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环形和圆形薄板屈曲后性态的非线性分析*

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摘 要

本文应用改进的多重尺度法，研究环形和圆形薄板屈曲后的性态.求出其渐近解和极限荷载，以及指出薄板屈曲后其皱纹和弯曲刚度间的关系.

一、引 言

至今，已有很多著作研究薄板的屈曲问题[1~2]，但只有少量著作涉及薄板屈曲后的性态[3]．虽然后一问题在工程上有着重要的意义，但由于数学上的困难，却发展很缓慢．有些数学家曾试用能量法来求其渐近解，由于其精确度依赖于坐标函数的选取，不能正确地描写薄板屈曲后的性态；因根据泰圣立的工作[4]知道，后一问题的解含有不同尺度的变量．下面，我们应用在研究薄板弯曲问题中很有成效的多重尺度法[5~7]，来研究薄板屈曲后的性态．

二、渐 近 解

今先考察环形薄板的屈曲问题．引进极坐标系(r,θ)，我们知道其挠度$w(r,\theta)$和应力函数$F(r,\theta)$确定于下面的von Kármán方程：

$$\left.\begin{aligned}\Delta^2 w&=\frac{h}{D}L(w,F)+\frac{q}{D}\\ \Delta^2 F&=\frac{-E}{2}L(w,w)\end{aligned}\right\}\tag{2.1}$$

其中E是弹性模数，h是板的厚度，$D=Eh^3/12(1-\nu^2)$是板的弯曲刚度，ν是泊松比，和

$$\Delta\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$L(w,F)\equiv\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)+\frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$$

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$$-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right).$$

以r_0表示内半径，r_1表示外半径．作为一个例子，考察内、外边缘是简支的情形，这时有下面的边界条件：

$$w\Big|_{r=r_0,r_1}=0,\ \left[\frac{\partial^2 w}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\Big|_{r=r_0,r_1}=0 \tag{2.2}$$

$$\left.\begin{aligned}&\left(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)\Big|_{r=r_0,r_1}=\frac{-T_0(\theta)}{h}<0\\&\left(\frac{1}{r^2}\frac{\partial F}{\partial\theta}-\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta}\right)\Big|_{r=r_0,r_1}=0\end{aligned}\right\} \tag{2.3}$$

其中$T_0(\theta)$是边缘上每单位长度所作用的轴向力．类似地可以处理其它类别的边界条件．

引进无量纲变量：

$$\tilde{w}=\frac{w}{r_1},\quad \tilde{r}=\frac{r}{r_1},\quad \tilde{F}=\frac{F}{Er_1^2},$$

$$\tilde{q}=\frac{r_1 q}{Eh},\quad \tilde{T}_0=\frac{T_0}{Eh}$$

边值问题(2.1)～(2.3)化为

$$\left.\begin{aligned}&\Pi_\varepsilon(\tilde{w},\tilde{F})\equiv\varepsilon^2\Delta^2\tilde{w}-L(\tilde{w},\tilde{F})-\tilde{q}=0\\&\Pi(\tilde{w},\tilde{F})\equiv\Delta^2\tilde{F}+\frac{1}{2}L(\tilde{w},\tilde{w})=0\end{aligned}\right\} \tag{2.4}$$

$$\tilde{w}\Big|_{\tilde{r}=b,1}=0,\ \left[\frac{\partial^2\tilde{w}}{\partial\tilde{r}^2}+\nu\left(\frac{1}{\tilde{r}}\frac{\partial\tilde{w}}{\partial\tilde{r}}+\frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{w}}{\partial\theta^2}\right)\right]\Big|_{\tilde{r}=b,1}=0 \tag{2.5}$$

$$\left.\begin{aligned}&\left(\frac{1}{r}\frac{\partial\tilde{F}}{\partial\tilde{r}}+\frac{1}{\tilde{r}^2}\frac{\partial^2\tilde{F}}{\partial\theta^2}\right)\Big|_{\tilde{r}=b,1}=-\tilde{T}_0(\theta)<0\\&\left(\frac{1}{\tilde{r}^2}\frac{\partial\tilde{F}}{\partial\theta}-\frac{1}{\tilde{r}}\frac{\partial^2\tilde{F}}{\partial\tilde{r}\partial\theta}\right)\Big|_{\tilde{r}=b,1}=0\end{aligned}\right\} \tag{2.6}$$

其中 $b=\frac{r_0}{r_1}$，$\varepsilon^2=\frac{h^2}{12(1-\nu^2)r_1^2}=\frac{D}{Ehr_1^2}\ll 1$．今后我们略去字母上的"～"号．

