(1-K^2)\frac{\partial^2 v_1}{\partial x\partial y}=\frac{\partial^2 u_1}{\partial t^2}-A\frac{\partial u_1}{\partial t}+\left(\frac{\partial u_0}{\partial t}\right)^3+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial x\partial y}\right)+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v_1}{\partial x\partial y}\right) \tag{3.5}$$

$$K^2\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+(1-K^2)\frac{\partial^2 u_1}{\partial x\partial y}=\frac{\partial^2 v_1}{\partial t^2}-A\frac{\partial v_1}{\partial t}+\left(\frac{\partial v_0}{\partial t}\right)^3+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 u_1}{\partial x\partial y}\right)+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u_1}{\partial x\partial y}\right) \tag{3.6}$$

二级近似:

$$\frac{\partial^2 u_2}{\partial x^2}+K^2\frac{\partial^2 u_2}{\partial y^2}+(1-K^2)\frac{\partial^2 v_2}{\partial x\partial y}=\frac{\partial^2 u_2}{\partial t^2}-A\frac{\partial u_2}{\partial t}+3\left(\frac{\partial u_0}{\partial t}\right)^2\frac{\partial u_1}{\partial t}+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial x\partial y}\right)$$

$$+\eta_2\frac{\partial}{\partial t}\left(\frac{4}{3}\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v_2}{\partial x\partial y}\right) \tag{3.7}$$

$$K^2\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}+(1-K^2)\frac{\partial^2 u_2}{\partial x\partial y}$$

$$=\frac{\partial^2 v_2}{\partial t^2}-A\frac{\partial v_2}{\partial t}+3\left(\frac{\partial v_0}{\partial t}\right)^2\frac{\partial v_1}{\partial t}+\eta_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v_2}{\partial y^2}+\frac{\partial^2 u_2}{\partial x\partial y}\right)$$

$$+\eta_2\frac{\partial}{\partial t}\left(\frac{\partial^2 v_2}{\partial x^2}+\frac{4}{3}\frac{\partial^2 v_2}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_2}{\partial x\partial y}\right) \tag{3.8}$$

. . . .

代(3.1)和(3.2)入初始条件和边界条件，则得零级近似的边界条件和初始条件：

$$\left.\frac{\partial \dot{u}_0}{\partial x}\right|_{y=0}=\left.\frac{\partial \dot{v}_0}{\partial y}\right|_{y=0}=0,\quad \sigma_y^{(0)}(x,0\pm,t)=0,\quad \tau_{xy}^{(0)}(x,0\pm,t)=\tau(x,t)\text{已知}$$

$$x\leqslant 1+a(t) \tag{3.9}$$

$$u_0(x,0\pm,t)=\sigma_y^{(0)}(x,0\pm,t)=0,\quad x<-1\text{ 或 }x>1+a(t) \tag{3.10}$$

$$u_0(x,y,0)=v_0(x,y,0)=0 \tag{3.11}$$

$$\left.\frac{\partial u_0}{\partial t}\right|_{t=0}=\left.\frac{\partial v_0}{\partial t}\right|_{t=0}=0 \tag{3.12}$$

对于一级和以后各级近似，都是零边界条件和零初始条件。

四、求解逐次逼近方程组

引入Galileo变换把静坐标系(x,y,t)变换为动坐标系(ξ,y',t')，其表达式是：

$$x=\xi+a(t),\quad y=y',\quad t=t' \tag{4.1}$$

$$a(t)=\frac{2\tau}{\pi}\dot{a}_M\left[1-\sin\frac{\pi}{2}\left(1-\frac{t}{\tau}\right)\right] \tag{4.2}$$

在(4.2)式中，$\dot{a}_M<K$为无量纲的最大破裂速度。τ为从开始破裂到破裂速度达$\dot{a}_M$时所需要的无量纲时间，$\dot{a}_M$和τ都是常数。

注意到：

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi},\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial t'}-\dot{a}\frac{\partial}{\partial \xi} \tag{4.3}$$

