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不对称的各向异性叠层矩形板的非线性弯曲（Ⅰ）*

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摘要

本文研究了不对称的各向异性叠层矩形板在多种支承条件下的非线性弯曲，利用文[1]中提出的奇异摄动方法，导出了板在横向载荷和边缘拉力的联合作用下，其挠度和应力函数的一致有效的N阶渐近解．因此，本文的研究对于这样一个复杂的问题提供了一个简单而又有效的方法．

一、引　言

在层合复合材料的许多实际应用中，需要利用不对称层合板以达到设计要求。Ashton和Whitney曾经用折减弯曲刚度方法、Zaghloul和Kennedy曾经用有限差分法研究过不对称层合板的大挠度问题[3,4]，Prabhakara和Chia、Giri和Simitses则分别利用多重富氏级数方法和改进了的Galerkin程序研究过此类问题[5,6]。本文则应用江福汝在文[1，2]中提出的奇异摄动方法研究了多种支承条件下，由多层各向异性单层组成的不对称层合矩形板（包括反对称正交铺设层合板和反对称角铺设层合板），在横向载荷和边缘拉力的联合作用下的非线性弯曲问题，使问题得到了简化。

二、基本方程

我们研究一薄矩形板，其长度为a(x方向)，宽度为b（y方向），厚度为t（z方向）；未变形前薄板的中面为x，y面。假设板系由N层各向异性单层粘合在一起，每层有任选的厚度和弹性性能，且单层的材料性能主方向可以与层合板轴成任选角度。

不对称的各向异性叠层矩形板的本构关系可用矩阵形式表为[7]

$$\begin{Bmatrix} N \\ M \end{Bmatrix}=\begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \varepsilon^o \\ k \end{Bmatrix} \tag{2.1}$$

式中

* 江福汝推荐．

$$
\left.\begin{array}{l}
\{N\}\text{：合力}=[N_x,\ N_y,\ N_{xy}]^T \\
\{M\}\text{：合力矩}=[M_x,\ M_y,\ M_{xy}]^T \\
\{\varepsilon^o\}\text{：中面应变}=[\varepsilon_x^o,\ \varepsilon_y^o,\ \gamma_{xy}^o]^T \\
\{k\}\text{：曲率}=[k_x,\ k_y,\ k_{xy}]^T \\
A_{ij}\text{：拉伸刚度}=\sum_{k=1}^{h}(\overline{Q}_{ij})_k(z_k-z_{k-1}) \\
B_{ij}\text{：耦合刚度}=\dfrac{1}{2}\sum_{k=1}^{n}(\overline{Q}_{ij})_k(z_k^2-z_{k-1}^2) \\
D_{ij}\text{：弯曲刚度}=\dfrac{1}{3}\sum_{k=1}^{n}(\overline{Q}_{ij})_k(z_k^3-z_{k-1}^3) \\
\qquad (i,\ j=1,\ 2,\ 6)
\end{array}\right\} \tag{2.2}
$$

为了方便起见，可将(2.1)写成“半逆”形式

$$
\begin{Bmatrix} \varepsilon^o \\ M \end{Bmatrix}=\begin{bmatrix} A^* & B^* \\ -(B^*)^T & D^* \end{bmatrix}\begin{Bmatrix} N \\ k \end{Bmatrix} \tag{2.3}
$$

式中 $A^*=A^{-1},\ B^*=-A^{-1}B,\quad D^*=D-BA^{-1}B$ (2.4)

且 $A^*_{ij}=a^*_{ij}/t,\ B^*_{ij}=b^*_{ij}\cdot t,\quad D^*_{ij}=d^*_{ij}\cdot t^3$ (2.5)

中面力可用应力函数 φ 表为

$$
N_x=\varphi_{,yy},\ N_y=\varphi_{,xx},\ N_{xy}=-\varphi_{,xy} \tag{2.6}
$$

则不对称叠层板非线性弯曲的 Von Kármán 方程为[8][12,13]

$$
\begin{bmatrix} L_1^* & L_2^* \\ L_3^* & L_4^* \end{bmatrix}\begin{Bmatrix} w \\ \varphi \end{Bmatrix}=\begin{Bmatrix} L(\varphi,\ w)+q(x,\ y) \\ -L(w,\ w)/2 \end{Bmatrix} \tag{2.7}
$$

式中

$$
\left.\begin{array}{l}
L_1^*=D_{11}^*\dfrac{\partial^4}{\partial x^4}+4D_{16}^*\dfrac{\partial^4}{\partial x^3\partial y}+2(D_{12}^*+2D_{66}^*)\dfrac{\partial^4}{\partial x^2\partial y^2} \\
\qquad +4D_{26}^*\dfrac{\partial^4}{\partial x\partial y^3}+D_{22}^*\dfrac{\partial^4}{\partial y^4} \\
L_2^*=B_{21}^*\dfrac{\partial^4}{\partial x^4}+\left(2B_{26}^*-B_{61}^*\right)\dfrac{\partial^4}{\partial x^3\partial y}+(B_{11}^*+B_{22}^*-2B_{66}^*)\dfrac{\partial^4}{\partial x^2\partial y^2} \\
\qquad +(2B_{16}^*-B_{62}^*)\dfrac{\partial^4}{\partial x\partial y^3}+B_{12}^*\dfrac{\partial^4}{\partial y^4} \\
L_3^*=-L_2^* \\
L_4^*=A_{22}^*\dfrac{\partial^4}{\partial x^4}-2A_{26}^*\dfrac{\partial^4}{\partial x^3\partial y}+(2A_{12}^*+A_{66}^*)\dfrac{\partial^4}{\partial x^2\partial y^2} \\
\qquad -2A_{16}^*\dfrac{\partial^4}{\partial x\partial y^3}+A_{11}^*\dfrac{\partial^4}{\partial y^4} \\
L=\dfrac{\partial^2}{\partial x^2}\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial y^2}\dfrac{\partial^2}{\partial x^2}-2\dfrac{\partial^2}{\partial x\partial y}\dfrac{\partial^2}{\partial x\partial y}
\end{array}\right\} \tag{2.8}
$$

引入无量纲量

$$\tilde{w}=w/a,\ \tilde{x}=x/a,\ \tilde{y}=y/a,\ \tilde{\varphi}=A_{11}^{*}\varphi/a^{2},\ \tilde{q}=aA_{11}^{*}q \tag{2.9}$$

和小参数 $\varepsilon=Kt/a,\quad (K=\sqrt{a_{11}^{*}d_{11}^{*}})$ (2.10)

方程(2.7)化为（略去字母上的"～"号）

$$\begin{bmatrix}\varepsilon^{2}L_{1} & \varepsilon L_{2}\\ \varepsilon L_{3} & L_{4}\end{bmatrix}\begin{Bmatrix}w\\ \varphi\end{Bmatrix}=\begin{Bmatrix}L(w,\ \varphi)+q(x,\ y)\\ -L(w,w)/2\end{Bmatrix} \tag{2.11}$$

式中

$$\left.\begin{aligned}&L_{i}=a_{i}\frac{\partial^{4}}{\partial x^{4}}+b_{i}\frac{\partial^{4}}{\partial x^{3}\partial y}+c_{i}\frac{\partial^{4}}{\partial x^{2}\partial y^{2}}+d_{i}\frac{\partial^{4}}{\partial x\partial y^{3}}+e_{i}\frac{\partial^{4}}{\partial y^{4}}\\&a_{1}=1,\quad b_{1}=\frac{4D_{16}^{*}}{D_{11}^{*}},\quad c_{1}=\frac{2(D_{12}^{*}+2D_{66}^{*})}{D_{11}^{*}},\quad d_{1}=\frac{4D_{26}^{*}}{D_{11}^{*}},\quad e_{1}=\frac{D_{22}^{*}}{D_{11}^{*}}\\&a_{2}=\frac{b_{21}^{*}}{K},\quad b_{2}=\frac{2b_{26}^{*}-b_{61}^{*}}{K},\quad c_{2}=\frac{b_{11}^{*}+b_{22}^{*}-2b_{66}^{*}}{K},\quad d_{2}=\frac{2b_{16}^{*}-b_{62}^{*}}{K},\\&e_{2}=\frac{b_{12}^{*}}{K};\ a_{3}=-a_{2},\ b_{3}=-b_{2},\ c_{3}=-c_{2},\ d_{3}=-d_{2},\ e_{3}=-e_{2}\\&a_{4}=\frac{A_{22}^{*}}{A_{11}^{*}},\ b_{4}=-\frac{2A_{26}^{*}}{A_{11}^{*}},\ c_{4}=\frac{2A_{12}^{*}+A_{66}^{*}}{A_{11}^{*}},\ d_{4}=-\frac{2A_{16}^{*}}{A_{11}^{*}},\ e_{4}=1\end{aligned}\right\} \tag{2.12}$$