假设边值问题(2.4)～(2.6)的解具有展开式：

$$\begin{aligned}w(r,\theta)=&w_0(r,\theta)+\varepsilon w_1(r,\theta)+\cdots\\&+\varepsilon^{\alpha_b}\left[v_0^{(b)}\left(\frac{u_b(r,\theta)}{\varepsilon},r,\theta\right)+\cdots\right]\\&+\varepsilon^{\alpha_1}\left[v_0^{(1)}\left(\frac{u_1(r,\theta)}{\varepsilon},r,\theta\right)+\cdots\right]\end{aligned} \tag{2.7}$$

$$\begin{aligned}F(r,\theta)=&f_0(r,\theta)+\varepsilon f_1(r,\theta)+\cdots\\&+\varepsilon^{\beta_b}\left[h_0^{(b)}\left(\frac{u_b(r,\theta)}{\varepsilon},r,\theta\right)+\cdots\right]\\&+\varepsilon^{\beta_1}\left[h_0^{(1)}\left(\frac{u_1(r,\theta)}{\varepsilon},r,\theta\right)+\cdots\right]\end{aligned} \tag{2.8}$$

其中 $\alpha_b,\cdots,\beta_1$ 是待定常数；u_b，u_1 是满足下面条件：

$$u_b(r,\theta)|_{r=b}=0,\qquad u_1(r,\theta)|_{r=1}=0 \tag{2.9}$$

$$u_b(r,\theta)>0,\ u_1(r,\theta)>0,\ \text{当}\ b<r<1 \tag{2.10}$$

的待定函数。$v_0^{(b)}, v_0^{(1)}, \cdots$用以校正$w_0$，$w_1, \cdots$使满足(2.5)~(2.6)中的全部边界条件。

为简单起见，这里我们只构造准确到$O(\varepsilon)$的项，和主要校正项。在研究薄板弯曲问题中[5-7]我们曾看到，以上的渐近式已具有足够的精确度。类似地可以构造准确到$O(\varepsilon^N)$的渐近式（N是任意正整数）。

引入多重尺变量：

$$\xi_p=\frac{u_p(r,\theta)}{\varepsilon},\quad \eta=r,\quad \zeta=\theta$$

从文献[6]我们知道对于一个含有多重尺度变量的函数$G(\xi_p,\ \eta,\ \zeta)$有

$$\left.\begin{aligned}
&\frac{\partial G}{\partial r}\approx\varepsilon^{-1}u_{p,r}\frac{\partial}{\partial\xi_p}, && \frac{\partial G}{\partial\theta}\approx\varepsilon^{-1}u_{p,\theta}\frac{\partial}{\partial\xi_p}\\
&\frac{\partial^2 G}{\partial r^2}\approx\varepsilon^{-2}u_{p,r}^2\frac{\partial^2}{\partial\xi_p^2}, && \frac{\partial^2 G}{\partial r\partial\theta}\approx\varepsilon^{-2}u_{p,r}u_{p,\theta}\frac{\partial^2}{\partial\xi_p^2}\\
&\frac{\partial^4 G}{\partial r^4}\approx\varepsilon^{-4}u_{p,r}^4\frac{\partial^4}{\partial\xi_p^4}, && \frac{\partial^4 G}{\partial r^2\partial\theta^2}\approx\varepsilon^{-4}u_{p,r}^2\ u_{p,\theta}^2\frac{\partial^4}{\partial\xi_p^4}\\
&\Delta^2 G\approx\varepsilon^{-4}\left(u_{p,r}^4+\frac{2}{\eta}u_{p,r}^2\ u_{p,\theta}^2+\frac{1}{\eta^4}u_{p,\theta}^4\right)\frac{\partial^4 G}{\partial\xi_p^4}
\end{aligned}\right\}\tag{2.11}$$

$$\begin{aligned}L(w(r,\theta),h(\xi_p,\eta,\zeta))\approx\varepsilon^{-2}\Big[&\frac{w_{rr}(\eta,\zeta)}{\eta^2}u_{p,\theta}^2+\left(\frac{w_r}{\eta}+\frac{w_{\theta\theta}}{\eta^2}\right)u_{p,r}^2\\
&-\frac{2}{\eta}\left(\frac{-w_\theta}{\eta^2}+\frac{w_{r\theta}}{\eta}\right)u_{p,r}\,u_{p,\theta}\Big]\frac{\partial^2 h}{\partial\xi_p^2}\end{aligned}\tag{2.12}$$