则在动坐标系中零级近似方程组(3.3)和(3.4)两式可以写成：

$$\dot{a}\left(\eta_1+\frac{4}{3}\eta_2\right)\frac{\partial^3 u_0}{\partial \xi^3}+\dot{a}\eta_2\frac{\partial^3 u_0}{\partial \xi\partial y^2}+\dot{a}\left(\eta_1+\frac{\eta_2}{3}\right)\frac{\partial^3 v_0}{\partial \xi^2\partial y}+\left(1-\dot{a}^2\right)\frac{\partial^2 u_0}{\partial \xi^2}$$

$$+(\ddot{a}-A\dot{a})\frac{\partial u_0}{\partial \xi}+K^2\frac{\partial^2 u_0}{\partial y^2}+(1-K^2)\frac{\partial^2 v_0}{\partial \xi\partial y}$$

$$=\frac{\partial^2 u_0}{\partial t^2}-2\dot{a}\frac{\partial^2 u_0}{\partial t\partial \xi}-A\frac{\partial u_0}{\partial t}+\left(\eta_1+\frac{4}{3}\eta_2\right)\frac{\partial^3 u_0}{\partial t\partial \xi^2}+\eta_2\frac{\partial^3 u_0}{\partial t\partial y^2}+\left(\eta_1+\frac{\eta_2}{3}\right)\frac{\partial^3 v_0}{\partial t\partial \xi\partial y} \tag{4.4}$$

$$\dot{a}\left(\eta_1+\frac{4}{3}\eta_2\right)\frac{\partial^3 v_0}{\partial \xi\partial y^2}+\dot{a}\eta_2\frac{\partial^3 v_0}{\partial \xi^3}+\dot{a}\left(\eta_1+\frac{\eta_2}{3}\right)\frac{\partial^3 u_0}{\partial \xi^2\partial y}+(K^2-\dot{a}^2)\frac{\partial^2 v_0}{\partial \xi^2}$$

$$+(\ddot{a}-A\dot{a})\frac{\partial v_0}{\partial\xi}+\frac{\partial^2 v_0}{\partial y^2}+(1-K^2)\frac{\partial^2 u_0}{\partial\xi\partial y}$$
$$=\frac{\partial^2 v_0}{\partial t^2}-2\dot{a}\frac{\partial^2 v_0}{\partial t\partial\xi}-A\frac{\partial v_0}{\partial t}+\eta_2\frac{\partial^3 v_0}{\partial t\partial\xi^2}+\left(\eta_1+\frac{4}{3}\eta_2\right)\frac{\partial^3 v_0}{\partial t\partial y^2}+\left(\eta_1+\frac{\eta_2}{3}\right)\frac{\partial^3 u_0}{\partial t\partial\xi\partial y} \tag{4.5}$$

以动坐标表示的边界条件和初始条件是:

$$\left.\begin{aligned}&\left(\frac{\partial^2 u_0}{\partial t\partial\xi}-\dot{a}\frac{\partial^2 u_0}{\partial\xi^2}\right)_{y=0}=\left(\frac{\partial^2 v_0}{\partial t\partial y}-\dot{a}\frac{\partial^2 v_0}{\partial\xi\partial y}\right)_{y=0}=0\\&\left(\frac{\partial^2 v_0}{\partial t\partial\xi}+\frac{\partial^2 u_0}{\partial t\partial y}-\dot{a}\frac{\partial^2 v_0}{\partial\xi^2}-\dot{a}\frac{\partial^2 u_0}{\partial\xi\partial y}\right)_{y=0}=\dot{\tau}(x,t)+\frac{\tau(x,t)}{\eta_2}\quad\text{已知}\end{aligned}\right\}-1\leqslant x\leqslant 1+a(t) \tag{4.6}$$

$$u_0(\xi,0\pm,t)=\sigma_y^{(0)}(\xi,0\pm,t)=0,\qquad x>1+a(t)\text{或}x<-1 \tag{4.7}$$

$$u_0(\xi,y,0)=v_0(\xi,y,0)=0 \tag{4.8}$$

$$\left(\dot{u}_0-\dot{a}\frac{\partial u_0}{\partial\xi}\right)_{t=0}=\left(\dot{v}_0-\dot{a}\frac{\partial v_0}{\partial\xi}\right)_{t=0}=0 \tag{4.9}$$