假设挠度w和应力函数φ的边界条件为

$$\left.\begin{aligned}&w|_{x=0}=f_{1}(y),\quad \left.\frac{\partial w}{\partial x}\right|_{x=0}=g_{1}(y)\\&w|_{x=1}=f_{2}(y),\quad \left.\frac{\partial w}{\partial x}\right|_{x=1}=g_{2}(y)\\&w|_{y=0}=f_{3}(x),\quad w|_{y=\frac{b}{a}}=f_{4}(x)\\&\Big[-b_{12}^{*}\varphi_{,yy}-b_{22}^{*}\varphi_{,xx}+b_{62}^{*}\varphi_{,xy}-a_{11}^{*}\frac{\varepsilon}{K}(d_{21}^{*}w_{,xx}\\&\qquad+d_{22}^{*}w_{,yy}+2d_{26}^{*}w_{,xy})\Big]_{y=0}=g_{3}(x)\\&\Big[-b_{12}^{*}\varphi_{,yy}-b_{22}^{*}\varphi_{,xx}+b_{62}^{*}\varphi_{,xy}-a_{11}^{*}\frac{\varepsilon}{K}(d_{21}^{*}w_{,xx}\\&\qquad+d_{22}^{*}w_{,yy}+2d_{26}^{*}w_{,xy})\Big]_{y=\frac{b}{a}}=g_{4}(x)\end{aligned}\right\} \tag{2.13}$$

$$\left.\begin{aligned}&\left.\frac{\partial^{2}\varphi}{\partial y^{2}}\right|_{x=0}=h_{1}(y),\quad \left.\frac{\partial^{2}\varphi}{\partial x\partial y}\right|_{x=0}=I_{1}(y)\\&\left.\frac{\partial^{2}\varphi}{\partial y^{2}}\right|_{x=1}=h_{2}(y),\quad \left.\frac{\partial^{2}\varphi}{\partial x\partial y}\right|_{x=1}=I_{2}(y)\\&\left.\frac{\partial^{2}\varphi}{\partial x^{2}}\right|_{y=0}=h_{3}(x),\quad \left.\frac{\partial^{2}\varphi}{\partial x\partial y}\right|_{y=0}=I_{3}(x)\\&\left.\frac{\partial^{2}\varphi}{\partial x^{2}}\right|_{y=\frac{b}{a}}=h_{4}(x),\quad \left.\frac{\partial^{2}\varphi}{\partial x\partial y}\right|_{y=\frac{b}{a}}=I_{4}(x)\end{aligned}\right\} \tag{2.14}$$

或

$$\int_0^{b/a} t\frac{\partial^2\varphi}{\partial y^2}dy=\bar{P}_x t\frac{b}{a} \qquad \int_0^1 t\frac{\partial^2\varphi}{\partial x^2}dx=\bar{P}_y t$$

$$\int_0^1\left\{\varphi_{,yy}+\frac{a_{12}^*}{a_{11}^*}\varphi_{,xx}-\frac{a_{16}^*}{a_{11}^*}\varphi_{,xy}-\frac{\varepsilon}{K}[b_{11}^*w_{,xx}+b_{12}^*w_{,yy}+2b_{16}^*w_{,xy}]-\frac{1}{2}(w_{,x})^2\right\}dx=\delta_x$$

$$\int_0^{b/a}\left\{\frac{a_{21}^*}{a_{11}^*}\varphi_{,yy}+\frac{a_{22}^*}{a_{11}^*}\varphi_{,xx}-\frac{a_{26}^*}{a_{11}^*}\varphi_{,xy}-\frac{\varepsilon}{K}[b_{21}^*w_{,xx}+b_{22}^*w_{,yy}+2b_{26}^*w_{,xy}]-\frac{1}{2}(w_{,y})^2\right\}dy=\delta_y$$

式中$\bar{P}_x tb/a$和$\bar{P}_y t$分别为在板平面内，在x方向和在y方向的载荷；δ_x和δ_y分别为在x方向和在y方向的伸长。

三、递推方程和边界条件

关于微分算子的展开式请参看文[11]。

在边界$x=0$和$x=1$的邻域分别引入w的边界层函数$v_n^{(1)}$和$v_n^{(2)}$，φ的边界层函数$h_n^{(1)}$和$h_n^{(2)}$；在边界$y=0$和$y=b/a$的邻域分别引入w的边界层函数$v_n^{(3)}$和$v_n^{(4)}$，φ的边界层函数$h_n^{(3)}$和$h_n^{(4)}$。假设挠度函数w和应力函数φ的N阶近似式为

$$W_N(x,y,\varepsilon)=\sum_{n=0}^N\varepsilon^n w_n(x,y)+\sum_{n=0}^N\varepsilon^{n+\alpha_1}v_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^N\varepsilon^{n+\alpha_2}v_n^{(2)}(\tilde{\xi},\tilde{\eta},y)+\sum_{n=0}^N\varepsilon^{n+\alpha_3}v_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^N\varepsilon^{n+\alpha_4}v_n^{(4)}(x,\tilde{\alpha},\tilde{\beta}) \tag{3.1}$$

$$\Phi_N(x,y,\varepsilon)=\sum_{n=0}^N\varepsilon^n\varphi_n(x,y)+\sum_{n=0}^N\varepsilon^{n+\beta_1}h_n^{(1)}(\xi,\eta,y)+\sum_{n=0}^N\varepsilon^{n+\beta_2}h_n^{(2)}(\tilde{\xi},\tilde{\eta},y)+\sum_{n=0}^N\varepsilon^{n+\beta_3}h_n^{(3)}(x,\alpha,\beta)+\sum_{n=0}^N\varepsilon^{n+\beta_4}h_n^{(4)}(x,\tilde{\alpha},\tilde{\beta}) \tag{3.2}$$

将(3.1)式和(3.2)式代入方程(2.11)和边界条件(2.13)、(2.14)，我们看到应取$\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$，$\beta_1=\beta_2=2$，$\beta_3=\beta_4=3$[9]。逐次地比较等式两端$\varepsilon$的各次幂的系数，我们得到关于$w_n$，$\varphi_n$和边界层函数$v_n^{(i)}$，$h_n^{(i)}(i=1,\cdots,4)$的递推方程和边界条件。关于$w_n$，$\varphi_n$的递推方程为

$$\begin{cases} L(w_0,\varphi_0)+q=0 & (3.3\text{a})\\ L_4\varphi_0+(w_0,w_0)L/2=0 & (3.3\text{b})\end{cases}$$

$$\begin{cases} L(w_0,\varphi_n)+L(w_n,\varphi_0)=L_1w_{n-2}+L_2\varphi_{n-1}-\sum_{i=1}^{n-1}L(w_i,\varphi_{n-i}) & (3.4\text{a})\\ \qquad (n=1,2,\cdots,N) \\ L_3w_{n-1}+L_4\varphi_n+L(w_0,w_n)=-\frac{1}{2}\sum_{i=1}^{n-1}L(w_i,w_{n-i}) & (3.4\text{b})\end{cases}$$

边界层函数$v_n^{(i)}(i=1,\ \cdots,\ 4)$的递推方程为

$$\left.\begin{aligned}&D_{10}v_0^{(1)}+D_{20}h_0^{(1)}-R_0^{(1)}(\varphi_0,\ v_0^{(1)})=0\\&\widetilde{D}_{10}v_0^{(2)}+\widetilde{D}_{20}h_0^{(2)}-R_0^{(2)}(\varphi_0,\ v_0^{(2)})=0\\&D'_{10}v_0^{(3)}+D'_{20}h_0^{(3)}-R_0^{(3)}(\varphi_0,v_0^{(3)})=0\\&\widetilde{D}'_{10}v_0^{(4)}+\widetilde{D}'_{20}h_0^{(4)}-R_0^{(4)}(\varphi_0,v_0^{(4)})=0\end{aligned}\right\}\tag{3.5}$$

$$\left.\begin{aligned}&D_{10}v_n^{(1)}+D_{20}h_n^{(1)}-R_0^{(1)}(\varphi_0,v_n^{(1)})=\sum_{\substack{j+k=n\\(j\neq0)}}R_0^{(1)}(\varphi_j,v_k^{(1)})\\&\quad+\sum_{i=1}^{2}\sum_{j+k=n-i}R_i^{(1)}(\varphi_j,v_k^{(1)})+\sum_{i=0}^{2}\sum_{j+k=n-1-i}R_i^{(1)}(w_j,\ h_k^{(1)})\\&\quad+\sum_{i=0}^{2}\sum_{j+k=n-2-i}M_i^{11}(v_j^{(1)},h_k^{(1)})-\sum_{i=1}^{4}D_{1i}v_{n-i}^{(1)}-\sum_{i=1}^{4}D_{2i}h_{n-i}^{(1)}\\&\widetilde{D}_{10}v_n^{(2)}+\widetilde{D}_{20}h_n^{(2)}-R_0^{(2)}(\varphi_0,\ v_n^{(2)})=\sum_{\substack{j+k=n\\(j\neq0)}}R_0^{(2)}(\varphi_j,v_k^{(2)})\\&\quad+\sum_{i=1}^{2}\sum_{j+k=n-i}R_i^{(2)}(\varphi_j,v_k^{(2)})+\sum_{i=0}^{2}\sum_{j+k=n-1-i}R_i^{(2)}(w_j,h_k^{(2)})\\&\quad+\sum_{i=0}^{2}\sum_{j+k=n-2-i}M_i^{22}(v_j^{(2)},h_k^{(2)})-\sum_{i=1}^{4}\widetilde{D}_{1i}v_{n-i}^{(2)}-\sum_{i=1}^{4}\widetilde{D}_{2i}h_{n-i}^{(2)}\\&D'_{10}v_n^{(3)}+D'_{20}h_n^{(3)}-R_0^{(3)}(\varphi_0,v_n^{(3)})=\sum_{\substack{j+k=n\\(j\neq0)}}R_0^{(3)}(\varphi_j,v_k^{(3)})\\&\quad+\sum_{i=1}^{2}\sum_{j+k=n-i}R_i^{(3)}(\varphi_j,v_k^{(3)})+\sum_{i=0}^{2}\sum_{j+k=n-1-i}R_i^{(3)}(w_j,h_k^{(3)})\\&\quad+\sum_{i=0}^{2}\sum_{j+k=n-3-i}N_i^{33}(v_j^{(3)},h_k^{(3)})-\sum_{i=1}^{4}D'_{1i}v_{n-i}^{(3)}-\sum_{i=1}^{4}D'_{2i}h_{n-i}^{(3)}\\&\widetilde{D}'_{10}v_n^{(4)}+\widetilde{D}'_{20}h_n^{(4)}-R_0^{(4)}(\varphi_0,v_0^{(4)})=\sum_{\substack{j+k=n\\(j\neq0)}}R_0^{(4)}(\varphi_j,v_k^{(4)})\\&\quad+\sum_{i=1}^{2}\sum_{j+k=n-i}R_i^{(4)}(\varphi_j,v_k^{(4)})+\sum_{i=0}^{2}\sum_{j+k=n-1-i}R_i^{(4)}(w_j,h_k^{(4)})\\&\quad+\sum_{i=0}^{2}\sum_{j+k=n-3-i}N_i^{44}(v_j^{(4)},h_k^{(4)})-\sum_{i=1}^{4}\widetilde{D}'_{1i}v_{n-i}^{(4)}-\sum_{i=1}^{4}\widetilde{D}'_{2i}h_{n-i}^{(4)}\\&\qquad\qquad\qquad\qquad(n=1,\ 2,\ \cdots,\ N)\end{aligned}\right\}\tag{3.6}$$