$$L(v(\xi_p,\ \eta,\ \zeta),\ h(\xi_q,\ \eta,\ \zeta))\approx 0\tag{2.13}$$

将(2.7)，(2.8)式代入边值问题(2.4)~(2.6)，再考虑到(2.9)~(2.11)式，我们有

$$\begin{aligned}\Pi_\varepsilon(w,F)\equiv&\varepsilon^2(\Delta^2 w_0(r,\theta)+\varepsilon\Delta^2 w_1(r,\theta)+\cdots)-\{L(w_0,f_0)\\
&+\varepsilon[L(w_0,f_1)+L(w_1,f_0)]+\cdots\}\\
&+\varepsilon^{\alpha_b-2}\left[\left(u_{b,r}^4+\frac{2}{\eta}u_{b,r}^2\ u_{b,\theta}^2+\frac{1}{\eta^4}u_{b,\theta}^4\right)\frac{\partial^4 v_0^{(b)}}{\partial\xi_b^4}+\cdots\right]\\
&+\varepsilon^{\alpha_1-2}\left[\left(u_{1,r}^4+\frac{2}{\eta}u_{1,r}^2u_{1,\theta}^2+\frac{1}{\eta^4}u_{1,\theta}^4\right)\frac{\partial^4 v_0^{(1)}}{\partial\xi_1^4}+\cdots\right]\\
&-\varepsilon^{\alpha_b-2}\left(M_b(f_0)\frac{\partial^2 v_0^{(b)}}{\partial\xi_b^2}+\cdots\right)-\varepsilon^{\alpha_1-2}\left(M_1(f_0)\frac{\partial^2 v_0^{(1)}}{\partial\xi_1^2}+\cdots\right)\\
&-\varepsilon^{\beta_b-2}\left(M_b(w_0)\frac{\partial^2 h_0^{(b)}}{\partial\xi_b^2}+\cdots\right)\\
&-\varepsilon^{\beta_1-2}\left(M_1(w_0)\frac{\partial^2 h_0^{(1)}}{\partial\xi_1^2}+\cdots\right)-q=0\end{aligned}\tag{2.14}$$

$$\begin{aligned}\Pi(w,F)\equiv&(\Delta^2 f_0+\varepsilon\Delta^2 f_1+\cdots)+\frac{1}{2}(L(w_0,\ w_0)+2\varepsilon L(w_0,\ w_1)+\cdots)\\
&+\varepsilon^{\beta_b-4}\left[\left(u_{b,r}^4+\frac{2}{\eta}u_{b,r}^2u_{b,\theta}^2+\frac{1}{\eta^4}u_{b,\theta}^4\right)\frac{\partial^4 h_0^{(b)}}{\partial\xi_b^4}+\cdots\right]\\
&+\varepsilon^{\beta_1-4}\left[\left(u_{1,r}^4+\frac{2}{\eta}u_{1,r}^2u_{1,\theta}^2+\frac{1}{\eta^4}u_{1,\theta}^4\right)\frac{\partial^4 h_0^{(1)}}{\partial\xi_1^4}+\cdots\right]\end{aligned}$$

$$+\varepsilon^{a_b-2}\left(M_b(w_0)\frac{\partial^2 v_0^{(b)}}{\partial\xi_b^2}+\cdots\right)$$

$$+\varepsilon^{a_1-2}\left(M_1(w_0)\frac{\partial^2 v_0^{(1)}}{\partial\xi_1^2}+\cdots\right)=0 \tag{2.15}$$

其中 $M_p(f)\equiv\frac{f_{rr}(\eta,\zeta)}{\eta^2}u_{p,\theta}^2+\left(\frac{f_r}{\eta}+\frac{f_{\theta\theta}}{\eta^2}\right)u_{p,r}^2-\frac{2}{\eta}\left(\frac{-f_\theta}{\eta^2}+\frac{f_{r\theta}}{\eta}\right)u_{p,r}u_{p,\theta}$