为求零级近似解，设:

$$u_0(x,y,t)=u_0(\xi,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{mn}^{(0)}(t)H(1-\xi)\cos m\pi\xi\sin n\pi y \tag{4.10}$$

$$v_0(x,y,t)=v_0(\xi,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}F_{mn}^{(0)}(t)H(1-\xi)\sin m\pi\xi\cos n\pi y \tag{4.11}$$

则(4.10)和(4.11)已经满足边界条件(4.6)的为零的两式和(4.7)式. 代(4.10)和(4.11)入(4.4)和(4.5)两式，利用富氏级数的正交性，得以下四式，其中(4.4)或(4.5)每一式变为相应的两个关于未知函数 $E_{mn}^{(0)}(t)$ 或 $F_{mn}^{(0)}(t)$ 的常微分方程:

$$\frac{dE_{mn}^{(0)}(t)}{dt}-\frac{1}{2}\left[\left(\eta_1+\frac{4}{3}\eta_2\right)m^2\pi^2+\eta_2n^2\pi^2-\left(\frac{\ddot{a}}{\dot{a}}-A\right)\right]E_{mn}^{(0)}(t)$$
$$=\frac{1}{2}mn\pi^2\left(\eta_1+\frac{\eta_2}{3}\right)F_{mn}^{(0)}(t) \tag{4.12}$$

$$\frac{d^2E_{mn}^{(0)}(t)}{dt^2}-\left[A+\left(\eta_1+\frac{4}{3}\eta_2\right)m^2\pi^2+\eta_2n^2\pi^2\right]\frac{dE_{mn}^{(0)}(t)}{dt}$$
$$+[(1-\dot{a}^2)m^2\pi^2+K^2n^2\pi^2]E_{mn}^{(0)}(t)$$
$$=\left(\eta_1+\frac{\eta_2}{3}\right)mn\pi^2\frac{dF_{mn}^{(0)}(t)}{dt}-(1-K^2)mn\pi^2F_{mn}^{(0)}(t) \tag{4.13}$$

$$\frac{dF_{mn}^{(0)}(t)}{dt}-\frac{1}{2}\left[\left(\eta_1+\frac{4}{3}\eta_2\right)n^2\pi^2+\eta_2m^2n^2-\left(\frac{\ddot{a}}{\dot{a}}-A\right)\right]F_{mn}^{(0)}(t)$$
$$=\frac{1}{2}mn\pi^2\left(\eta_1+\frac{\eta_2}{3}\right)E_{mn}^{(0)}(t) \tag{4.14}$$

$$\frac{d^2F_{mn}^{(0)}(t)}{dt^2}-\left[A+\left(\eta_1+\frac{4}{3}\eta_2\right)n^2\pi^2+\eta_2m^2\pi^2\right]\frac{dF_{mn}^{(0)}(t)}{dt}$$

$$+[(K^2-\dot{a}^2)m^2\pi^2+n^2\pi^2]F_{mn}^{(0)}(t)$$

$$=\left(\eta_1+\frac{\eta_2}{3}\right)mn\pi^2\frac{dE_{mn}^{(0)}(t)}{dt}-(1-K^2)mn\pi^2E_{mn}^{(0)}(t) \tag{4.15}$$