$h_n^{(i)}$ 的递推方程为

$$\left.\begin{array}{l} D_{30}v_0^{(1)}+D_{40}h_0^{(1)}=0, \quad \widetilde{D}_{30}v_0^{(2)}+\widetilde{D}_{40}h_0^{(2)}=0 \\ D'_{30}v_0^{(3)}+D'_{40}h_0^{(3)}=0, \quad \widetilde{D}'_{30}v_0^{(4)}+\widetilde{D}'_{40}h_0^{(4)}=0 \end{array}\right\} \tag{3.7}$$

$$\left.\begin{array}{l} D_{30}v_n^{(1)}+D_{40}h_n^{(1)}=-\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-1-i}R_i^{(1)}(w_j,v_k^{(1)}) \\ \qquad -\frac{1}{2}\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-2-i}M_i^{11}(v_j^{(1)},\ v_k^{(1)}) \ -\sum\limits_{i=1}^{4}D_{3i}v_{n-i}^{(1)} \ -\sum\limits_{i=1}^{4}D_{4i}h_{n-i}^{(1)} \\ \widetilde{D}_{30}v_n^{(2)}+\widetilde{D}_{40}h_n^{(2)}=-\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-1-i}R_i^{(2)}(w_j,\ v_k^{(2)}) \\ \qquad -\sum\limits_{i=1}^{4}\widetilde{D}_{3i}v_{n-i}^{(2)}-\sum\limits_{i=1}^{4}\widetilde{D}_{4i}h_{n-i}^{(2)} \ -\frac{1}{2}\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-2-i}M_i^{22}(v_j^{(2)},v_k^{(2)}) \\ D'_{30}v_n^{(3)}+D'_{40}h_n^{(3)}=-\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-1-i}R_i^{(3)}(w_j,v_k^{(3)}) \\ \qquad -\sum\limits_{i=1}^{4}D'_{3i}v_{n-i}^{(3)}-\sum\limits_{i=1}^{4}D'_{4i}h_{n-i}^{(3)} \ -\frac{1}{2}\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-3-i}N_i^{33}(v_j^{(3)},v_k^{(3)}) \\ \widetilde{D}'_{30}v_n^{(4)}+\widetilde{D}'_{40}h_n^{(4)}=-\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-1-i}R_i^{(4)}(w_j,v_k^{(4)}) \\ \qquad -\sum\limits_{i=1}^{4}\widetilde{D}'_{3i}v_{n-i}^{(4)}-\sum\limits_{i=1}^{4}\widetilde{D}'_{4i}h_{n-i}^{(4)} \ -\frac{1}{2}\sum\limits_{i=0}^{2}\sum\limits_{j+k=n-3-i}N_i^{44}(v_j^{(4)},v_k^{(4)}) \\ \qquad\qquad (n=1,\ 2,\ \cdots,\ N) \end{array}\right\} \tag{3.8}$$

注意在上式和以后的计算中，我们都将带负下标的量取为零。对应的边界条件为

$$\left.\begin{array}{ll} w_0|_{x=0}=f_1(y), & w_0|_{x=1}=f_2(y) \\ w_0|_{y=0}=f_3(x), & w_0|_{y=\frac{b}{a}}=f_4(y) \\ \multicolumn{2}{l}{w_{0,x}|_{x=0}+\delta_{1,0}v_0^{(1)}|_{\eta=0}=g_1(y)} \\ \multicolumn{2}{l}{w_{0,x}|_{x=1}+\widetilde{\delta}_{1,0}v_0^{(2)}|_{\widetilde{\eta}=1}=g_2(y)} \\ \multicolumn{2}{l}{-b_{12}^*\varphi_{0,yy}-b_{22}^*\varphi_{0,xx}+b_{62}^*\varphi_{0,xy}|_{y=0}-\frac{a_{11}^*}{K}d_{22}^*\gamma_{2,0}v_0^{(3)}|_{\beta=0}=g_3(x)} \\ \multicolumn{2}{l}{-b_{12}^*\varphi_{0,yy}-b_{22}^*\varphi_{0,xx}+b_{62}^*\varphi_{0,xy}|_{y=\frac{b}{a}}-\frac{a_{11}^*}{K}d_{22}^*\widetilde{\gamma}_{2,0}v_0^{(4)}|_{\widetilde{\beta}=\frac{b}{a}}=g_4(x)} \end{array}\right\} \tag{3.9}$$

$$\left.\begin{aligned}
&w_n|_{x=0}+v_{n-1}^{(1)}|_{\eta=0}=0 \qquad w_n|_{x=1}+v_{n-1}^{(2)}|_{\bar\eta=1}=0\\
&w_n|_{y=0}+v_{n-1}^{(3)}|_{\beta=0}=0 \qquad w_n|_{y=\frac{b}{a}}+v_{n-1}^{(4)}|_{\bar\beta=\frac{b}{a}}=0\\
&w_{n,x}|_{x=0}+[\delta_{1,0}v_n^{(1)}+\delta_{1,1}v_{n-1}^{(1)}]_{\eta=0}=0\\
&w_{n,x}|_{x=1}+[\bar\delta_{1,0}v_n^{(2)}+\bar\delta_{1,1}v_{n-1}^{(2)}]_{\bar\eta=1}=0\\
&\Big\{-b_{12}^*\varphi_{n,yy}-b_{22}^*\varphi_{n,xx}+b_{62}^*\varphi_{n,xy}-\frac{a_{11}^*}{K}[d_{21}^*w_{n-1,xx}+d_{22}^*w_{n-1,yy}\\
&\qquad+2d_{26}^*w_{n-1,xy}]\Big\}_{y=0}+\Big\{-b_{12}^*(\gamma_{2,0}h_{n-1}^{(3)}+\gamma_{2,1}h_{n-2}^{(3)}+\gamma_{2,2}h_{n-3}^{(3)})\\
&\qquad-b_{22}^*h_{n-3,xx}^{(3)}+b_{62}^*(\gamma_{1,0}h_{n-2,x}^{(3)}+\gamma_{1,1}h_{n-3,x}^{(3)})-\frac{a_{11}^*}{K}[d_{21}^*v_{n-2,xx}^{(3)}\\
&\qquad+d_{22}^*(\gamma_{2,0}v_n^{(3)}+\gamma_{2,1}v_{n-1}^{(3)}+\gamma_{2,2}v_{n-2}^{(3)})\\
&\qquad+2d_{26}^*(\gamma_{1,0}v_{n-1,x}^{(3)}+\gamma_{1,1}v_{n-2,x}^{(3)})]\Big\}_{\beta=0}=0\\
&\Big\{-b_{12}^*\varphi_{n,yy}-b_{22}^*\varphi_{n,xx}+b_{62}^*\varphi_{n,xy}-\frac{a_{11}^*}{K}[d_{21}^*w_{n-1,xx}\\
&\qquad+d_{22}^*w_{n-1,yy}+2d_{26}^*w_{n-1,xy}]\Big\}_{y=\frac{b}{a}}+\Big\{-b_{12}^*(\bar\gamma_{2,0}h_{n-1}^{(4)}\\
&\qquad+\bar\gamma_{2,1}h_{n-2}^{(4)}+\bar\gamma_{2,2}h_{n-3}^{(4)})-b_{22}^*h_{n-3,xx}^{(4)}+b_{62}^*(\bar\gamma_{1,0}h_{n-2,x}^{(4)}\\
&\qquad+\bar\gamma_{1,1}h_{n-3,x}^{(4)})-\frac{a_{11}^*}{K}[d_{21}^*v_{n-2,xx}^{(4)}+d_{22}^*(\bar\gamma_{2,0}v_n^{(4)}\\
&\qquad+\bar\gamma_{2,1}v_{n-1}^{(4)}+\bar\gamma_{2,2}v_{n-2}^{(4)})+2d_{26}^*(\bar\gamma_{1,0}v_{n-1,x}^{(4)}+\bar\gamma_{1,1}v_{n-2,x}^{(4)})]\Big\}_{\bar\beta=\frac{b}{a}}=0
\end{aligned}\right\}\tag{3.10}$$

$$\left.\begin{aligned}
&\varphi_{0,yy}|_{x=0}=h_1(y) \qquad \varphi_{0,xy}|_{x=0}=I_1(y)\\
&\varphi_{0,yy}|_{x=1}=h_2(y) \qquad \varphi_{0,xy}|_{x=1}=I_2(y)\\
&\varphi_{0,xx}|_{y=0}=h_3(x) \qquad \varphi_{0,xy}|_{y=0}=I_3(x)\\
&\varphi_{0,xx}|_{y=\frac{b}{a}}=h_4(x) \qquad \varphi_{0,xy}|_{y=\frac{b}{a}}=I_4(x)
\end{aligned}\right\}\tag{3.11}$$