$p=b,1$，$f=f_0,w_0$；和得到下面的边界条件：

$$(w_0+\varepsilon w_1+\cdots)\Big|_{r=b}+\varepsilon^{a_b}(v_0^{(b)}(0,b,\zeta)+\cdots)$$

$$+\varepsilon^{a_1}\left(v_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)+\cdots\right)=0 \tag{2.16}$$

$$(w_0+\varepsilon w_1+\cdots)\Big|_{r=1}+\varepsilon^{a_b}\left(v_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)+\cdots\right)$$

$$+\varepsilon^{a_1}(v_0^{(1)}(0,1,\zeta)+\cdots)=0 \tag{2.17}$$

$$\left\{\left[\frac{\partial^2 w_0}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]+\cdots\right\}\Big|_{r=b}$$

$$+\varepsilon^{a_b-2}\left(u_{b,r}^2\frac{\partial^2 v_0^{(b)}(0,b,\zeta)}{\partial\xi_b^2}+\cdots\right)$$

$$+\varepsilon^{a_1-2}\left(u_{1,r}^2\frac{\partial^2 v_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\xi_1^2}+\cdots\right)$$

$$+\frac{\nu}{b}\left[\varepsilon^{a_b-1}\left(u_{b,r}\frac{\partial v_0^{(b)}(0,b,\zeta)}{\partial\xi_b}+\cdots\right)\right.$$

$$\left.+\varepsilon^{a_1-1}\left(u_{1,r}\frac{\partial v_0^{(1)}\left(\frac{u_1(b,\zeta)}{\partial\xi_1},b,\zeta\right)}{\partial\xi_1}+\cdots\right)\right]$$

$$+\frac{\nu}{b^2}\left[\varepsilon^{a_b}-1\left(u_{b,\theta}^2\frac{\partial^2 v_0^{(b)}(0,b,\zeta)}{\partial\theta^2}+\cdots\right)\right.$$

$$\left.+\varepsilon^{a_1-1}\left(u_{1,\theta}^2\frac{\partial^2 v_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\theta^2}+\cdots\right)\right]=0 \tag{2.18}$$

$$\left\{\left[\frac{\partial^2 w_0}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]+\cdots\right\}\Big|_{r=1}$$

$$+\varepsilon^{a_b-2}\left(u_{b,r}^2\frac{\partial^2 v_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b^2}\right.$$

$$\left.+\cdots\right)+\varepsilon^{a_1-2}\left(u_{1,r}^2\frac{\partial^2 v_0^{(1)}(0,1,\zeta)}{\partial\xi_1^2}+\cdots\right)$$

$$+\nu\left[\varepsilon^{a_b-1}\left(u_{b,r}\frac{\partial v_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b}+\cdots\right)\right.$$

$$+\varepsilon^{\alpha_1-1}\left(u_{1,r}\frac{\partial v_0^{(1)}(0,1,\zeta)}{\partial\xi_1}+\cdots\right)\Bigg]$$

$$+\nu\Bigg[\varepsilon^{\alpha_b-2}\left(u_{b,\theta}^2\frac{\partial^2 v_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b^2}+\cdots\right)$$

$$+\varepsilon^{\alpha_1-2}\left(u_{1,\theta}^2\frac{\partial^2 v_0^{(1)}(0,1,\zeta)}{\partial\xi_1^2}+\cdots\right)\Bigg]=0 \tag{2.19}$$

$$\left[\frac{1}{r}\left(\frac{\partial f_0}{\partial r}+\varepsilon\frac{\partial f_1}{\partial r}+\cdots\right)+\frac{1}{r^2}\left(\frac{\partial^2 f_0}{\partial\theta^2}+\varepsilon\frac{\partial^2 f_1}{\partial\theta^2}+\cdots\right)\right]\Bigg|_{r=b}$$

$$+\frac{1}{b}\Bigg[\varepsilon^{\beta_b-1}\left(u_{b,r}\frac{\partial h_0^{(b)}(0,b,\zeta)}{\partial\xi_b}+\cdots\right)$$

$$+\varepsilon^{\beta_1-1}\left(u_{1,r}\frac{\partial h_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\xi_1}+\cdots\right)\Bigg]$$

$$+\frac{1}{b^2}\Bigg[\varepsilon^{\beta_b-2}\left(u_{b,\theta}^2\frac{\partial^2 h_0^{(b)}(0,b,\zeta)}{\partial\xi_b^2}+\cdots\right)$$

$$+\varepsilon^{\beta_1-2}\left(u_{1,\theta}^2\frac{\partial^2 h_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\xi_1^2}+\cdots\right)\Bigg]=-T_0(\theta) \tag{2.20}$$

$$\left[\frac{1}{r}\left(\frac{\partial f_0}{\partial r}+\varepsilon\frac{\partial f_1}{\partial r}+\cdots\right)+\frac{1}{r^2}\left(\frac{\partial^2 f_0}{\partial\theta^2}+\varepsilon\frac{\partial^2 f_1}{\partial\theta^2}+\cdots\right)\right]\Bigg|_{r=1}$$

