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均布载荷下矩形板大挠度问题的摄动变分解*

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摘　要

本文以中心挠度为摄动参数，将矩形板大挠度问题的非线性偏微分方程组转化成几个线性偏微分方程，然后用变分法求解，得出了具有任意长宽比的板的解答，给出了位移、挠度及各内力的解析表达式；并给出中心点和边界中心的应力数值计算公式．本文还以长宽比λ为参数，作出了最大挠度——载荷曲线及最大应力曲线．其结果与实验进行了比较，表明二者是一致的．

一、引　言

薄板在工程中使用非常广泛。如果挠度比厚度小，那么在 Kirchhoff 理论基础上计算出的结果是非常满意的。在金属飞机的情况下，控制飞机的重量是非常重要的，所以使用的金属板必须很薄，此时挠度就比其厚度大。因此，要得到这样的金属板的设计公式及图表，必须运用大挠度理论。关于在法向均布载荷下，四边固定的矩形板大挠度问题，许多学者已用各种方法计算过。如S. Way[1]用里兹能量法，S. Levy[2]用双富里叶级数法，C. T. Wang[3,4]用差分法。此外，钱伟长、叶开沅等使用摄动法，以中心挠度为摄动参数，解决了各种载荷及各种边界条件下圆薄板的大挠度问题[5,6,7,8,9]。

本文用摄动变分法，求解了任意长宽比的矩形板大挠度问题。首先，以中心挠度为摄动参数进行摄动，将卡门大挠度非线性方程组分解为三组线性方程。然后使用变分法确定假设位移形式中的待定系数，从而得出各级摄动解。并将结果与实验[11]进行了比较。

二、基本方程及边界条件

图1表示均布载荷 q 作用下的四边固定的矩形板中面。板长为$2a$，宽为$2b$，厚为h（常数）。用$U(x, y)$，$V(x, y)$和$W(x, y)$分别表示x，y和z方向的位移，z方向的位移$W(x, y)$又称挠度。

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* 本文摘要载《国际非线性力学会议论文集》，上海（1985）。

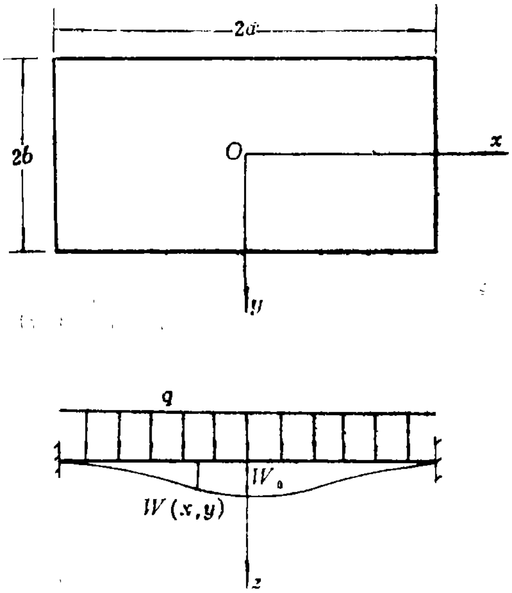

图 1

应力应变关系：

$$\left.\begin{aligned}
\sigma_x &= \frac{E}{1-\mu^2}\left\{\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2+\mu\left[\frac{\partial V}{\partial y}+\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2\right]\right\} \\
\sigma_y &= \frac{E}{1-\mu^2}\left\{\frac{\partial V}{\partial y}+\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2+\mu\left[\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]\right\} \\
\tau_{xy} &= \frac{E}{2(1+\mu)}\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial y}\right)
\end{aligned}\right\} \qquad (2.1)$$

平衡方程：

$$\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0 \\
&\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \\
&D\nabla^2\nabla^2 W=q+h\left(\sigma_x\frac{\partial^2 W}{\partial x^2}+\sigma_y\frac{\partial^2 W}{\partial y^2}+2\tau_{xy}\frac{\partial^2 W}{\partial x\partial y}\right)
\end{aligned}\right\} \qquad (2.2)$$

其中：$D=\dfrac{Eh^3}{12(1-\mu^2)}$，$\nabla^2\nabla^2(\ \ )=\dfrac{\partial^4(\ \ )}{\partial x^4}+2\dfrac{\partial^4(\ \ )}{\partial x^2\partial y^2}+\dfrac{\partial^4(\ \ )}{\partial y^4}$

E，μ 分别为板的杨氏模量和泊松比。

将(2.1)代入(2.2)，便得矩形板位移形式的卡门大挠度方程组：

$$
\left.\begin{aligned}
&2\frac{\partial^2 U}{\partial x^2}+(1-\mu)\frac{\partial^2 U}{\partial y^2}+(1+\mu)\frac{\partial^2 V}{\partial x\partial y}\\
&\quad=-(1-\mu)\frac{\partial W}{\partial x}\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)-\frac{1+\mu}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial W}{\partial x}\right)^2\right.\\
&\quad\left.+\left(\frac{\partial W}{\partial y}\right)^2\right](1+\mu)\frac{\partial^2 U}{\partial x\partial y}+(1-\mu)\frac{\partial^2 V}{\partial x^2}+2\frac{\partial^2 V}{\partial y^2}\\
&\quad=-(1-\mu)\frac{\partial W}{\partial y}\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)-\frac{1+\mu}{2}\frac{\partial}{\partial y}\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right]\\
&D\nabla^2\nabla^2 W=\frac{Eh}{1-\mu^2}\left\{\frac{\partial^2 W}{\partial x^2}\left(\frac{\partial U}{\partial x}+\mu\frac{\partial V}{\partial y}\right)\right.\\
&\quad\left.+\frac{\partial^2 W}{\partial y^2}\left(\frac{\partial V}{\partial y}+\mu\frac{\partial U}{\partial x}\right)+(1-\mu)\left(\frac{\partial U}{\partial y}+\mu\frac{\partial V}{\partial x}\right)\frac{\partial^2 W}{\partial x\partial y}\right\}\\
&\quad+q+\frac{Eh}{2(1-\mu^2)}\left\{\left[\left(\frac{\partial W}{\partial x}\right)^2+\mu\left(\frac{\partial W}{\partial y}\right)^2\right]\frac{\partial^2 W}{\partial x^2}\right.\\
&\quad+\left[\left(\frac{\partial W}{\partial y}\right)^2+\mu\left(\frac{\partial W}{\partial x}\right)^2\right]\frac{\partial^2 W}{\partial y^2}\\
&\quad\left.+2(1-\mu)\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial y}\cdot\frac{\partial^2 W}{\partial x\partial y}\right\}
\end{aligned}\right\}\quad(2.3)
$$