$$\left.\begin{aligned}
&\varphi_{n,yy}|_{x=0}+h_{n-2,yy}^{(1)}|_{\eta=0}=0\\
&\varphi_{n,xy}|_{x=0}+(\delta_{1,0}h_{n-1,y}^{(1)}+\delta_{1,1}h_{n-2,y}^{(1)})|_{\eta=0}=0\\
&\varphi_{n,yy}|_{x=1}+h_{n-2,yy}^{(2)}|_{\bar\eta=1}=0\\
&\varphi_{n,xy}|_{x=1}+(\bar\delta_{1,0}h_{n-1,y}^{(2)}+\bar\delta_{1,1}h_{n-2,y}^{(2)})|_{\bar\eta=1}=0\\
&\varphi_{n,xx}|_{y=0}+h_{n-3,xx}^{(3)}|_{\beta=0}=0\\
&\varphi_{n,xy}|_{y=0}+(\gamma_{1,0}h_{n-2,x}^{(3)}+\gamma_{1,1}h_{n-3,x}^{(3)})|_{\beta=0}=0\\
&\varphi_{n,xx}|_{y=\frac{b}{a}}+h_{n-3,xx}^{(4)}|_{\bar\beta=\frac{b}{a}}=0\\
&\varphi_{n,xy}|_{y=\frac{b}{a}}+(\bar\gamma_{1,0}h_{n-2,x}^{(4)}+\bar\gamma_{1,1}h_{n-3,x}^{(4)})|_{\bar\beta=\frac{b}{a}}=0\\
&\qquad\qquad(n=1,2,\cdots,N)
\end{aligned}\right\}\tag{3.12}$$

或

$$\left.\begin{aligned}
&\int_0^1 t\varphi_{0,xx}dx=\bar P_y t \qquad \int_0^{b/a}t\varphi_{0,yy}dy=\bar P_x t\frac{b}{a}\\
&\int_0^1\Big[\varphi_{0,yy}+\frac{a_{12}^*}{a_{11}^*}\varphi_{0,xx}-\frac{a_{16}^*}{a_{11}^*}\varphi_{0,xy}-\frac{1}{2}(w_{0,x})^2\Big]dx=\delta_x\\
&\int_0^{b/a}\Big[\frac{a_{21}^*}{a_{11}^*}\varphi_{0,yy}+\frac{a_{22}^*}{a_{11}^*}\varphi_{0,xx}-\frac{a_{26}^*}{a_{11}^*}\varphi_{0,xy}-\frac{1}{2}(w_{0,y})^2\Big]dy=\delta_y
\end{aligned}\right\}\tag{3.13}$$

$$
\left.
\begin{aligned}
&\int_0^1 t[\varphi_{n,xx}+h^{(3)}_{n-3,xx}]dx=0, \quad \int_0^1 t[\varphi_{n,xx}+h^{(4)}_{n-3,xx}]dx=0\\
&\int_0^{b/a} t[\varphi_{n,yy}+h^{(1)}_{n-2,yy}]dy=0, \quad \int_0^{b/a} t[\varphi_{n,yy}+h^{(2)}_{n-2,yy}]dy=0\\
&\int_0^1\Big\{\Big[\varphi_{n,yy}+h^{(3)}_{n-3,yy}+\frac{a^*_{12}}{a^*_{11}}(\varphi_{n,xx}+h^{(3)}_{n-3,xx})-\frac{a^*_{16}}{a^*_{11}}(\varphi_{n,xy}+h^{(3)}_{n-3,xy})\Big]\\
&\quad-\frac{1}{K}[b^*_{11}(w_{n-1,xx}+v^{(3)}_{n-2,xx})+b^*_{12}(w_{n-1,yy}+v^{(3)}_{n-2,yy})\\
&\quad+2b^*_{16}(w_{n-1,xy}+v^{(3)}_{n-2,xy})\Big]-\frac{1}{2}\Big[w^2_{\frac{n}{2},x}\\
&\quad+2\sum_{i=0}^{n}(w_{i,x}w_{n-i,x}+w_{i,x}v^{(3)}_{n-1-i,x})\Big]\Big\}dx=0\\
&\int_0^1\Big\{\Big[\varphi_{n,yy}+h^{(4)}_{n-3,yy}+\frac{a^*_{12}}{a^*_{11}}(\varphi_{n,xx}+h^{(4)}_{n-3,xx})-\frac{a^*_{16}}{a^*_{11}}(\varphi_{n,xy}+h^{(4)}_{n-3,xy})\Big]\\
&\quad-\frac{1}{K}[b^*_{11}(w_{n-1,xx}+v^{(4)}_{n-2,xx})+b^*_{12}(w_{n-1,yy}+v^{(4)}_{n-2,yy})\\
&\quad+2b^*_{16}(w_{n-1,xy}+v^{(4)}_{n-2,xy})]-\frac{1}{2}\Big[w^2_{\frac{n}{2},x}\\
&\quad+2\sum_{i=0}^{n}(w_{i,x}w_{n-i,x}+w_{i,x}v^{(4)}_{n-1-i,x})\Big]\Big\}dx=0\\
&\int_0^{b/a}\Big\{\Big[\frac{a^*_{21}}{a^*_{11}}(\varphi_{n,yy}+h^{(1)}_{n-2,yy})+\frac{a^*_{22}}{a^*_{11}}(\varphi_{n,xx}+h^{(1)}_{n-2,xx})\\
&\quad-\frac{a^*_{26}}{a^*_{11}}(\varphi_{n,xy}+h^{(1)}_{n-2,xy})\Big]-\frac{1}{K}\Big[b^*_{21}(w_{n-1,xx}+v^{(1)}_{n-2,xx})\\
&\quad+b^*_{22}(w_{n-1,yy}+v^{(1)}_{n-2,yy})+2b^*_{26}(w_{n-1,xy}+v^{(1)}_{n-2,xy})\Big]\\
&\quad-\frac{1}{2}\Big[w^2_{\frac{n}{2},y}+2\sum_{i=0}^{n}(w_{i,y}w_{n-i,y}+w_{i,y}v^{(1)}_{n-1-i,y})\Big\}dy=0\\
&\int_0^{b/a}\Big\{\Big[\frac{a^*_{21}}{a^*_{11}}(\varphi_{n,yy}+h^{(2)}_{n-2,yy})+\frac{a^*_{22}}{a^*_{11}}(\varphi_{n,xx}+h^{(2)}_{n-2,xx})\\
&\quad-\frac{a^*_{26}}{a^*_{11}}(\varphi_{n,xy}+h^{(2)}_{n-2,xy})\Big]-\frac{1}{K}[b^*_{21}(w_{n-1,xx}+v^{(2)}_{n-1,xx})\\
&\quad+b^*_{22}(w_{n-1,yy}+v^{(2)}_{n-2,yy})+2b^*_{26}(w_{n-1,xy}+v^{(2)}_{n-2,xy})]\\
&\quad-\frac{1}{2}\Big[w^2_{\frac{n}{2},y}+2\sum_{i=0}^{n}(w_{i,y}w_{n-i,y}+w_{i,y}v^{(2)}_{n-1-i,y})\Big]\Big\}dy=0\\
&\qquad\qquad\qquad\qquad(n=1,2,\cdots,N)
\end{aligned}
\right\}\quad(3.14)
$$

四、形式渐近解的导出

方程(3.3a,b)和边界条件(3.9)、(3.11)给出退化边值问题

$$L(w_0,\varphi_0)+q=0 \qquad L_4\varphi_0+(w_0,w_0)L/2=0 \tag{4.1a,b}$$

$$\left.\begin{aligned} &w_0|_{x=0}=f_1(y), && w_0|_{x=1}=f_2(y) \\ &w_0|_{y=0}=f_3(x), && w_0|_{y=\frac{b}{a}}=f_4(x) \end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned} &\varphi_{0,yy}|_{x=0}=h_1(y), && \varphi_{0,yy}|_{x=1}=h_2(y) \\ &\varphi_{0,xx}|_{y=0}=h_3(x), && \varphi_{0,xx}|_{y=\frac{b}{a}}=h_4(x) \end{aligned}\right\} \tag{4.3}$$

将所求得的薄膜解w_0,φ_0代入(3.5)中各式，取待定系数

$$\left.\begin{aligned} &u(x,y)=\sqrt{\frac{a_4}{a_4+a_2^2}}\int_0^x\sqrt{\varphi_{0,yy}(x,y)}dx,\ \tilde{u}(x,y)=\sqrt{\frac{a_4}{a_4+a_2^2}}\int_x^1\sqrt{\varphi_{0,yy}(x,y)}dx \\ &p(x,y)=(e_1+e_2^2)^{-\frac{1}{2}}\int_0^y\sqrt{\varphi_{0,xx}(x,y)}dy,\ \tilde{p}(x,y)=(e_1+e_2^2)^{-\frac{1}{2}}\int_y^{b/a}\sqrt{\varphi_{0,xx}(x,y)}\,dy \end{aligned}\right\} \tag{4.4}$$

得边界层函数

$$\left.\begin{aligned} v_0^{(1)}(\xi,\eta,y)&=C_0^{(1)}(\eta,y)\exp[-\xi] \\ &=C_0^{(1)}(x,y)\exp\left[-\frac{1}{\varepsilon}\sqrt{\frac{a_4}{a_4+a_2^2}}\int_0^x\sqrt{\varphi_{0,yy}(x,y)}\,dx\right] \\ v_0^{(2)}(\tilde{\xi},\tilde{\eta},y)&=C_0^{(2)}(\tilde{\eta},y)\exp[-\tilde{\xi}] \\ &=C_0^{(2)}(x,y)\exp\left[-\frac{1}{\varepsilon}\sqrt{\frac{a_4}{a_4+a_2^2}}\int_x^1\sqrt{\varphi_{0,yy}(x,y)}\,dx\right] \\ v_0^{(3)}(x,\alpha,\beta)&=C_0^{(3)}(x,\beta)\exp[-\alpha] \\ &=C_0^{(3)}(x,y)\exp\left[-\frac{1}{\varepsilon}(e_1+e_2^2)^{-\frac{1}{2}}\int_0^y\sqrt{\varphi_{0,xx}(x,y)}\,dy\right] \\ v_0^{(4)}(x,\tilde{\alpha},\tilde{\beta})&=C_0^{(4)}(x,\tilde{\beta})\exp[-\tilde{\alpha}] \\ &=C_0^{(4)}(x,y)\exp\left[-\frac{1}{\varepsilon}(e_1+e_2^2)^{-\frac{1}{2}}\int_y^{b/a}\sqrt{\varphi_{0,xx}(x,y)}\,dy\right] \end{aligned}\right\} \tag{4.5}$$