$$+\Bigg[\varepsilon^{\beta_b-1}\left(u_{b,r}\frac{\partial h_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b}+\cdots\right)$$

$$+\varepsilon^{\beta_1-1}\left(u_{1,r}\frac{\partial h_0^{(1)}(0,1,\zeta)}{\partial\xi_1}+\cdots\right)\Bigg]$$

$$+\Bigg[\varepsilon^{\beta_b-2}\left(u_{b,\theta}^2\frac{\partial^2 h_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b^2}+\cdots\right)$$

$$+\varepsilon^{\beta_1-2}\left(u_{1,\theta}^2\frac{\partial^2 h_0^{(1)}(0,1,\zeta)}{\partial\xi_1^2}+\cdots\right)\Bigg]=-T_0(\theta) \tag{2.21}$$

$$\left[\frac{1}{r^2}\left(\frac{\partial f_0}{\partial\theta}+\varepsilon\frac{\partial f_1}{\partial\theta}+\cdots\right)-\frac{1}{r}\left(\frac{\partial^2 f_0}{\partial r\,\partial\theta}+\varepsilon\frac{\partial^2 f_1}{\partial r\,\partial\theta}+\cdots\right)\right]\Bigg|_{r=b}$$

$$+\frac{1}{b^2}\Bigg[\varepsilon^{\beta_b-1}\left(u_{b,\theta}\frac{\partial h_0^{(b)}(0,b,\zeta)}{\partial\xi_b}+\cdots\right)$$

$$+\varepsilon^{\beta_1-1}\left(u_{1,\theta}\frac{\partial h_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\xi_1}+\cdots\right)\Bigg]$$

$$-\frac{1}{b}\Bigg[\varepsilon^{\beta_b-2}\left(u_{b,r}u_{b,\theta}\frac{\partial^2 h_0^{(b)}(0,b,\zeta)}{\partial\xi_b^2}+\cdots\right)$$

$$+\varepsilon^{\beta_1-2}\left(u_{1,r}u_{1,\theta}\frac{\partial^2 h_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\xi_1^2}+\cdots\right)\Bigg]=0 \tag{2.22}$$

$$\left[\frac{1}{r^2}\left(\frac{\partial f_0}{\partial\theta}+\varepsilon\frac{\partial f_1}{\partial\theta}+\cdots\right)-\frac{1}{r}\left(\frac{\partial^2 f_0}{\partial r\,\partial\theta}+\varepsilon\frac{\partial^2 f_1}{\partial r\,\partial\theta}+\cdots\right)\right]\Bigg|_{r=1}$$

$$+\left[\varepsilon^{\beta_b-1}\left(u_{b,\theta}\frac{\partial h_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b}+\cdots\right)+\varepsilon^{\beta_1-1}\left(u_{1,\theta}\frac{\partial h_0^{(1)}(0,1,\zeta)}{\partial\xi_1}\right.\right.$$

$$\left.\left.+\cdots\right)\right]-\left[\varepsilon^{\beta_b-2}\left(u_{b,r}\,u_{b,\theta}\frac{\partial^2 h_0^{(b)}\left(\frac{u_b(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b^2}+\cdots\right)\right.$$

$$\left.+\varepsilon^{\beta_1-2}\left(u_{b,r}\,u_{b,\theta}\frac{\partial^2 h_0^{(1)}(0,1,\zeta)}{\partial\xi_1^2}+\cdots\right)\right]=0 \tag{2.23}$$

从以上各式可以看出，为了得到确定w_0，f_0，w_1，f_1，$v_0^{(b)}$，$v_0^{(1)}$…等的递推公式，我们应取$\alpha_b=\alpha_1=2$，$\beta_b=\beta_1=4$。再比较ε的同次幂的系数，我们得到确定w_0，f_0的边值问题：

$$\left.\begin{aligned}
&L(w_0,f_0)-q=0\\
&\Delta^2 f_0+\frac{1}{2}L(w_0,w_0)=0\\
&w_0(r,\theta)\big|_{r=b,1}=0\\
&\left(\frac{1}{r}\frac{\partial f_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f_0}{\partial\theta^2}\right)\Bigg|_{r=b,1}=-T_0(\theta)<0\\
&\left(\frac{1}{r^2}\frac{\partial f_0}{\partial\theta}-\frac{1}{r}\frac{\partial^2 f_0}{\partial r\,\partial\theta}\right)\Bigg|_{r=b,1}=0
\end{aligned}\right\} \tag{2.24}$$