由于矩形板四边固定，其边界条件为：

$$\text{在 } x=\pm a \text{ 时，}\qquad U=V=W=\frac{\partial W}{\partial x}=0 \qquad (2.4)$$

$$\text{在 } y=\pm b \text{ 时，}\qquad U=V=W=\frac{\partial W}{\partial y}=0 \qquad (2.5)$$

三、摄 动 变 分 法

首先引入下列无量纲符号：

$$
\left.\begin{aligned}
&\lambda=\frac{a}{b},\ \xi=\frac{x}{a},\ \eta=\frac{y}{b},\ u=\frac{12aU}{h^2}\\
&v=\frac{12aV}{h^2},\quad w=\frac{2\sqrt{3}\,W}{h},\quad Q=24\sqrt{3}\,(1-\mu^2)a^4q/Eh^4
\end{aligned}\right\}\quad(3.1)
$$

将(3.1)代入(2.3)，得卡门大挠度方程组的无量纲形式：

$$
\begin{aligned}
&2\frac{\partial^2 u}{\partial \xi^2}+(1-\mu)\lambda^2\frac{\partial^2 u}{\partial \eta^2}+(1+\mu)\lambda\frac{\partial^2 v}{\partial \xi\partial \eta}\\
&\quad=-(1-\mu)\frac{\partial w}{\partial \xi}\left(\frac{\partial^2 w}{\partial \xi^2}+\lambda^2\frac{\partial^2 w}{\partial \eta^2}\right)-\frac{1+\mu}{2}\cdot\frac{\partial}{\partial \xi}\left[\left(\frac{\partial w}{\partial \xi}\right)^2\right.\\
&\quad\left.+\lambda^2\left(\frac{\partial w}{\partial \eta}\right)^2\right]\cdot(1+\mu)\lambda\frac{\partial^2 u}{\partial \xi\partial \eta}+(1-\mu)\frac{\partial^2 v}{\partial \xi^2}+2\lambda^2\frac{\partial^2 v}{\partial \eta^2}\\
&\quad=-(1-\mu)\lambda\frac{\partial w}{\partial \eta}\left(\frac{\partial^2 w}{\partial \xi^2}+\lambda^2\frac{\partial^2 w}{\partial \eta^2}\right)-\frac{1+\mu}{2}\lambda\frac{\partial}{\partial \eta}\left[\left(\frac{\partial w}{\partial \xi}\right)^2\right.
\end{aligned}
$$

$$\left.\begin{aligned}&+\lambda^2\left(\frac{\partial w}{\partial\eta}\right)^2\Bigg]\frac{\partial^4 w}{\partial\xi^4}+2\lambda^2\frac{\partial^4 w}{\partial\xi^2\partial\eta^2}+\lambda^4\frac{\partial^4 w}{\partial\eta^4}\\&=Q+\frac{\partial^2 w}{\partial\xi^2}\left(\frac{\partial u}{\partial\xi}+\lambda\mu\frac{\partial v}{\partial\eta}\right)+\lambda^2\frac{\partial^2 w}{\partial\eta^2}\left(\lambda\frac{\partial v}{\partial\eta}+\mu\frac{\partial u}{\partial\xi}\right)\\&+\lambda(1-\mu)\left(\lambda\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right)\cdot\frac{\partial^2 w}{\partial\xi\partial\eta}+\frac{1}{2}\frac{\partial^2 w}{\partial\xi^2}\left[\left(\frac{\partial w}{\partial\xi}\right)^2\right.\\&\left.+\mu\lambda^2\left(\frac{\partial w}{\partial\eta}\right)^2\right]+\frac{\lambda^2}{2}\cdot\frac{\partial^2 w}{\partial\eta^2}\left[\lambda^2\left(\frac{\partial w}{\partial\eta}\right)^2+\mu\left(\frac{\partial w}{\partial\eta}\right)^2\right]\\&+\lambda^2(1-\mu)\frac{\partial w}{\partial\xi}\cdot\frac{\partial w}{\partial\eta}\cdot\frac{\partial^2 w}{\partial\xi\partial\eta}\end{aligned}\right\}\tag{3.2}$$

边界条件的无量纲形式为：

在 $\xi=\pm1$ 时：　$u=v=w=\dfrac{\partial w}{\partial\xi}=0$　(3.3)

在 $\eta=\pm1$ 时：　$u=v=w=\dfrac{dw}{d\eta}=0$　(3.4)

板中心挠度为：$w_0=w(0,\ 0)=2\sqrt{3}\ W_0/h$。今将$Q$，$u$，$v$，$w$各量展成$w_0$的幂级数形式如下（考虑对称性）：

$$\left.\begin{aligned}&Q=\alpha_1w_0+\alpha_3w_0^3+\alpha_5w_0^5+\cdots\\&u=s_2(\xi,\ \eta)w_0^2+s_4(\xi,\ \eta)w_0^4+\cdots\\&v=t_2(\xi,\eta)w_0^2+t_4(\xi,\ \eta)w_0^4+\cdots\\&w=w_1(\xi,\eta)w_0+w_3(\xi,\eta)w_0^3+\cdots\end{aligned}\right\}\tag{3.5}$$

其中 w 的展开式中的系数要满足如下条件：

$$w_1(0,\ 0)=1,\qquad w_3(0,\ 0)=w_5(0,\ 0)=\cdots=0\tag{3.6}$$

将(3.5)代入(3.2)，然后分别比较每个等式两边w_0的同次幂系数并使之相等，便得到各级摄动方程组：

（1）第一级摄动方程：

$$\frac{\partial^4 w_1}{\partial\xi^4}+2\lambda^2\frac{\partial^4 w_1}{\partial\xi^2\partial\eta^2}+\lambda^4\frac{\partial^4 w_1}{\partial\eta^4}=\alpha_1\tag{3.7}$$