利用边界条件(3.9)，可得到$C_0^{(1)}(\eta,y)$，$C_0^{(2)}(\tilde{\eta},y)$，$C_0^{(3)}(x,\beta)$和$C_0^{(4)}(x,\tilde{\beta})$的边值条件为

$$\begin{aligned} &C_0^{(1)}(\eta y)\Big|_{\eta=0}=-\sqrt{\frac{a_4+a_2^2}{a_4h_1(y)}}\,[g_1(y)-w_{0,x}(0,y)] \\ &C_0^{(2)}(\tilde{\eta},y)|_{\tilde{\eta}=1}=-\sqrt{\frac{a_4+a_2^2}{a_4h_2(y)}}\,[g_2(y)-w_{0,x}(1,y)] \end{aligned}$$

$$\left.\begin{aligned}C_0^{(3)}(x,\beta)|_{\beta=0}=&-\frac{K(e_1+e_2^2)}{a_{11}^*d_{22}^*h_3(x)}[g_3(x)+b_{12}^*\varphi_{0,yy}(x,0)\\&+b_{22}^*\varphi_{0,xx}(x,0)-b_{62}^*\varphi_{0,xy}(x,0)]\\C_0^{(4)}(x,\tilde\beta)|_{\tilde\beta=\frac{b}{a}}=&-\frac{K(e_1+e_2^2)}{a_{11}^*d_{22}^*h_4(x)}\left[g_4(x)+b_{12}^*\varphi_{0,yy}\left(x,\frac{b}{a}\right)\right.\\&\left.+b_{22}^*\varphi_{0,xx}\left(x,\frac{b}{a}\right)-b_{62}^*\varphi_{0,xy}\left(x,\frac{b}{a}\right)\right]\end{aligned}\right\}\quad(4.6)$$

然后从方程(3.7)可得到边界层函数

$$\left.\begin{aligned}&h_0^{(1)}(\xi,\eta,y)=\frac{a_2}{a_4}C_0^{(1)}(\eta,y)\exp[-\xi]\\&h_0^{(2)}(\tilde\xi,\tilde\eta,y)=\frac{a_2}{a_4}C_0^{(2)}(\tilde\eta,y)\exp[-\tilde\xi]\\&h_0^{(3)}(x,\alpha,\beta)=e_2C_0^{(3)}(x,\beta)\exp[-\alpha]\\&h_0^{(4)}(x,\tilde\alpha,\tilde\beta)=e_2C_0^{(4)}(x,\tilde\beta)\exp[-\tilde\alpha]\end{aligned}\right\}\quad(4.7)$$

在方程(3.4)和边界条件(3.10)和(3.12)中取$n=1$，得到w_1和φ_1的线性边值问题

$$\begin{cases}L(w_0,\varphi_1)+L(w_1,\varphi_0)=L_2\varphi_0 & (4.8\text{a})\\L_3w_0+L_4\varphi_1+L(w_0,w_1)=0 & (4.8\text{b})\end{cases}$$

$$\left.\begin{aligned}&w_1|_{x=0}=-v_0^{(1)}|_{\eta=0}=-C_0^{(1)}(0,y)\\&w_1|_{x=1}=-v_0^{(2)}|_{\tilde\eta=1}=-C_0^{(2)}(1,y)\\&w_1|_{y=0}=-v_0^{(3)}|_{\beta=0}=-C_0^{(3)}(x,0)\\&w_1|_{y=\frac{b}{a}}=-v_0^{(4)}\Big|_{\tilde\beta=\frac{b}{a}}=-C_0^{(4)}\left(x,\frac{b}{a}\right)\end{aligned}\right\}\quad(4.9)$$

$$\left.\begin{aligned}&\varphi_{1,yy}|_{x=0}=0 &&\varphi_{1,yy}|_{x=1}=0\\&\varphi_{1,xx}|_{y=0}=0 &&\varphi_{1,xx}|_{y=\frac{b}{a}}=0\end{aligned}\right\}\quad(4.10)$$

将所求得的w_0, φ_0, $v_0^{(i)}$和$h_0^{(i)}$ ($i=1,\cdots,4$) 代入以上方程，得到w_1和φ_1后，再代入方程(3.6) (取$n=1$)，为消除长期项，令各等式右端为零，得到确定$C_0^{(1)}(\eta,y),\cdots,C_0^{(4)}(x,\tilde\beta)$的一阶线性偏微分方程

$$\left.\begin{aligned}&2A\frac{\partial C_0^{(1)}}{\partial\eta}+\left[2\varphi_{0,xy}+\frac{a_4}{a_4+a_2^2}\left(b_1+\frac{2a_2b_2}{a_4}-\frac{a_2^2b_4}{a_4^2}\right)A\right]\frac{\partial C_0^{(1)}}{\partial y}\\&\quad+\left[\left(\varphi_{1,yy}+\frac{2a_2}{a_4}w_{0,yy}\right)\sqrt{\frac{a_4A}{a_4+a_2^2}}+\frac{5}{2}A_{,x}\right]C_0^{(1)}=0\\&2A\frac{\partial C_0^{(2)}}{\partial\tilde\eta}+\left[2\varphi_{0,xy}+\frac{a_4}{a_4+a_2^2}\left(b_1+\frac{2a_2b_2}{a_4}-\frac{a_2^2b_4}{a_4^2}\right)A\right]\frac{\partial C_0^{(2)}}{\partial y}\\&\quad+\left[\left(\varphi_{1,yy}+\frac{2a_2}{a_4}w_{0,yy}\right)\sqrt{\frac{a_4A}{a_4+a_2^2}}+\frac{5}{2}A_{,x}\right]C_0^{(2)}=0\\&2B\frac{\partial C_0^{(3)}}{\partial\beta}+\left[2\varphi_{0,xy}+\frac{B}{e_1+e_2^2}\left(d_1+2d_2e_2-d_4e_2^2\right)\right]\frac{\partial C_0^{(3)}}{\partial x}\\&\quad+\left[\left(\varphi_{1,xx}+2e_2w_{0,xx}\right)\sqrt{\frac{B}{e_1+e_2^2}}+\frac{5}{2}B_{,y}\right]C_0^{(3)}=0\end{aligned}\right\}\quad(4.11)$$

$$2B\frac{\partial C_0^{(4)}}{\alpha\tilde{\beta}}+\left[2\varphi_{0,xy}+\frac{B}{e_1+e_2^2}(d_1+2d_2e_2-d_4e_2^2)\right]\frac{\partial C_0^{(4)}}{\partial\eta}$$
$$+\left[\left(\varphi_{1,xx}+2e_2w_{0,xx}\right)\sqrt{\frac{B}{e_1+e_2^2}}+\frac{5}{2}B_{,y}\right]C_0^{(4)}=0$$

式中$A(x,y)=\varphi_{0,yy}$，$B(x,y)=\varphi_{0,xx}$，只要$A(x,y)>0$，$B(x,y)>0$，便可以根据 Cauchy 条件（4.6）唯一地解得$C_0^{(1)}(\eta,y)$，…，$C_0^{(4)}(x,\tilde{\beta})$．至此，便完全确定了边界层型函数$v_0^{(i)}$，$h_0^{(i)}(i=1,\ \cdots,\ 4)$．方程(3.6)中各式(取$n=1$)则化为齐次方程，可求得

$$\left.\begin{aligned}
v_1^{(1)}(\xi,\eta,y)&=C_1^{(1)}(\eta,y)\text{xep}[-\xi]\\
v_1^{(2)}(\tilde{\xi},\tilde{\eta},y)&=C_1^{(2)}(\tilde{\eta},y)\exp[-\tilde{\xi}]\\
v_1^{(3)}(x,\alpha,\beta)&=C_1^{(3)}(x,\beta)\exp[-\alpha]\\
v_1^{(4)}(x,\tilde{\alpha},\tilde{\beta})&=C_1^{(4)}(x,\tilde{\beta})\exp[-\tilde{\alpha}]
\end{aligned}\right\}\tag{4.12}$$