和关于$v_0^{(p)}(p=b,1)$的边值问题：

$$\left.\begin{aligned}
&\left(u_{p,r}^4+\frac{2}{\eta}u_{p,r}^2u_{p,\theta}^2+\frac{1}{\eta^4}u_{p,\theta}^4\right)\frac{\partial^4 v_0^{(p)}}{\partial\xi_p^4}\\
&\quad-\left[\frac{f_{0,rr}}{\eta^2}u_{p,\theta}^2+\left(\frac{f_{0,r}}{\eta}+\frac{f_{0,\theta\theta}}{\eta^2}\right)u_{p,r}^2\right.\\
&\quad\left.-\frac{2}{\eta}\left(\frac{-f_{0,\theta}}{\eta^2}+\frac{f_{0,r\theta}}{\eta}\right)u_{p,r}u_{p,\theta}\right]\frac{\partial^4 v_0^{(p)}}{\partial\xi_p^4}=0 \qquad (p=b,1)\\
&u_{b,r}^2\frac{\partial^2 v_0^{(b)}(0,b,\zeta)}{\partial\xi_b^2}+u_{1,r}^2\frac{\partial^2 v_0^{(1)}\left(\frac{u_1(b,\zeta)}{\varepsilon},b,\zeta\right)}{\partial\xi_1^2}\\
&\quad=-\left[\frac{\partial^2 w_0}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]\Bigg|_{r=b}\\
&u_{b,r}^2\frac{\partial^2 v_0^{(b)}\left(\frac{u_1(1,\zeta)}{\varepsilon},1,\zeta\right)}{\partial\xi_b^2}+u_{1,r}^2\frac{\partial^2 v_0^{(1)}(0,1,\zeta)}{\partial\xi_1^2}\\
&\quad=-\left[\frac{\partial^2 w_0}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w_0}{\partial r}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]\Bigg|_{r=1}
\end{aligned}\right\} \tag{2.25}$$

和关于$h_0^{(p)}(p=b,1)$的控制方程：

$$\left(u_{p,r}^4+\frac{2}{\eta}u_{p,r}^2u_{p,\theta}^2+\frac{1}{\eta^4}u_{p,\theta}^4\right)\frac{\partial^4 h_0^{(p)}}{\partial\xi_p^4}$$

$$-\left[\frac{w_{0,rr}}{\eta^2}u_{p,\theta}^2+\left(\frac{w_{0,r}}{\eta}+\frac{w_{0,\theta\theta}}{\eta^2}\right)u_{p,r}^2-\frac{2}{\eta}\left(\frac{-w_{0,\theta}}{\eta^2}\right.\right.$$

$$\left.\left.+\frac{w_{0,r\theta}}{\eta}\right)u_{p,r}u_{p,\theta}\right]\frac{\partial^2 v_0^{(p)}}{\partial\xi_p^2}=0 \qquad (p=b,1) \tag{2.26}$$

和关于w_1，f_1的边值问题：

$$\left.\begin{aligned}&L(w_0,f_1)+L(w_1,f_0)=0\\&\Delta^2 f_0+L(w_0,w_1)=0\\&w_1(r,\theta)|_{r=b,1}=0\\&\left(\frac{1}{r}\frac{\partial f_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f_1}{\partial\theta^2}\right)\Big|_{r=b,1}=0\\&\left(\frac{1}{r^2}\frac{\partial f_1}{\partial\theta}-\frac{1}{r}\frac{\partial^2 f_1}{\partial r\partial\theta}\right)\Big|_{r=b,1}=0\end{aligned}\right\} \tag{2.27}$$

和关于$v_1^{(p)}(p=b,1)$的控制方程：

$$\left(u_{p,r}^4+\frac{2}{\eta}u_{p,r}^2u_{p,\theta}^2+\frac{1}{\eta^4}u_{p,\theta}^4\right)\frac{\partial^4 v_1^{(p)}}{\partial\xi_p^4}$$

$$-\left[\frac{f_{0,rr}}{\eta^2}u_{p,\theta}^2+\left(\frac{f_{0,r}}{\eta}+\frac{f_{0,\theta\theta}}{\eta^2}\right)u_{p,r}^2\right.$$

$$\left.-\frac{2}{\eta}\left(\frac{-f_{0,\theta}}{\eta^2}+\frac{f_{0,r\theta}}{\eta}\right)u_{p,r}u_{p,\theta}\right]\frac{\partial^4 v_1^{(p)}}{\partial\xi_p^4}=-D^{(1,p)}v_0^{(p)}$$