边界条件：

$$w_1=\frac{\partial w_1}{\partial\xi}=0,\quad \xi=\pm1\tag{3.8}$$

$$w_1=\frac{\partial w_1}{\partial\eta}=0,\quad \eta=\pm1\tag{3.9}$$

（2）第二级摄动方程组：

$$\left.\begin{aligned}&2\frac{\partial^2 s_2}{\partial\xi^2}+(1-\mu)\lambda^2\frac{\partial^2 s_2}{\partial\eta^2}+(1+\mu)\lambda\frac{\partial^2 t_2}{\partial\xi\partial\eta}=-(1-\mu)\frac{\partial w_1}{\partial\xi}\left(\frac{\partial^2 w_1}{\partial\xi^2}\right.\\&\left.+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)-\frac{1+\mu}{2}\frac{\partial}{\partial\xi}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\\&(1+\mu)\lambda\frac{\partial^2 s_2}{\partial\xi\partial\eta}+(1-\mu)\frac{\partial^2 t_2}{\partial\xi^2}+2\lambda^2\frac{\partial^2 t_2}{\partial\eta^2}=-(1-\mu)\lambda\frac{\partial w_1}{\partial\eta}\left(\frac{\partial^2 w_1}{\partial\xi^2}\right.\\&\left.+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)-\frac{1+\mu}{2}\frac{\partial}{\partial\eta}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\end{aligned}\right\}\tag{3.10}$$

边界条件：

$$s_2=t_2=0,\quad \xi=\pm 1 \tag{3.11}$$

$$s_2=t_2=0,\quad \eta=\pm 1 \tag{3.12}$$

（3）第三级摄动方程：

$$\left.\begin{aligned}&\frac{\partial^4 w_3}{\partial\xi^4}+2\lambda^2\frac{\partial^4 w_3}{\partial\xi^2\partial\eta^2}+\lambda^4\frac{\partial^4 w_3}{\partial\eta^4}\\&\quad=\alpha_3+\left(\frac{\partial s_2}{\partial\xi}+\mu\lambda\frac{\partial t_2}{\partial\eta}\right)\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\left(\lambda\frac{\partial t_2}{\partial\eta}+\mu\frac{\partial s_2}{\partial\xi}\right)\frac{\partial^2 w_1}{\partial\eta^2}\\&\quad+(1-\mu)\lambda\left(\lambda\frac{\partial s_2}{\partial\eta}+\frac{\partial t_2}{\partial\xi}\right)+\frac{1}{2}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2+\mu\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\frac{\partial^2 w_1}{\partial\xi^2}\\&\quad+\frac{\lambda^2}{2}\left[\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2+\mu\left(\frac{\partial w_1}{\partial\xi}\right)^2\right]\frac{\partial^2 w_1}{\partial\eta^2}\\&\quad+(1-\mu)\lambda^2\frac{\partial w_1}{\partial\xi}\frac{\partial w_1}{\partial\eta}\frac{\partial^2 w_1}{\partial\xi\partial\eta}\end{aligned}\right\} \tag{3.13}$$

边界条件：

$$w_3=\frac{\partial w_3}{\partial\xi}=0,\quad \xi=\pm 1 \tag{3.14}$$

$$w_3=\frac{\partial w_3}{\partial\eta}=0,\quad \eta=\pm 1 \tag{3.15}$$

同样可得到四、五阶方程，本文只计算到三级解。对于各级摄动方程，可推导出与之相应的泛函如下：

一级泛函：

$$\left.\begin{aligned}&\Pi_1=\int_{-1}^{+1}\int_{-1}^{+1}\left[\frac{1}{2}\left(\frac{\partial^2 w_1}{\partial\xi^2}\right)^2+\lambda^2\left(\frac{\partial^2 w_1}{\partial\xi\partial\eta}\right)^2+\frac{1}{2}\lambda^4\left(\frac{\partial^2 w_1}{\partial\eta^2}\right)^2\right.\\&\qquad\left.-\alpha_1 w_1\right]d\xi d\eta\\&w_1=\frac{\partial w_1}{\partial\xi}=0,\quad \xi=\pm 1\\&w_1=\frac{\partial w_1}{\partial\eta}=0,\quad \eta=\pm 1\end{aligned}\right\} \tag{3.16}$$

二级泛函：

$$\left.\begin{aligned}&\Pi_2=-\int_{-1}^{+1}\int_{-1}^{+1}\left\{\left(\frac{\partial s_2}{\partial\xi}\right)^2+\frac{1}{2}(1-\mu)\lambda^2\left(\frac{\partial s_2}{\partial\eta}\right)^2\right.\\&\quad+(1+\mu)\lambda\frac{\partial t_2}{\partial\eta}\frac{\partial s_2}{\partial\xi}+\frac{1}{2}(1-\mu)\left(\frac{\partial t_2}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial t_2}{\partial\eta}\right)^2\\&\quad+\left[(1-\mu)\frac{\partial w_1}{\partial\xi}\left(\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)+\frac{1+\mu}{2}\frac{\partial}{\partial\xi}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2\right.\right.\\&\quad\left.\left.+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\right]s_2+\left[(1-\mu)\lambda\frac{\partial w_1}{\partial\eta}\left(\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)\right.\\&\quad\left.\left.+\frac{1+\mu}{2}\lambda\frac{\partial}{\partial\eta}\left[\left(\frac{\partial w_1}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\right]\right]t_2\right\}d\xi d\eta\\&s_2=t_2=0,\quad \xi=\pm 1,\ \text{或}\ \eta=\pm 1\end{aligned}\right\} \tag{3.17}$$