由边界条件（3.10）（取$n=1$），得到$C_1^{(1)}(\eta,y)$，…，$C_1^{(4)}(x,\tilde{\beta})$的边值条件为

$$\left.\begin{aligned}
&C_1^{(1)}(\eta,y)|_{\eta=0}=\sqrt{\frac{a_4+a_2^2}{a_4h_1(y)}}\left[w_{1,x}(0,y)+\frac{\partial C_0^{(1)}}{\partial\eta}(0,y)\right]\\
&C_1^{(2)}(\tilde{\eta},y)|_{\tilde{\eta}=1}=\sqrt{\frac{a_4+a_2^2}{a_4h_2(h)}}\left[w_{1,x}(1,y)+\frac{\partial C_0^{(2)}}{\partial\tilde{\eta}}(1,y)\right]\\
&C_1^{(3)}(x,\beta)|_{\beta=0}=-\frac{e_1+e_2^2}{d_{22}^*h_3(x)}\left[d_{21}^*w_{0,xx}(x,0)\right.\\
&\qquad\left.+d_{22}^*w_{0,yy}(x,0)+2d_{26}w_{0,xy}(x,0)\right]\\
&\qquad-\frac{Ke_2b_{12}^*}{a_{11}^*d_{22}^*}C_0^{(3)}(x,0)+\frac{2\sqrt{e_1+e_2^2}}{d_{22}^*}\left[\left(\frac{\partial C_0^{(3)}}{\partial\beta}\right.\right.\\
&\qquad\left.\left.+d_{26}^*\frac{\partial C_0^{(3)}}{\partial x}\right)h_3^{-\frac{1}{2}}(x)+\frac{1}{4}h_3^{-\frac{3}{2}}(x)B_{,y}C_0^{(3)}\right]_{\beta=0}\\
&C_1^{(4)}(x,\tilde{\beta})\Big|_{\tilde{\beta}=\frac{b}{a}}=-\frac{e_1+e_2^2}{d_{22}^*h_4(x)}\left[d_{21}^*w_{0,xx}\left(x,\frac{b}{a}\right)+d_{22}^*w_{0,yy}\left(x,\frac{b}{a}\right)\right.\\
&\qquad\left.+2d_{26}^*w_{0,xy}\left(x,\frac{b}{a}\right)\right]-\frac{Ke_2b_{12}^*}{a_{11}^*d_{22}^*}C_0^{(4)}\left(x,\frac{b}{a}\right)\\
&\qquad+\frac{2\sqrt{e_1+e_2^2}}{d_{22}^*}\left[\left(\frac{\partial C_0^{(4)}}{\alpha\tilde{\beta}}+d_{26}^*\frac{\partial C_0^{(4)}}{\partial x}\right)h_4^{-\frac{1}{2}}(x)\right.\\
&\qquad\left.+\frac{1}{4}h_4^{-\frac{3}{2}}(x)B_{,y}C_0^{(4)}\right]_{\tilde{\beta}=\frac{b}{a}}
\end{aligned}\right\}\tag{4.13}$$

然后，由方程（3.8）（取$n=1$），我们得到

$$h_1^{(1)}(\xi,\eta,y)=\frac{1}{a_4^2}\left[a_2a_4C_1^{(1)}(\eta,y)-(a_4+a_2^2)A^{-1}w_{0,yy}C_0^{(1)}(\eta,y)\right.$$
$$\left.+(b_3a_4+b_4a_2)\sqrt{\frac{a_4+a_2^2}{a_4A}}\frac{\partial C_0^{(1)}}{\partial y}\right]\exp[-\xi]$$

$$\left.\begin{aligned}
h_1^{(2)}(\xi,\tilde{\eta},y)&=\frac{1}{a_4^2}\Big[a_2a_4C_1^{(2)}(\tilde{\eta},y)-(a_4+a_2^2)A^{-1}w_{0,yy}C_0^{(2)}(\tilde{\eta},y)\\
&\quad+(b_3a_4+b_4a_2)\sqrt{\frac{a_4+a_2^2}{a_4A}}\frac{\partial C_0^{(2)}}{\partial y}\Big]\exp[-\xi]\\
h_1^{(3)}(x,\alpha,\beta)&=\Big[e_2C_1^{(3)}(x,\beta)-(e_1+e_2^2)B^{-1}w_{0,xx}C_0^{(3)}(x,\beta)\\
&\quad+(d_3+d_4e_2)\sqrt{\frac{e_1+e_2^2}{B}}\frac{\partial C_0^{(3)}}{\partial x}\Big]\exp[-\alpha]\\
h_1^{(4)}(x,\tilde{\alpha},\tilde{\beta})&=\Big[e_2C_1^{(4)}(x,\tilde{\beta})-(e_1+e_2^2)B^{-1}w_{0,xx}C_0^{(4)}(x,\tilde{\beta})\\
&\quad+(d_3+d_4e_2)\sqrt{\frac{e_1+e_2^2}{B}}\frac{\partial C_0^{(4)}}{\partial x}\Big]\exp[-\tilde{\alpha}]
\end{aligned}\right\}\qquad(4.14)$$

如此继续下去，便可逐次求得展开式（3.1）和（3.2）中的 w_n，$v_n^{(i)}$ 和 φ_n，$h_n^{(i)}$（$i=1$，…，4；$n=1$，2，…，N）。

五、横向载荷和边缘拉力联合作用下的矩形板

对于不对称的各向异性叠层板，由于存在弯曲-拉伸耦合，方程较之对称的各向异性叠层板更为复杂[1]。这里，结合伽辽金方法（一种加权残数法），以四边固定和四边简支的矩形板为例作具体分析。首先，讨论四边固定的矩形板。

基本方程为

$$\begin{cases}\varepsilon^2L_1w+\varepsilon L_2\varphi=L(w,\varphi)+q(x,y) & (5.1a)\\ \varepsilon L_3w+L_4\varphi=-L(w,w)/2 & (5.1b)\end{cases}$$

边界条件为

$$\left.\begin{aligned}x=0,1:\quad & w=0,\quad w_{,x}=0\\ y=0,b/a:\quad & w=0,\quad w_{,y}=0\end{aligned}\right\}\qquad(5.2)$$

$$\int_0^1 t\varphi_{,xx}dx=\bar{P}_yt,\quad \int_0^{b/a}t\varphi_{,yy}dy=\bar{P}_xt\frac{b}{a}\qquad(5.3)$$

零级解（$\varepsilon=0$）确定于下列方程和边界条件

$$\begin{cases}L(w_0,\varphi_0)+q=0 & (5.4a)\\ L_4\varphi_0+(w_0,w_0)L/2=0 & (5.4b)\end{cases}$$

$$w_0(0,y)=w_0(1,y)=w_0(x,0)=w_0(x,b/a)=0\qquad(5.5)$$

$$\int_0^1 t\varphi_{0,xx}dx=\bar{P}_yt,\qquad \int_0^{b/a}t\varphi_{0,yy}dy=\bar{P}_xt\frac{b}{a}\qquad(5.6)$$

我们注意到其薄膜解w_0,φ_0与对称的各向异性叠层矩形板的相同（请读者参看[11]）。边界层函数分别为

$$\left.\begin{aligned}
v_0^{(1)}&=C_0^{(1)}(\eta,y)\exp[-\xi]=C_0^{(1)}(\eta,y)\exp\left[-\frac{1}{\varepsilon}\sqrt{\frac{a_4}{a_4+a_2^2}}\int_0^x\sqrt{\varphi_{0,yy}(x,y)}dx\right]\\
v_0^{(2)}&=C_0^{(2)}(\tilde{\eta},y)\exp[-\tilde{\xi}]=C_0^{(2)}(\tilde{\eta},y)\exp\left[-\frac{1}{\varepsilon}\sqrt{\frac{a_4}{a_4+a_2^2}}\int_x^1\sqrt{\varphi_{0,yy}(x,y)}dx\right]\\
v_0^{(3)}&=C_0^{(3)}(x,\beta)\exp[-\alpha]=C_0^{(3)}(x,\beta)\exp\left[-\frac{1}{\varepsilon}(e_1+e_2^2)^{-\frac{1}{2}}\int_0^y\sqrt{\varphi_{0,xx}(x,y)}dy\right]\\
v_0^{(4)}&=C_0^{(4)}(x,\tilde{\beta})\exp[-\tilde{\alpha}]=C_0^{(4)}(x,\tilde{\beta})\exp\left[-\frac{1}{\varepsilon}(e_1+e_2^2)^{-\frac{1}{2}}\int_y^{b/a}\sqrt{\varphi_{0,xx}(x,y)}dy\right]
\end{aligned}\right\} \tag{5.7}$$

$$\left.\begin{aligned}
&h_0^{(1)}=\frac{a_2}{a_4}C_0^{(1)}(\eta,y)\exp[-\xi],\quad h_0^{(2)}=\frac{a_2}{a_4}C_0^{(2)}(\tilde{\eta},y)\exp[-\tilde{\xi}]\\
&h_0^{(3)}=e_2C_0^{(3)}(x,\beta)\exp[-\alpha],\quad h_0^{(4)}=e_2C_0^{(4)}(x,\tilde{\beta})\exp[-\tilde{\alpha}]
\end{aligned}\right\} \tag{5.8}$$

式中$C_0^{(i)}(i=1,\ \cdots,\ 4)$为待定系数（确定$C_0^{(i)}$的边界条件参考[11]）。一级渐近解($\varepsilon=1$)确定于下列方程和边界条件

$$L(w_0,\varphi_1)+L(w_1,\varphi_0)=L_2\varphi_0 \tag{5.9a}$$

$$L_3w_0+L_4\varphi_1+L(w_0,w_1)=0 \tag{5.9b}$$

$$\left.\begin{aligned}
w_1(0,y)&=-C_0^{(1)}(0,y)=-\sqrt{\frac{\pi}{\bar{P}_x}}\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(r)\sin\frac{sa\pi}{b}y\\
w_1(1,y)&=-C_0^{(3)}(1,y)=-\sqrt{\frac{\pi}{\bar{P}_x}}\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(r)\,(-1)^r\sin\frac{sa\pi}{b}y\\
w_1(x,0)&=-C_0^{(3)}(x,0)=-\sqrt{\frac{e_1}{\bar{P}_y}}\left(\frac{a\pi}{b}\right)\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(s)\sin r\pi x\\
w_1\left(x,\frac{b}{a}\right)&=-C_0^{(4)}\left(x,\frac{b}{a}\right)=-\sqrt{\frac{e_1}{\bar{P}_y}}\left(\frac{a\pi}{b}\right)\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(s)(-1)^s\sin r\pi x
\end{aligned}\right\} \tag{5.10}$$

$$\int_0^1 t\varphi_{1,xx}dx=0,\quad \int_0^{b/a}t\varphi_{1,yy}dy=0 \tag{5.11}$$