$$+M_0(v_0^{(p)},\ f_1)+M_1(v_0^{(p)},\ f_0) \qquad (p=b,1) \tag{2.28}$$

其中 $D^{(1,p)}$，M_0，M_1是形如[5]中所定义的微分算子。

从关于$v_0^{(p)}$，$v_1^{(p)}$的控制方程可以看出，若取$u_p(r,\theta)$是下面方程的解

$$\left(u_{p,r}^4+\frac{2}{r}u_{p,r}^2u_{p,\theta}^2+\frac{1}{r^4}u_{p,\theta}^4\right)=-\left[\frac{f_{0,rr}}{r^2}u_{p,\theta}^2+\left(\frac{f_{0,r}}{r}+\frac{f_{0,\theta\theta}}{r^2}\right)u_{p,r}^2\right.$$

$$\left.-\frac{2}{r}\left(\frac{-f_{0,\theta}}{r^2}+\frac{f_{0,r\theta}}{r}\right)u_{p,r}u_{p,\theta}\right] \qquad (p=b,1) \tag{2.29}$$

则取最简单的形式：

$$\frac{\partial^4 v_i^{(p)}}{\partial\xi_p^4}+\frac{\partial^2 v_i^{(p)}}{\partial\xi_p^2}=0 \qquad (i=0,1;p=b,1)$$

很容易求解。

在一般情况下成立$\frac{\partial w}{\partial r}\gg\frac{\partial w}{\partial\theta}$，$\frac{\partial F}{\partial r}\gg\frac{\partial F}{\partial\theta}$，此时可以认为$u_{p,r}\gg u_{p,\theta}$，关于$u_p$的控制方程(2.29)可以近似地代之以

$$u_{p,r}^4=-\left(\frac{f_{0,r}}{r}-\frac{f_{0,\theta\theta}}{r^2}\right)u_{p,r}^2 \qquad (p=b,1) \tag{2.30}$$

它们的满足条件(2.9)，(2.10)的解为

$$u_b(r,\theta)=\int_b^r\sqrt{-\left(\frac{f_{0,r}}{r}-\frac{f_{0,\theta\theta}}{r^2}\right)}\,dr \tag{2.31}$$

$$u_1(r,\theta)=\int_r^1\sqrt{-\left(\frac{f_{0,r}}{r}-\frac{f_{0,\theta\theta}}{r^2}\right)}\,dr \tag{2.32}$$

从上面的讨论中可以看到，从(2.4)式求出w_0、f_0后，将它们代入(2.25)式可以确定出$v_0^{(p)}(p=b,1)$。将w_0，f_0，$v_0^{(p)}(p=b,1)$代入(2.26)，又可确定出$h_0^{(p)}(p=b,1)$；从(2.27)接着又可定出w_1，f_1等等。

下面考察几个例子。

三、均匀压力作用下的环板的屈曲后性态

此时控制w，F的边值问题化为

$$\left.\begin{aligned}&\varepsilon^2\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}-\frac{1}{r}\frac{d}{dr}\left(\frac{dw}{dr}\frac{dF}{dr}\right)-q=0\\&\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)\right]\right\}+\frac{1}{2r}\frac{d}{dr}\left(\frac{dw}{dr}\right)^2=0\end{aligned}\right\} \tag{3.1}$$

$$w|_{r=b,1}=0,\qquad \left.\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)\right|_{r=b,1}=0 \tag{3.2}$$

$$\left.\left(\frac{1}{r}\frac{dF}{dr}\right)\right|_{r=b,1}=-T_0<0,\quad \frac{1}{r}\frac{dF}{dr}\text{当}r\to0\text{时是有界.} \tag{3.3}$$

假设

$$\left.\begin{aligned}w=&w_0+\varepsilon w_1+O(\varepsilon^2)+\varepsilon^2v_0^{(b)}(\xi_b,\eta,\zeta)\\&+\varepsilon^2v_0^{(1)}(\xi_1,\eta,\zeta)+O(\varepsilon^3)\\F=&f_0+\varepsilon f_1+O(\varepsilon^2)+\varepsilon^4h_0^{(b)}(\xi_b,\eta,\zeta)\\&+\varepsilon^4h_0^{(1)}(\xi_1,\eta,\zeta)+O(\varepsilon^5)\end{aligned}\right\} \tag{3.4}$$