三级泛函

$$
\left.\begin{aligned}
\Pi_3=&\int_{-1}^{+1}\int_{-1}^{+1}\Big\{\frac{1}{2}\left(\frac{\partial^2 w_3}{\partial\xi^2}\right)^2+\lambda^2\left(\frac{\partial^2 w_3}{\partial\xi\partial\eta}\right)^2+\frac{1}{2}\lambda^4\left(\frac{\partial^2 w_3}{\partial\eta^2}\right)^2-\alpha_3 w_3\\
&-\Big[\left(\frac{\partial s_2}{\partial\xi}+\mu\lambda\frac{\partial t_2}{\partial\eta}\right)\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\left(\lambda\frac{\partial t_2}{\partial\eta}+\mu\frac{\partial s_2}{\partial\xi}\right)\frac{\partial^2 w_1}{\partial\eta^2}\\
&+(1-\mu)\lambda\left(\lambda\frac{\partial s_2}{\partial\eta}+\frac{\partial t_2}{\partial\xi}\right)\frac{\partial^2 w_1}{\partial\xi\partial\eta}+\frac{1}{2}\Big[\left(\frac{\partial w_1}{\partial\xi}\right)^2\\
&+\mu\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\Big]\frac{\partial^2 w_1}{\partial\xi^2}+\frac{\lambda^2}{2}\left[\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2+\mu\left(\frac{\partial w_1}{\partial\xi}\right)^2\right]\frac{\partial^2 w_1}{\partial\eta^2}\\
&+(1-\mu)\lambda^2\frac{\partial w_1}{\partial\xi}\frac{\partial w_1}{\partial\eta}\frac{\partial^2 w_1}{\partial\xi\partial\eta}\Big]w_3\Big\}d\xi d\eta\\
w_3=&\frac{\partial w_3}{\partial\xi}=0,\quad \xi=\pm1\\
w_3=&\frac{\partial w_3}{\partial\eta}=0,\quad \eta=\pm1
\end{aligned}\right\}\tag{3.18}
$$

对于泛函(3.16)中的自变函数w_1，我们取下列多项式形式的函数：

$$
w_1(\xi,\ \eta)=(1-\xi^2)^2(1-\eta^2)^2(A_1+B_1\xi^2+C_1\eta^2+D_1\xi^4+E_1\eta^4+F_1\xi^2\eta^2)\tag{3.19}
$$

显然(3.19)满足(3.16)中的边界条件，将$w_1(\xi,\ \eta)$代入(3.16)泛函中，通过泛函取极值的条件，得到关于A_1，B_1，C_1，D_1，E_1，F_1和α_1为未知数的线性代数方程组如下：

$$
\left.\begin{array}{l}
a_{11}A_1+a_{12}B_1+a_{13}C_1+a_{14}D_1+a_{15}E_1+a_{16}F_1+a_{17}\alpha_1=0\\
a_{21}A_1+a_{22}B_1+\quad\cdots\quad\cdots\quad\cdots\quad+a_{27}\alpha_1=0\\
\quad\vdots\qquad\qquad\vdots\qquad\vdots\qquad\vdots\qquad\qquad\vdots\\
a_{61}A_1+a_{62}B_1+\quad\cdots\quad\cdots\quad\cdots\quad+a_{67}\alpha_1=0
\end{array}\right\}\tag{3.20}
$$

加上(3.6)式中条件$w_1(0,\ 0)=1$，得

$$
A_1=1\tag{3.21}
$$

(3.20)和(3.21)联立共七个方程，可解出问题的一级解。其中的系数a_{ij}是长宽比λ的函数，其具体形式如下：

$$
a_{ij}=a_{ij}^{(1)}+a_{ij}^{(2)}\lambda^2+a_{ij}^{(3)}\lambda^4\tag{3.22}
$$

其中$a_{ij}^{(1)}$，$a_{ij}^{(2)}$和$a_{ij}^{(3)}$见表1，2，3。

表 1

$a_{ij}^{(1)}$　j / i	1	2	3	4	5	6	7
1	20.81	2.972	1.891	0.9907	0.4365	0.2702	−1.138
2	2.972	8.916	0.27020	4.773	0.06235	0.81059	−0.1625
3	1.891	0.27020	0.43647	0.09007	0.14549	0.06235	−0.1625
4	0.99072	4.773	0.09007	4.454	0.02078	0.43395	−0.0542
5	0.43647	0.06235	0.14549	0.02078	0.05872	0.02078	−0.0542
6	0.27020	0.81059	0.06235	0.43395	0.02078	0.18705	−0.0232

表 2

$a_{ij}^{(2)}$ i \ j	1	2	3	4	5	6	7
1	11.89	0.00000	0.00000	−0.3603	−0.3603	−0.0000	0.00000
2	−0.0000	1.081	−0.0000	0.38797	0.00000	−0.0000	0.00000
3	−0.000	−0.0000	1.081	0.00000	0.38797	−0.0000	0.00000
4	−0.3603	0.38797	0.00000	0.24941	0.01092	−0.0000	0.00000
5	−0.3602	0.00000	0.38797	0.01092	0.24941	−0.0000	0.00000
6	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	0.09825	0.00000

表 3

$a_{ij}^{(3)}$ i \ j	1	2	3	4	5	6	7
1	20.81	1.891	2.972	0.43647	0.99072	0.27020	0.00000
2	1.891	0.43647	0.27020	0.14549	0.09007	0.06235	0.00000
3	2.972	0.27020	8.9165	0.06235	4.773	0.81059	0.00000
4	0.43647	0.14549	0.06235	0.05872	0.02078	0.02078	0.00000
5	0.99072	0.09007	4.773	0.02078	4.455	0.43395	0.00000
6	0.27020	0.06235	0.81059	0.02078	0.43395	0.18706	0.00000

同样，对于泛函(3.17)中的自变函数s_2，t_2，我们分别取下列多项式形式的函数：

$$s_2(\xi,\ \eta)=(1-\xi^2)(1-\eta^2)\xi(A_2+B_2\xi^2+C_2\eta^2+D_2\xi^4+E_2\eta^4+F_2\xi^2\eta^2) \tag{3.23}$$

$$t_2(\xi,\ \eta)=(1-\xi^2)(1-\eta^2)\eta(A_3+B_3\xi^2+C_3\eta^2+D_3\xi^4+E_3\eta^4+F_3\xi^2\eta^2) \tag{3.24}$$