假设

$$\begin{aligned}
w_1(x,y)=&\sum_{r=1}^{R}\sum_{s=1}^{S}w_1^{rs}\sin r\pi x\sin\frac{sa\pi}{b}y\\
&+\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}(\pi)\left(C\cos r\pi x\sin\frac{sa\pi}{b}y+D\sin r\pi x\cos\frac{sa\pi}{b}y\right)
\end{aligned} \tag{5.12}$$

$$\varphi_1(x,y)=\sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_1^{pq}(1-\cos 2p\pi x)\left(1-\cos\frac{2qa\pi}{b}y\right) \tag{5.13}$$

式中

$$C=-\sqrt{\frac{r}{\bar{P}_x}}\qquad D=-\sqrt{\frac{e_1}{\bar{P}_y}}\left(\frac{as}{b}\right) \tag{5.14}$$

则边界条件 (5.10) 和 (5.11) 自然满足。方程 (5.9a) 和 (5.9b) 分别以

$$\sin r'\pi x\sin\frac{s'a\pi}{b}y\qquad (r'=1,2,\cdots,R;\ s'=1,2,\cdots,S)\tag{5.15}$$

和

$$(1-\cos 2p'\pi x)\left(1-\cos\frac{2q'a\pi}{b}y\right)\qquad (p'=1,2,\cdots,P;\ q'=1,2,\cdots,Q)\tag{5.16}$$

作为权函数，并将表达式 (5.12) 和 (5.13) 代入其内，得到运算方程

$$-\sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_0^{pq}\begin{cases}\dfrac{64}{9\pi^2r's'}\left(4a_2p^4+c_2p^2q^2\dfrac{a^2}{b^2}+4e_2q^4\dfrac{a^4}{b^4}\right) & (r'=p=\text{奇数},s'=q=\text{奇数})\\ -\dfrac{64}{3\pi r's'(s'^2-4q^2)}\left(4a_2p^4q^2+c_2p^2q^2s'^2\dfrac{a^2}{b^2}+4e_2q^4s'^2\dfrac{a^4}{b^4}\right) & \begin{pmatrix}r'=p=\text{奇数}\\ s'=\text{奇数},s'\neq q\end{pmatrix}\\ -\dfrac{64}{3\pi^2r's'(r'^2-4p^2)}\left(4a_2r'^2p^4+c_2p^2q^2r'^2\dfrac{a^2}{b^2}+4e_2q^4p^2\dfrac{a^4}{b^4}\right) & \begin{pmatrix}s'=q=\text{奇数}\\ r'=\text{奇数},r'\neq p\end{pmatrix}\\ \dfrac{64p^2q^2}{\pi^2(r'^2-4p^2)(s'^2-4q^2)}\left(4a_2p^2\dfrac{r'}{s'}+c_2r's'\dfrac{a^2}{b^2}+4e_2q^2\dfrac{s'}{r'}\dfrac{a^4}{b^4}\right) & \\ & \begin{pmatrix}r',s'=\text{奇数}\\ r'\neq p,s'\neq q\end{pmatrix}\\ -4pq\dfrac{a}{b}\left(b_2p^2+d_2q^2\dfrac{a^2}{b^2}\right) & (r'=2p,s'=2q)\\ 0 & \text{其它}\end{cases}$$

$$-\sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{r=1}^{R}\sum_{s=1}^{S}\left(\varphi_1^{pq}w_0^{rs}+\varphi_0^{pq}w_1^{rs}\right)\frac{a^2}{b^2}\left\{\frac{p^2s^2}{4}\begin{bmatrix}-1 & r'=r=p\\ -1 & r'=2p-r\\ & (r'\neq r)\\ 1 & r'=r\pm 2p\\ 0 & \text{其它}\end{bmatrix}\right.$$

$$\times\begin{bmatrix}3 & s'=s=q\\ 2 & s'=s\neq q\\ 1 & s'=2q-s\\ & (s'\neq s)\\ -1 & s'=s\pm 2q\\ 0 & \text{其它}\end{bmatrix}+\frac{pqrs}{2}\begin{bmatrix}1 & r'=r=p\\ 1 & r'=2p-r\\ & (r'\neq r)\\ 1 & r'=r+2p\\ -1 & r'=r-2p\\ 0 & \text{其它}\end{bmatrix}\times\begin{bmatrix}1 & s'=s=q\\ 1 & s'=2q-s\\ & (s'\neq s)\\ 1 & s'=s+2q\\ -1 & s'=s-2q\\ 0 & \text{其它}\end{bmatrix}$$

$$\left.+\frac{q^2r^2}{4}\begin{bmatrix}3 & r'=r=p\\ 2 & r'=r\neq p\\ 1 & r'=2p-r\\ & (r'\neq r)\\ -1 & r'=r\pm 2p\\ 0 & \text{其它}\end{bmatrix}\times\begin{bmatrix}-1 & s'=s=q\\ -1 & s'=2q-s\\ & (s'\neq s)\\ 1 & s'=s\pm 2q\\ 0 & \text{其它}\end{bmatrix}\right\}$$

$$+\sum_{p=1}^{P}\sum_{q=1}^{Q}\sum_{r=1}^{R}\sum_{s=1}^{S}\varphi_0^{pq}w_0^{rs}\frac{4a^2}{b^2}\left\{p^2s^2\left[-\frac{D}{2}\left(\frac{s'}{s'^2-s^2}-\frac{N_1}{2}\right)\begin{pmatrix}1 & r'+r=2p\\ -1 & r=r'\mp 2p\\ 0 & \text{其它}\end{pmatrix}\right.\right.$$

$$+CM_1\begin{pmatrix}3/4 & s'+s=2q\\ 1/4 & s'=s\mp 2q\\ 0 & \text{其它}\end{pmatrix}\Bigg]\Bigg]-2pqrs\Bigg[CM_2\begin{pmatrix}1 & s'=s-2q\\ -1 & s'=s\mp 2q\\ 0 & \text{其它}\end{pmatrix}$$

$$+DN_2\begin{pmatrix}1 & r'=r-2p\\ -1 & r'=r\mp 2p\\ 0 & \text{其它}\end{pmatrix}\Bigg]+q^2r^2\Bigg[-\frac{C}{2}\left(\frac{r'}{r'^2-r^2}-\frac{M_1}{2}\right)\begin{pmatrix}1 & s'+s=2q\\ -1 & s'=s\mp 2p\\ 0 & \text{其它}\end{pmatrix}$$

$$+DN_1\begin{pmatrix}3/4 & r'+r=2p\\ 1/4 & r=r'\mp 2p\\ 0 & \text{其它}\end{pmatrix}\Bigg]\Bigg]\Bigg\}=0\qquad(r'=1,2,\cdots,R;s'=1,2,\cdots,S)\qquad(5.17\text{a})$$

式中 $M_1=\frac{r'+r}{(r'+r)^2-4p^2}+\frac{r'-r}{(r'-r)^2-4p^2}$, $N_1=\frac{s'+s}{(s'+s)^2-4q^2}+\frac{s'-s}{(s'-s)^2-4q^2}$

$$M_2=\frac{2prr'}{[(r'+r)^2-4p^2][(r'-r)^2-4p^2]},\quad N_2=\frac{2qss'}{[(s'+s)^2-4q^2][(s'-s)^2-4q^2]}$$

$$\sum_{r=1}^{R}\sum_{s=1}^{S}w_0^{rs}\begin{Bmatrix}64H/9\pi^2 & r=p'=\text{奇数}\quad s=q'=\text{奇数}\\ -64q'^2H/3\pi^2(s^2-4q'^2) & r=p'=\text{奇数};s=\text{奇数},s\neq q'\\ -64p'^2H/3\pi^2(r^2-4p'^2) & r=\text{奇数},r\neq p';s=q'=\text{奇数}\\ 64p'^2q'^2H/\pi^2(r^2-4p'^2)(s^2-4q'^2) & r,s=\text{奇数},r\neq p',s\neq q'\\ -rs(b_3r^2+d_3s^2)/4 & r=2p',s=2q'\\ 0 & \text{其　它}\end{Bmatrix}$$

$$+\sum_{p=1}^{P}\sum_{q=1}^{Q}\varphi_1^{pq}\begin{Bmatrix}12a_4p^4+4c_4p^2q^2a^2/b^2+12q^4a^4/b^4 & p=p',q=q'\\ 8a_4p^4 & p=p',q\neq q'\\ 8q^4a^4/b^4 & p\neq p',q=q'\\ 0 & p\neq p',q\neq q'\end{Bmatrix}$$

$$-\sum_{r=1}^{R}\sum_{s=1}^{S}\sum_{r'=1}^{R}\sum_{s'=1}^{S}w_0^{rs}w_1^{r's'}\cdot\frac{a^2}{b^2}\cdot\frac{rs'}{8}\left\{r's\begin{bmatrix}2 & r'=r\neq p'\\ 1 & r'=r=p'\\ -1 & r\pm r'=2p'\\ & (r\neq r')\\ -1 & r+2p'=r'\\ 0 & \text{其它}\end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix}2 & s'=s\neq q'\\ 1 & s'=s=q'\\ -1 & s\pm s'=2q'\\ & (s\neq s')\\ -1 & s+2q'=s'\\ 0 & \text{其它}\end{bmatrix}-rs'\begin{bmatrix}2 & r'=r\neq p'\\ 3 & r'=r=p'\\ 1 & r+r'=2p'\\ & (r'\neq r)\\ -1 & r-r'=2p'\\ -1 & r+2p'=r'\\ 0 & \text{其它}\end{bmatrix}\times\begin{bmatrix}2 & s'=s\neq q'\\ 3 & s'=s=q'\\ 1 & s+s'=2q'\\ & (s'\neq s)\\ -1 & s-s'=2q'\\ -1 & s+2q'=s'\\ 0 & \text{其它}\end{bmatrix}\right\}$$