首先求解(4.12)和(4.14)两式得：

$$\left.\begin{aligned}
E_{mn}^{(0)}(t)=&\frac{D_1\exp\left[\frac{t}{2}(\psi_{mn}^{(0)}+A)\right]}{\sqrt{\sin\frac{\pi}{2\tau}t}}+\frac{1}{2}mn\pi^2\left(\eta_1+\frac{\eta_2}{3}\right)\frac{\exp\left[\frac{t}{2}(\psi_{mn}^{(0)}+A)\right]}{\sqrt{\sin\frac{\pi}{2\tau}t}}\\
&\cdot\int\exp\left[-\frac{t}{2}(\psi_{mn}^{(0)}+A)\right]\sqrt{\sin\frac{\pi}{2\tau}t}\quad F_{mn}^{(0)}(t)dt\\
F_{mn}^{(0)}(t)=&\frac{D_2\exp\left[\frac{t}{2}(\psi_{nm}^{(0)}+A)\right]}{\sqrt{\sin\frac{\pi}{2\tau}t}}+\frac{1}{2}mn\pi^2\left(\eta_1+\frac{\eta_2}{3}\right)\frac{\exp\left[\frac{t}{2}(\psi_{nm}^{(0)}+A)\right]}{\sqrt{\sin\frac{\pi}{2\tau}t}}\\
&\cdot\int\exp\left[-\frac{t}{2}(\psi_{nm}^{(0)}+A)\right]\sqrt{\sin\frac{\pi}{2\tau}t}\quad E_{mn}^{(0)}(t)dt\\
\psi_{mn}^{(0)}=&\left(\eta_1+\frac{4}{3}\eta_2\right)m^2\pi^2+\eta_2n^2\pi^2,\qquad \psi_{nm}^{(0)}=\left(\eta_1+\frac{4}{3}\eta_2\right)n^2\pi^2+\eta_2m^2\pi^2
\end{aligned}\right\} \tag{A}$$

在(A)式中的 D_1 和 D_2 是积分常数，易知必须有 $D_1=D_2=0$，否则在 t 的定义区间两端点 $t=0$ 及 $t=2\tau$ 时(A)的解就有奇异值。此外在(A)式中积分号前的 $1/\sqrt{\sin\pi t/2\tau}$ 也同样有上述奇异值。为消除这些永年项(secular term)，必须使积分值恒为零。所以有：

$$E_{mn}^{(0)}(t)=F_{mn}^{(0)}(t)=0 \tag{B}$$

现在我们只需求解(4.13)和(4.15)两式即可，为此引入下列变换：

$$\left.\begin{aligned}
&E_{mn}^{(0)}(t)=\phi_{mn}^{(0)}(t)\exp\left[\frac{t}{2}(A+\psi_{mn}^{(0)})\right],\quad F_{mn}^{(0)}(t)=\Gamma_{mn}^{(0)}(t)\exp\left[\frac{t}{2}(A+\psi_{nm}^{(0)})\right]\\
&T=\frac{\pi}{2}\left(1-\frac{t}{\tau}\right),\quad \phi_{mn}^{(0)}(t)=\phi_{mn}^{(0)}(T),\quad \Gamma_{mn}^{(0)}(t)=\Gamma_{mn}^{(0)}(T)
\end{aligned}\right\} \tag{4.16}$$

则(4.13)和(4.15)分别化为下列非齐次 Mathieu 方程式：

$$\frac{d^2\phi_{mn}^{(0)}(T)}{dT^2}+(\zeta_{mn}-2q_m\cos2T)\,\phi_{mn}^{(0)}(T)$$

$$=-\left[\frac{2\tau}{\pi}\frac{d\Gamma_{mn}^{(0)}(T)}{dT}+(1-K^2)mn\pi^2\Gamma_{mn}^{(0)}(T)\right]\exp\left[\frac{\tau}{2}(\psi_{nm}^{(0)}-\psi_{mn}^{(0)})\left(1-\frac{2T}{\pi}\right)\right] \tag{4.17}$$

$$\frac{d^2\Gamma_{mn}^{(0)}(T)}{dT^2}+(\bar{\zeta}_{mn}-2q_m\cos2T)\Gamma_{mn}^{(0)}(T)$$

$$=-\left[\frac{2\tau}{\pi}\frac{d\phi_{mn}^{(0)}(T)}{dT}+(1-K^2)mn\pi^2\phi_{mn}^{(0)}(T)\right]\exp\left[\frac{\tau}{2}(\psi_{mn}^{(0)}-\psi_{nm}^{(0)})\left(1-\frac{2T}{\pi}\right)\right] \tag{4.18}$$