显然(3.23)和(3.24)满足(3.17)式中边界条件和对称中心条件，$s_2(0,\ 0)=t_2(0,\ 0)=0$。将$s_2(\xi,\ \eta)$，$t_2(\xi,\ \eta)$代入(3.17)泛函中，同样得到关于A_2，B_2，C_2，D_2，E_2，F_2和A_3，B_3，C_3，D_3，E_3，F_3为未知数的线性方程组如下：

$$\left.\begin{array}{l}
b_{11}A_2+b_{12}B_2+b_{13}C_2+b_{14}D_2+b_{15}E_2+b_{16}F_2\\
\qquad +b_{17}A_3+b_{18}B_3+b_{19}C_3+b_{1,10}D_3+b_{1,11}E_3+b_{1,12}F_3=b_{10}\\
b_{21}A_2+b_{22}B_2+\quad\cdots\quad\cdots\quad\cdots\\
\qquad +b_{27}A_3+\quad\cdots\quad\cdots\quad\cdots\qquad +b_{2,12}F_3=b_{20}\\
\qquad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
b_{12,1}A_2+b_{12,2}B_2+b_{12,3}C_2+b_{12,4}D_2+b_{12,5}E_2+b_{12,6}F_2\\
\qquad +b_{12,7}A_3+b_{12,8}B_3+b_{12,9}C_3+b_{12,10}D_3+b_{12,11}E_3+b_{12,12}F_3=b_{12,0}\\
\qquad\qquad\qquad\qquad\cdots\cdots\cdots\cdots\cdots
\end{array}\right\} \tag{3.25}$$

其中系数b_{ij}是λ的函数，且相互间有下列关系：

$$b_{i7}=b_{i1},\quad b_{i8}=b_{i3},\quad b_{i9}=b_{i2},\quad b_{i10}=b_{i6},\quad b_{i11}=b_{i4},\quad b_{i12}=b_{i5} \tag{3.26}$$

对于角标 j 也存在上列相似关系，因此只要计算$b_{ij}(i,j=1,2,\cdots,12)$中一半即可，其表达式为：

$$\left.\begin{aligned} & b_{ij}=b_{ij}^{(1)}+b_{ij}^{(2)}\lambda^2 \qquad (i=1,2,\cdots,6;\ j=0,1,\cdots,6) \\ & b_{ij}=b_{ij}^{(1)}\lambda \qquad (i=1,2,\cdots,6;\ j=7,8,\cdots,12) \end{aligned}\right\} \tag{3.27}$$

其中系数$b_{ij}^{(1)}$和$b_{ij}^{(2)}$见表4，5。对给定的λ解，上述方程即得二级解。

对于三级解，取如下形式：

$$w_3(\xi,\ \eta)=(1-\xi^2)^2(1-\eta^2)^2(A_4+B_4\xi^2+C_4\eta^2+D_4\xi^4+E_4\eta^4+F_4\xi^2\eta^2) \tag{3.28}$$

代入泛函(3.18)，得：

表 4a

$b_{ij}^{(1)}$ i \ j	1	2	3	4	5	6
1	−3.413	−1.463	−0.4876	−0.8127	−0.1625	−0.2090
2	−1.463	−1.246	−0.2090	−0.9112	−0.0697	−0.1780
3	−0.4876	−0.2090	−0.1625	−0.1161	−7.388	−0.0697
4	−0.8127	−0.9112	−0.1161	−0.7824	−0.0387	−0.1302
5	−0.1625	−0.0697	−0.0739	−0.0387	−0.0398	−0.0317
6	−0.2090	−0.1782	−0.0697	−0.1302	−0.0317	−0.0593

表 4b

$b_{ij}^{(1)}$ i \ j	7	8	9	10	11	12
1	−0.3793	−0.1625	0.05418	−0.0903	0.05418	0.02322
2	0.05418	−0.0181	−0.0077	−0.0246	−0.0077	0.00258
3	−0.1625	−0.0697	−0.0181	−0.0387	0.00821	−0.0077
4	0.05418	0.00821	−0.0077	−0.0044	−0.0077	−0.0012
5	−0.0903	−0.0387	−0.0246	−0.0215	−0.0044	−0.0106
6	0.02322	0.0077	−0.0026	−0.0106	−0.0012	−0.00086

$$\left.\begin{aligned} & c_{11}A_4+c_{12}B_4+c_{13}C_4+c_{14}D_4+c_{15}E_4+c_{16}F_4+c_{17}\alpha_3=c_{10} \\ & c_{21}A_4+c_{22}B_4+ \quad \cdots \quad \cdots \quad +c_{27}\alpha_3=c_{20} \\ & \cdots \quad \cdots \quad \cdots \quad \cdots \\ & c_{61}A_4+c_{62}B_4+c_{63}C_4+c_{64}D_4+c_{65}E_4+c_{66}F_4+c_{67}\alpha_3=c_{60} \end{aligned}\right\} \tag{3.29}$$

表 5a

$b_{ij}^{(2)}$ i \ j	1	2	3	4	5	6
1	−0.2709	−0.0903	−0.0542	−0.0410	−0.0232	−0.0181
2	−0.0903	−0.0410	−0.0181	−0.0221	−0.0077	−0.0082
3	−0.0542	−0.0181	−0.0851	−0.0082	−0.0542	−0.0284
4	−0.0410	−0.0221	−0.0082	−0.0133	−0.0035	−0.0044
5	−0.0232	−0.0077	−0.0542	−0.0035	−0.0457	−0.0181
6	−0.0181	−0.0082	−0.0284	−0.0044	−0.0181	−0.0129

表 5b

i	1	2	3	4	5	6
$b_{i0}^{(1)}$	−0 0000	0.57811	0.00000	0.44470	0.00000	0.11333
$b_{i0}^{(2)}$	−0.0525	−0.1459	0.14489	−0.0806	0.07320	−0.0411
i	7	8	9	10	11	12
$b_{i0}^{(1)}$	−0.0381	0.14957	−0.1335	0.06571	0.0717	0.01128
$b_{i0}^{(2)}$	−0.0144	−0.0047	0.56568	0.00749	0.43571	0.06099

表 6

i	1	2	3	4	5	6
$c_{i0}^{(1)}$	−1.007	0.00148	−0.1459	0.03131	−0.0481	−0.00068
$c_{i0}^{(2)}$	−7.455	−0.7782	0.00494	−0.1510	0.20072	−0.0368
$c_{i0}^{(3)}$	12.01	1.044	0.17894	0.14411	−0.2325	0.07015
$c_{i0}^{(4)}$	−5.004	−0.1803	−0.1466	0.05463	0.05627	−0.0341
$c_{i0}^{(5)}$	−0.7777	−0.1728	0.02293	−0.0646	0.03798	0.00400