$$+\sum_{r=1}^{R}\sum_{s=1}^{S}\sum_{r'=1}^{R}\sum_{s'=1}^{S}w_0^{rs}w_0^{r's'}\cdot\frac{a^2}{b^2}$$

$$\cdot\left\{(rs'+r's)^2\left[\begin{pmatrix} CM_3 & \begin{matrix} s'=s \\ r\pm r'=\text{奇数}\end{matrix} \\ 0 & \text{其它}\end{pmatrix}+\begin{pmatrix} DN_3 & \begin{matrix} r'=r \\ s\pm s'=\text{奇数}\end{matrix} \\ 0 & \text{其它}\end{pmatrix}-\begin{pmatrix} \frac{C}{2}M_3 & s=s'\mp 2q' \\ 0 & \text{其它}\end{pmatrix}\right.\right.$$

$$\left.\left.-\begin{pmatrix} \frac{D}{2}N_3 & r=r'\mp 2p' \\ 0 & \text{其它}\end{pmatrix}+(rs'-r's)^2\left[\begin{pmatrix} \frac{C}{2}M_3 & s+s'=2q' \\ 0 & \text{其它}\end{pmatrix}+\begin{pmatrix} \frac{D}{2}N_3 & r+r'=2p' \\ 0 & \text{其它}\end{pmatrix}\right]\right]\right\}=0$$

$$(p'=1,2,\cdots,P;\ q'=1,2,\cdots,Q) \tag{5.17b}$$

式中
$$H=a_3\frac{r^3}{s}+c_3 rs+e_3\frac{s^3}{r}$$

$$M_3=\frac{r}{r^2-r'^2}-\frac{1}{2}\left[\frac{r+r'}{(r+r')^2-4p'^2}+\frac{r-r'}{(r-r')^2-4p^2}\right]$$

$$N_3=\frac{s}{s^2-s'^2}-\frac{1}{2}\left[\frac{s+s'}{(s+s')^2-4q'^2}+\frac{s-s'}{(s-s')^2-4q'^2}\right]$$

联解方程（5.17a）和（5.17b），可解得w_1，φ_1。确定系数$C_0^{(i)}$（$i=1,\cdots,4$）的偏微分方程为（4.11），结合边界条件，即可求得$C_0^{(i)}$。例如，w_0取一项计算时，$C_0^{(i)}$为

$$C_0^{(1)}(\eta,y)=\sqrt{\frac{\pi}{\bar{P}_x}}\,w_0^{11}\sin\frac{a\pi y}{b}\exp\left[-\left(F_1\left(\frac{a\pi}{b}\right)\operatorname{ctg}\frac{a\pi}{b}y+\frac{K_1}{2A}\right)\eta\right]$$

$$C_0^{(2)}(\tilde{\eta},y)=-\sqrt{\frac{\pi}{\bar{P}_x}}\,w_0^{11}\sin\frac{a\pi y}{b}\exp\left[-\left(F_1\left(\frac{a\pi}{b}\right)\operatorname{ctg}\frac{a\pi y}{b}+\frac{K_1}{2A}\right)(1-\tilde{\eta})\right]$$

$$C_0^{(3)}(x,\beta)=\sqrt{\frac{e_1}{\bar{P}_y}}\left(\frac{a\pi}{b}\right)w_0^{11}\sin\pi x\exp\left[-\left(F_2(\pi)\operatorname{ctg}\pi x+\frac{K_2}{2B}\right)\beta\right]$$

$$C_0^{(4)}(x,\tilde{\beta})=-\sqrt{\frac{e_1}{\bar{P}_y}}\left(\frac{a\pi}{b}\right)w_0^{11}\sin\pi x\exp\left[-\left(F_2(\pi)\operatorname{ctg}\pi x+\frac{K_2}{2B}\right)\left(\frac{b}{a}-\tilde{\beta}\right)\right]$$

$$\tag{5.18}$$

式中
$$F_1=\varphi_{0,xy}A^{-1}+\frac{a_4}{2(a_4+a_2^2)}\left(b_1+\frac{2a_2b_2}{a_4}-\frac{a_2^2b_4}{a_4^2}\right)$$

$$K_1=\left(\varphi_{1,yy}+\frac{2a_2}{a_4}w_{0,yy}\right)\sqrt{\frac{a_4A}{a_4+a_2^2}}+\frac{5}{2}A_{,x}$$

$$F_2=\varphi_{0,xy}B^{-1}+\frac{1}{2(e_1+e_2^2)}(d_1+2d_2e_2-d_4e_2^2)$$

$$K_2=(\varphi_{1,xx}+2e_2w_{0,xx})\sqrt{\frac{B}{e_1+e_2^2}}+\frac{5}{2}B_{,y}$$

对于四边简支情形，其薄膜解与四边固支的相同。零级边界层函数$v_0^{(i)}$和$h_0^{(i)}$（$i=1,\cdots,4$）仍分别为（5.7）和（5.8）。只是这里确定系数$C_0^{(i)}$（$i=1,\cdots,4$）的边界条件为

$$\left.\begin{aligned}C_0^{(1)}(\eta,y)|_{\eta=0}=C_0^{(2)}(\tilde{\eta},y)|_{\tilde{\eta}=1}=-\frac{(a_4+a_2^2)Kb_{11}^*}{a_4a_{11}^*d_{11}^*}\\ C_0^{(3)}(x,\beta)|_{\beta=0}=C_0^{(4)}(x,\tilde{\beta})|_{\tilde{\beta}=\frac{b}{a}}=-\frac{(e_1+e_2^2)Kb_{22}^*}{a_{11}^*d_{22}^*}\end{aligned}\right\} \tag{5.19}$$

对于反对称角铺设叠层板，$b_{11}^*=b_{22}^*=0$；由方程（4.9），有

$$w_1(0,y)=w_1(1,y)=w_1(x,0)=w_1\left(x,\frac{b}{a}\right)=0 \tag{5.20}$$

于是，$w_1\equiv0$，$\varphi_1\equiv0$；且由于 $C_0^{(i)}$ 在相应的边界上取值为零，所以，$v_0^{(i)}\equiv0$，$h_0^{(i)}\equiv0$ $(i=1,\cdots,4)$。

至于其它情形，皆可类似制作，不再赘述。

下面，我们以承受均布横向载荷和边缘拉力联合作用的四边简支矩形板，在边界位移为零的情况下为例，取$R=S=P=Q=2$；$\varepsilon=0,1$作具体数值计算。

首先，将高模量石墨/环氧反对称正交铺设的二层（0°/90°）和六层（0°/90°/0°/90°/0°/90°）叠层板的无量纲载荷与中心挠度的关系曲线图分别作于图1中。由图可见，当层数N增加时，耦合刚度的影响减小了。与对称正交铺设相比[10]，由于耦合刚度的影响，挠度增加了。

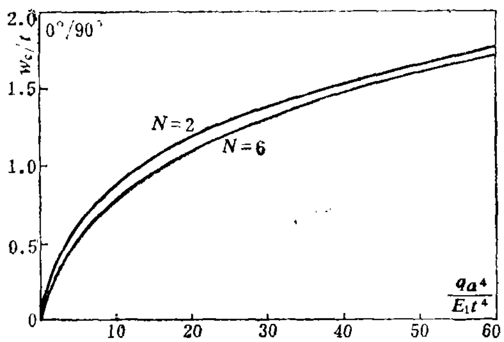

图 1

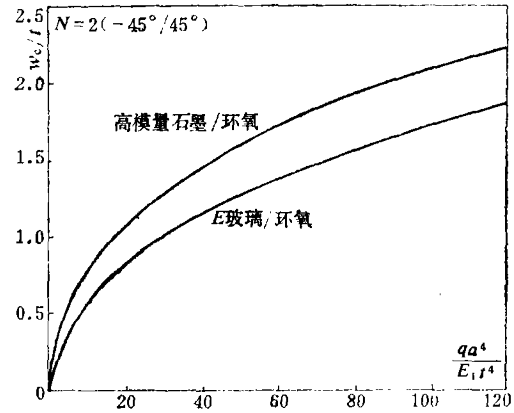

图 2

再将高模量石墨/环氧和玻璃/环氧的反对称角铺设的二层（−45°/45°）叠层板的无量纲载荷与中心挠度的关系曲线图分别作于图2中。E玻璃/环氧单层板性能是

$$E_1=53.78\times10^6\ \text{kPa} \qquad G_{12}=7.32\times10^6\ \text{kPa}$$

$$E_2=E_3=17.93\times10^6\ \text{kPa} \qquad \gamma_{12}=0.25$$

由图2可见，E_1/E_2越高，耦合刚度B_{ij}的影响就越大。

对于高模量石墨/环氧以−45°/45°叠合顺序的二层层合板，$\varepsilon=0.3260t/a$；以正交铺设的叠层板，当$N=6$时，$\varepsilon=0.2808t/a$；当$N=2$时，$\varepsilon=0.2057t/a$。因为参数ε很小，所以，W_N和Φ_N很快收敛。事实上，零级解（薄膜解）和一级解已足够描述板的非线性特性。因此，可以说，本文的研究对如此复杂的问题提供了一个简单而又有效的方法。

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Nonlinear Bendings of Unsymmetrically Layered Anisotropic Rectangular Plates

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Abstract

An analysis for the nonlinear bendings of unsymmetrically layered anisotropic rectangular plates subjected to combined edge tensions and lateral loading under various supports is presented. The uniformly valid N-order asymptotic solutions of the transverse deflection and stress function are derived by the singular perturbation method offered in [1]. The present investigation may provide a simple and convenient method for such a complex problem.