在(4.17)和(4.18)两式中：

$$
\left.
\begin{aligned}
q_m &= \dot{a}_M^2 m^2 \tau^2 \\
\zeta_{mn} &= 4\tau^2 K^2 n^2 + 4\tau^2 m^2 - 2\dot{a}_M^2 \tau^2 m^2 - \frac{\tau^2}{\pi^2}\left(A+\psi_{mn}^{(0)}\right)^2 \\
&= \partial t^2 - 2\dot{a}\ \partial t\partial\xi - A\ \partial t + \eta_2\ \partial t\partial\xi^2 + \left(\eta_1 + \frac{}{3}\eta_2\right)\partial t\partial y^2 + \left(\eta_1 + \frac{}{3}\right)\partial t\partial\xi\partial y \\
\bar{\zeta}_{mn} &= 4\tau^2 n^2 + 4\tau^2 K^2 m^2 - 2\dot{a}_M^2 \tau^2 m^2 - \frac{\tau^2}{\pi^2}\left(A+\psi_{nm}^{(0)}\right)^2
\end{aligned}
\right\} \tag{4.19}
$$

类似文献[1]的步骤，我们用 WKBJ 法[9,10]，得到(4.17)和(4.18)两式在稳定区域的渐近解为：

$$
\begin{aligned}
\phi_{mn}^{(0)}(T) = &\sum_{r=-r_0}^{r_0} c_r^{(0)} J_r\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right)\cos(\sqrt{\zeta_{mn}}-2r)T \\
&+ \sum_{r=-r_0}^{r_0}\sum_{s=-r_0}^{r_0} a_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right)\cos 2(r-s)T
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
\Gamma_{mn}^{(0)}(T) = &\sum_{r=-r_0}^{r_0} d_r^{(0)} J_r\left(\frac{q_m}{2\sqrt{\bar{\zeta}_{mn}}}\right)\cos(\sqrt{\bar{\zeta}_{mn}}-2r)T \\
&+ \sum_{r=-r_0}^{r_0}\sum_{s=-r_0}^{r_0} b_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\bar{\zeta}_{mn}}}\right)\cos 2(r-s)T
\end{aligned} \tag{4.21}
$$

在(4.20)和(4.21)两式中的待定系数 $a_r^{(0)}$，$b_r^{(0)}$，$c_r^{(0)}$ 和 $d_r^{(0)}$ 由下面四式确定：

$$
\sum_{r=-r_0}^{r_0} c_r^{(0)} J_r\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right) + \sum_{r=-r_0}^{r_0}\sum_{s=-r_0}^{r_0} a_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right) = 0 \tag{4.22}
$$

$$
\sum_{r=-r_0}^{r_0} d_r^{(0)} J_r\left(\frac{q_m}{2\sqrt{\bar{\zeta}_{mn}}}\right) + \sum_{r=-r_0}^{r_0}\sum_{s=-r_0}^{r_0} b_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\bar{\zeta}_{mn}}}\right) = 0 \tag{4.23}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\pi[m\dot{F}_{mn}^{(0)}(t)+n\dot{E}_{mn}^{(0)}(t)]\cos m\pi a(t) - m\pi^2\dot{a}(t)[mF_{mn}^{(0)}(t) + nE_{mn}^{(0)}(t)]\sin m\pi a(t)\right\} \\
&= 2\int_0^1 \left[\dot{\tau}(x,t)+\frac{1}{\eta_2}\tau(x,t)\right]\cos m\pi x\,dx
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\pi[m\dot{F}_{mn}^{(0)}(t)+n\dot{E}_{mn}^{(0)}(t)]\sin m\pi a(t) + m\pi^2\dot{a}(t)[mF_{mn}^{(0)}(t) + nE_{mn}^{(0)}(t)]\cos m\pi a(t)\right\} \\
&= 2\int_0^1 \left[\dot{\tau}(x,t)+\frac{1}{\eta_2}\tau(x,t)\right]\sin m\pi x\,dx \qquad (m=1,2,3,\cdots)
\end{aligned} \tag{4.25}
$$