加上(3.6)式中条件 $w_3(0,\ 0)=0$，得：

$$A_4=0$$

其中系数 $c_{ij}=a_{ij}$ $(i=1,\ 2,\ \cdots,\ 6;\ j=1,2,\cdots,7)$，$c_{i0}$ $(i=1,2,\ \cdots,\ 6)$ 是λ的函数，其表达式如下：

$$c_{i0}=c_{i0}^{(1)}+c_{i0}^{(2)}\lambda+c_{i0}^{(3)}\lambda^2+c_{i0}^{(4)}\lambda^3+c_{i0}^{(5)}\lambda^4 \tag{3.30}$$

系数$c_{i0}^{(1)}$，$c_{i0}^{(2)}$，$c_{i0}^{(3)}$，$c_{i0}^{(4)}$和$c_{i0}^{(5)}$见附表6。

四、挠度及应力公式

通过上面的求解，得到了载荷系数α_1，α_3，再由(3.1)和(3.5)，可得：

$$24\sqrt{3}\ (1-\mu^2)a^4q/Eh^4=\alpha_1\left(\frac{2\sqrt{3}\ W_0}{h}\right)+\alpha_3\left(\frac{2\sqrt{3}\ W_0}{h}\right)^3 \tag{4.1}$$

整理可得：

$$\frac{b^4q}{Eh^4}=\frac{\alpha_1}{12(1-\mu^2)\lambda^4}\left(\frac{W_0}{h}\right)+\frac{\alpha_3}{(1-\mu^2)\lambda^4}\left(\frac{W_0}{h}\right)^3 \tag{4.2}$$

公式(4.2)即为最大挠度与载荷之间的非线性关系。

距中曲面为Z的弯曲应力为：

$$\left.\begin{aligned}\sigma''_x&=-\frac{EZ}{1-\mu^2}\left(\frac{\partial^2W}{\partial x^2}+\mu\frac{\partial^2W}{\partial y^2}\right)\\ \sigma''_y&=-\frac{EZ}{1-\mu^2}\left(\frac{\partial^2W}{\partial y^2}+\mu\frac{\partial^2W}{\partial x^2}\right)\\ \tau''_{xy}&=-\frac{EZ}{1+\mu}\cdot\frac{\partial^2W}{\partial x\partial y}\end{aligned}\right\} \tag{4.3}$$

取$Z=-h/2$，则由(4.3)得矩形板表面的弯曲应力为

$$\left.\begin{aligned}\sigma''_x&=\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2W}{\partial x^2}+\mu\frac{\partial^2W}{\partial y^2}\right)\\ \sigma''_y&=\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2W}{\partial y^2}+\mu\frac{\partial^2W}{\partial x^2}\right)\\ \tau''_{xy}&=\frac{Eh}{2(1+\mu)}\frac{\partial^2W}{\partial x\partial y}\end{aligned}\right\} \tag{4.4}$$

将各应力无量纲化，令：

$$\left.\begin{aligned}\Sigma'_x(\xi,\ \eta)&=\frac{12(1-\mu^2)a^2}{Eh^2}\sigma_x\\ \Sigma'_y(\xi,\ \eta)&=\frac{12(1-\mu^2)a^2}{Eh^2}\sigma_y\\ \Sigma'_{xy}(\xi,\ \eta)&=\frac{4\sqrt{3}\ (1+\mu)a\cdot b}{Eh^2}\tau_{xy}\end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned}\Sigma''_x(\xi,\ \eta)&=\frac{4\sqrt{3}\ (1-\mu^2)a^2}{Eh^2}\sigma''_x\\ \Sigma''_y(\xi,\ \eta)&=\frac{4\sqrt{3}\ (1-\mu^2)a^2}{Eh^2}\sigma''_y\\ \Sigma''_{xy}(\xi,\ \eta)&=\frac{4\sqrt{3}\ (1+\mu)a\cdot b}{Eh^2}\tau''_{xy}\end{aligned}\right\} \tag{4.6}$$

公式(4.5)为中曲面内应力无量纲式，(4.6)为板表面纯弯曲应力的无量纲式。将(3.1)代入(2.1)和(4.4)，便有下列关系：

$$\left.\begin{aligned}\Sigma'_x&=\frac{\partial u}{\partial\xi}+\mu\lambda\frac{\partial v}{\partial\eta}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial\xi}\right)^2+\lambda^2\mu\left(\frac{\partial w}{\partial\eta}\right)^2\right]\\\Sigma'_y&=\lambda\frac{\partial v}{\partial\eta}+\mu\frac{\partial u}{\partial\xi}+\frac{1}{2}\left[\mu\left(\frac{\partial w}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w}{\partial\eta}\right)^2\right]\\\Sigma'_{xy}&=\lambda\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}+\lambda\frac{\partial w}{\partial\xi}\cdot\frac{\partial w}{\partial\eta}\end{aligned}\right\}\tag{4.7}$$

$$\left.\begin{aligned}\Sigma''_x&=\frac{\partial^2 w}{\partial\xi^2}+\mu\lambda^2\frac{\partial^2 w}{\partial\eta^2}\\\Sigma''_y&=\lambda^2\frac{\partial^2 w}{\partial\eta^2}+\mu\frac{\partial^2 w}{\partial\xi^2}\\\Sigma''_{xy}&=\lambda\frac{\partial^2 w}{\partial\xi\partial\eta}\end{aligned}\right\}\tag{4.8}$$

再将前面所得的三次结果代入(4.7)和(4.8)中，便可得板中曲面应力及板表面弯曲应力的关系式：

$$\begin{aligned}\Sigma'_x=w_0^2\left(\frac{\partial s_2}{\partial\xi}+\mu\lambda\frac{\partial t_2}{\partial\eta}\right)+\frac{w_0^2}{2}\Bigg\{\left(\frac{\partial w_1}{\partial\xi}\right)^2+\mu\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\\+2w_0\left(\frac{\partial w_1}{\partial\xi}\frac{\partial w_3}{\partial\xi}+\mu\lambda^2\frac{\partial w_1}{\partial\eta}\frac{\partial w_3}{\partial\eta}\right)+w_0^4\left[\left(\frac{\partial w_3}{\partial\xi}\right)^2+\mu\lambda^2\left(\frac{\partial w_3}{\partial\eta}\right)^2\right]\Bigg\}\end{aligned}\tag{4.9a}$$

$$\begin{aligned}\Sigma'_y=w_0^2\left(\mu\frac{\partial s_2}{\partial\xi}+\lambda\frac{\partial t_2}{\partial\eta}\right)+\frac{w_0^2}{2}\Bigg\{\mu\left(\frac{\partial w_1}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w_1}{\partial\eta}\right)^2\\+2w_0^2\left(\mu\frac{\partial w_1}{\partial\xi}\cdot\frac{\partial w_3}{\partial\xi}+\lambda^2\frac{\partial w_1}{\partial\eta}\frac{\partial w_3}{\partial\eta}\right)+w_0^4\left[\mu\left(\frac{\partial w_3}{\partial\xi}\right)^2+\lambda^2\left(\frac{\partial w_3}{\partial\eta}\right)^2\right]\Bigg\}\end{aligned}\tag{4.9b}$$

$$\begin{aligned}\Sigma'_{xy}=w_0^2\left(\lambda\frac{\partial s_2}{\partial\eta}+\frac{\partial t_2}{\partial\xi}\right)+\lambda w_0^2\Bigg[\frac{\partial w_1}{\partial\xi}\cdot\frac{\partial w_1}{\partial\eta}\\+w_0^2\left(\frac{\partial w_1}{\partial\eta}\frac{\partial w_2}{\partial\eta}+\frac{\partial w_1}{\partial\eta}\frac{\partial w_3}{\partial\xi}\right)+\frac{\partial w_3}{\partial\xi}\frac{\partial w_3}{\partial\eta}w_0^4\Bigg]\end{aligned}\tag{4.9c}$$

$$\left.\begin{aligned}\Sigma''_x&=w_0\left(\frac{\partial^2 w_1}{\partial\xi^2}+\mu\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)+w_0^3\left(\frac{\partial^2 w_3}{\partial\xi^2}+\mu\lambda\frac{\partial^2 w_3}{\partial\eta^2}\right)\\\Sigma''_y&=w_0\left(\mu\frac{\partial^2 w_1}{\partial\xi^2}+\lambda^2\frac{\partial^2 w_1}{\partial\eta^2}\right)+w_0^3\left(\mu\frac{\partial^2 w_3}{\partial\xi^2}+\lambda\frac{\partial^2 w_3}{\partial\eta^2}\right)\\\Sigma''_{xy}&=\lambda w_0\frac{\partial^2 w_1}{\partial\xi\partial\eta}+\lambda w_0^3\frac{\partial^2 w_3}{\partial\xi\partial\eta}\end{aligned}\right\}\tag{4.10}$$

所以，板表面任一点总应力分量为：

$$\left.\begin{aligned}s_x(\xi,\ \eta)&=\Sigma'_x+\Sigma''_x\\s_y(\xi,\ \eta)&=\Sigma'_y+\Sigma''_y\\s_{xy}(\xi,\ \eta)&=\Sigma'_{xy}+\Sigma''_{xy}\end{aligned}\right\}\tag{4.11}$$

将三次解 $w_1(\xi,\ \eta)$，$w_3(\xi,\ \eta)$，$s_2(\xi,\ \eta)$，$t_2(\xi,\ \eta)$代入(4.9)，(4.10)和(4.11)，便可得板表面任意一点的总应力分量。

对于 $\lambda=a/b$趋于无穷时，相当于 $\lambda=0$ 时，将 ξ 和 η 相互交换位置的情况。

五、结 果 讨 论

板中心的最大挠度与文[11]中实验结果相比，是基本一致的，如图 2 所示。本文计算了大量不同长宽比的矩形板，其挠度～载荷关系曲线绘在图 3 中，表明对于相同宽度的矩形板，长度越长（λ越大)，则挠度越大（受相同均布力），且当 $\lambda>2$ 时，其曲线都很靠近。

矩形板中最大应力发生在板长边中点，即(0, 1)处，该点总应力为：$s_y(0,1)=\Sigma'_y(0,1)+\Sigma''_y(0,1)$。通过数值计算，对于各种长宽比的板，绘出曲线如图 4。对相同宽度、相同挠度的矩形板，其最大应力值是随长宽比的不同而不同的。具体地说，当 λ 从 1.0 增大到 2.0或2.5时，最大应力值也随之增大，也就是说应力分布趋于不均布。当 λ 从 2.5 左右增大到无穷时，最大应力值随之减小，也就是说应力分布趋于均匀。了解这一点对矩形板的优化设计是有益的。

本文对具有不同泊松比材料的矩形薄板也进行了计算。结果表明，当泊松比减小时，挠度和应力相应增加和减小，其中位移的增加很少，如图 5 所示，最大应力随泊松比减小的幅度如图 6 所示。

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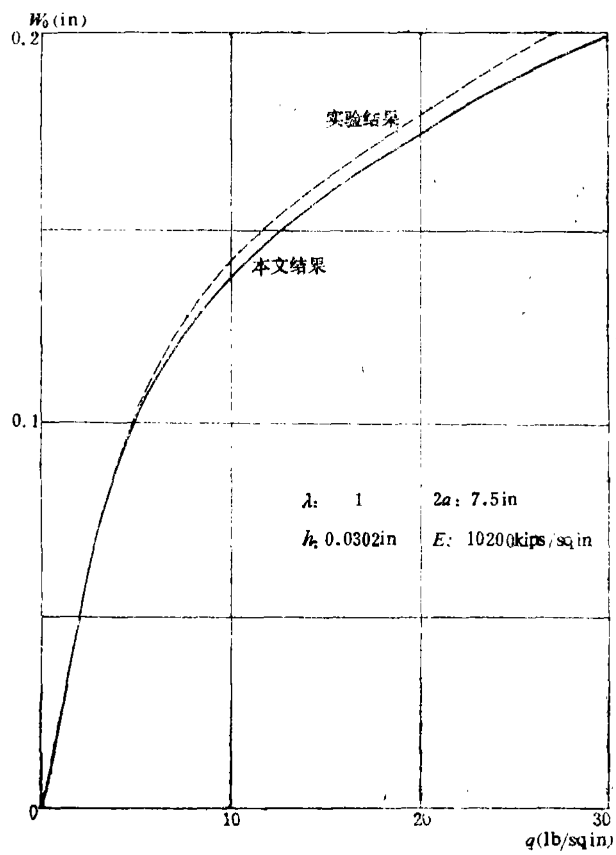