至此零级近似解已经得出。若把以上有关零级近似中的 $A+\psi_{mn}^{(0)}$ 和 $A+\psi_{nm}^{(0)}$ 以及 $\psi_{mn}^{(0)}-\psi_{nm}^{(0)}$ 等均改为相应的负值，就是考虑线性阻尼力的粘弹性介质Ⅰ型破裂扩展问题的解。

还有用“常数变更法”求非齐次 Mathieu 方程的特解时，下列两式亦需成立，从而在 r 和 s 取和之值加以下列限制，即：

$$a_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right)(\sqrt{\zeta_{mn}}-2r)(\sqrt{\zeta_{mn}}-2s)$$

$$=4\left[\frac{4\tau^4}{\pi^4}(\psi_{nm}^{(0)}-\psi_{mn}^{(0)})-(1-K^2)mn\tau^2\right]b_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right) \tag{4.26}$$

$$b_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right)(\sqrt{\zeta_{mn}}-2r)(\sqrt{\zeta_{mn}}-2s)$$

$$=4\left[\frac{4\tau^4}{\pi^4}(\psi_{mn}^{(0)}-\psi_{nm}^{(0)})-(1-K^2)mn\tau^2\right]a_r^{(0)} J_s\left(\frac{q_m}{2\sqrt{\zeta_{mn}}}\right) \tag{4.27}$$

至于一级近似和以后各级近似的解，既类同于以上零级近似解，何况我们已经在文献[1]中给出弹性介质的一级近似解，所以粘弹性介质的一级近似解也可仿此照办，为精简计，不再在本文中对此述及。

五、结 束 语

线性弹性理论的断裂力学解，要求在裂缝尖端的应力分量表达式中有坐标的1/2阶奇异性，因而使此点的应力分量为无穷大。其实裂缝尖端并没有无穷大的应力，只是存在相当程度的应力集中。这是线性弹性理论本身的假定不合理而产生的缺点。因为此理论假定物体是由无穷小的质点组成，故所取微分自由体只能有正应力和剪应力作用，应力张量对称，而不能有分布的面力偶和体力偶作用。因此才产生了裂缝尖端应力表达式中有坐标1/2阶奇异性。如果考虑偶应力的弹性理论，上面所述奇异性就不存在了。所以即使在弹性范围内放弃经典的线弹性理论不合理的假定，裂缝尖端只应该存在有限度的应力集中[11]。本文已经考虑粘滞性与弹性并联的Voigt模型，就不必考虑裂缝尖端应力奇异性。最后本文所用广义富氏级数解三阶偏微分方程组，如弹性问题的二阶方程组一样[1]，用不着如[4]那样建立用Hermite函数表示的子波函数。

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Research of Visco-Elastic Type II Rupture with Exciting and Attenuation Process

Fan Ja-shen

(Yunnan Institute of Technology, Kunming)

Abstract

With non-linear Rayleigh damping formula we describe the exciting process when the rupture velocity is low and the attenuation process when the rupture velocity reaches a certain high value. Assuming the medium of the earth crust is homogeneous and isotropic linear Voigt visco-elastic body, with small parameter perturbation method to deduce the non-linear governing partial differential equations into a system of asymptotic linear ones, we solve them by means of generalized Fourier series with moving coordinates as its variables, thus we transform them into non-homogeneous Mathieu equations. At last Mathieu equations are solved by WKBJ method.