图 2

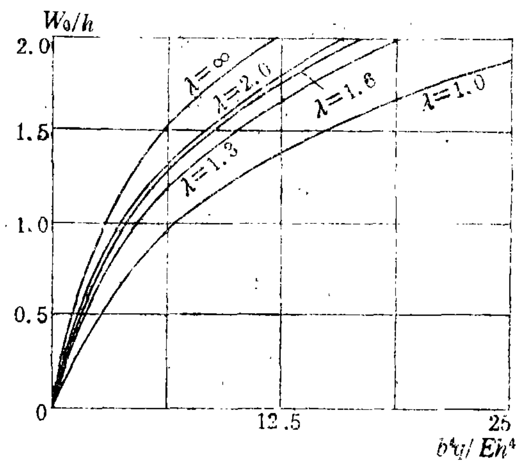

图 3

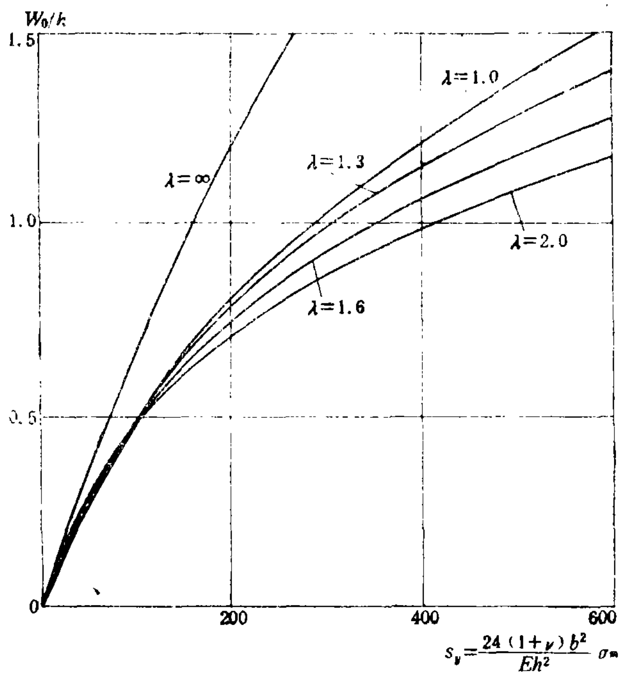
W_0/h
1.5
1.0
0.5
0
200
400
600
$\lambda=1.0$
$\lambda=1.3$
$\lambda=\infty$
$\lambda=2.0$
$\lambda=1.6$
$S_y=\frac{24(1+\nu)b^2}{Eh^2}\sigma_m$

图 4

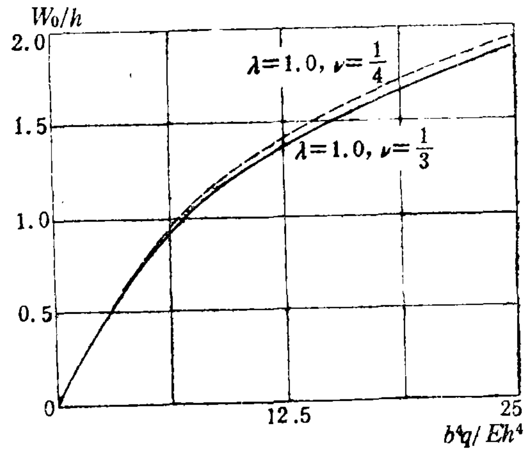
W_0/h
2.0
1.5
1.0
0.5
0
12.5
25
$\lambda=1.0,\ \nu=\frac{1}{4}$
$\lambda=1.0,\ \nu=\frac{1}{3}$
b^4q/Eh^4

图 5

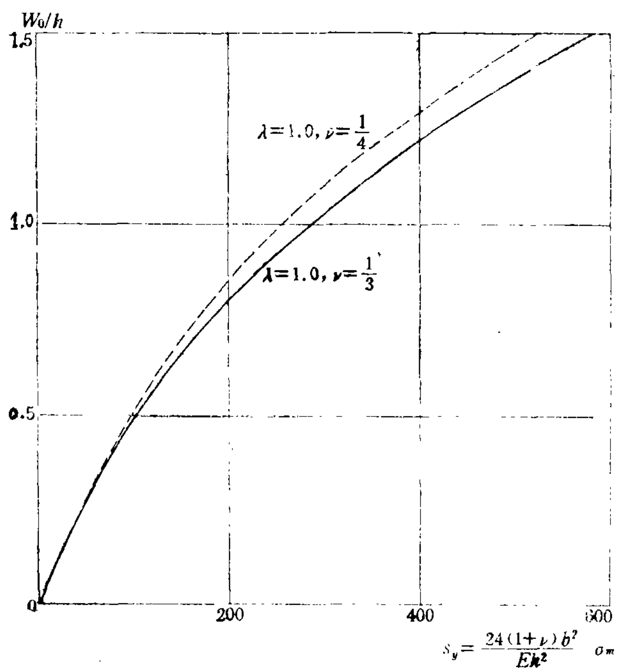
W_0/h
1.5
1.0
0.5
0
200
400
600
$\lambda=1.0,\ \nu=\frac{1}{4}$
$\lambda=1.0,\ \nu=\frac{1}{3}$
$S_y=\frac{24(1+\nu)b^2}{Eh^2}\sigma_m$

图 6

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A Perturbation-Variational Solution of the Large Deflection of Rectangular Plates under Uniform Load

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Abstract

In this paper, von Kármán's set of nonlinear equation for large deflection of rectangular plates is at first converted into several sets of linear equations by taking central dimensionless deflection as perturbation parameter, and then, the sets of linear equations for plates with various ratios of length to width are solved with application of variational method. The analytical expressions for displacements and stresses as well as formulas for numerical calculation are worked out. The figures of maximum deflection-load and maximum stress with ratio λ of length to width as a parameter are given in this paper. Through comparison, it is found that the results of this paper are quite in accord with experiments.