从控制w_0，f_0的边值问题(2.24)有

$$\left.\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left(\frac{dw_0}{dr}\frac{df_0}{dr}\right)=-q\\&\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df_0}{dr}\right)\right]\right\}+\frac{1}{r}\frac{d^2w_0}{dr^2}\frac{dw_0}{dr}=0\end{aligned}\right\} \tag{3.5}$$

$$w_0|_{r=b,1}=0 \tag{3.6}$$

$$\left.\left(\frac{1}{r}\frac{df_0}{dr}\right)\right|_{r=b,1}=-T_0,\qquad \frac{1}{r}\frac{df_0}{dr}\text{当}r\to0\text{有界} \tag{3.7}$$

将(3.5)式经一次积分得

$$\left.\begin{aligned}&\frac{dw_0}{dr}\frac{df_0}{dr}=-\frac{q}{2}r^2+C_0\\&\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df_0}{dr}\right)\right]=-\frac{1}{2r}\left(\frac{dw_0}{dr}\right)^2+\frac{D_0}{r}\end{aligned}\right\}\tag{3.8}$$

其中C_0和D_0是待定常数。若令

$$y=r^2,\qquad z=y\frac{df_0}{dy}\tag{3.9}$$

则边值问题(3.5)～(3.7)变换成

$$\left.\begin{aligned}&\frac{dw_0}{dy}=\frac{1}{4z}\left(C_0-\frac{q}{2}y\right)\\&\frac{d^2z}{dy^2}=\frac{-1}{64z^2}\left(C_0^2-C_0qy+\frac{q^2}{4}y^2\right)+\frac{D_0}{y}\end{aligned}\right\}\tag{3.10}$$

$$w_0\big|_{y=b^2,1}=0\tag{3.11}$$

$$z\Big|_{y=b^2,1}=\left(\frac{-1}{2}yT_0\right)\Big|_{y=b^2,1}\tag{3.12}$$

根据边界条件(2.7)知应取$D_0=0$，和假设

$$z=a_0+a_1y+a_2y^2+a_3y^3+\cdots\tag{3.13}$$

将(3.13)式代入(3.10)的第二方程，有

$$(a_0+a_1y+a_2y^2+a_3y^3+\cdots)^2(2a_2+6a_3y+\cdots)$$

$$=\frac{-1}{64}\left(C_0^2+C_0qy+\frac{q^2}{4}y^2\right)$$

比较上式两端y的同次幂的系数，得

$$2a_0^2a_2=\frac{-1}{64}C_0^2,\qquad 6a_0^2a_3+4a_0a_1a_2=\frac{1}{64}C_0q,\cdots$$

求解以上方程得

$$a_2=\frac{-C_0^2}{128}\frac{1}{a_0^2},\qquad a_3=\frac{C_0q}{384a_0^2}-\frac{2}{3}\frac{a_1}{a_0}a_2,\cdots\tag{3.14}$$

其中a_0，a_1是任意常数，将由边界条件(3.12)确定。

将(3.13)代入(3.10)中的第一方程，经积分得

$$w_0=\int_{b^2}^{r^2}\frac{C_0-\frac{q}{2}y}{4(a_0+a_1y+a_2y^2+\cdots)}dy$$

$$=\frac{1}{4a_0}\left[C_0(r^2-b^2)+\frac{1}{2}\left(\frac{-a_1}{a_0}C_0-\frac{q}{2}\right)(r^4-b^2)+\cdots\right]\tag{3.15}$$

其中C_0是待定常数。

如果我们只要求得到准确到$O(y^2)$的近似解，可以在(3.14)中令$a_3=a_4=\cdots=0$，此时根据边界条件(3.12)求解

$$a_0=\frac{-C_0^{2/3}b^{2/3}}{4(2^{1/3})},\quad a_1=\frac{-T_0}{2}+\frac{C_0^{2/3}}{4(2^{1/3})}\left(\frac{1+b^2}{b^{4/3}}\right)\tag{3.16}$$

其中C_0根据边界条件(3.11)知是下面方程的根

$$C_0(1-b^2)+\frac{1}{2}\left(\frac{-a_1}{a_0}C_0-\frac{q}{2}\right)(1-b^4)+\cdots=0\tag{3.17}$$

求得w_0, f_0后，从(2.25)得到关于$v_0^{(0)}$和$v_0^{(1)}$的边值问题：

$$u_{p,r}^2 \frac{\partial^4 v_0^{(p)}}{\partial \xi_p^4} - \frac{f_{0,r}}{\eta} \frac{\partial^2 v_0^{(p)}}{\partial \xi_p^2} = 0 \qquad (p=b,1) \tag{3.18}$$

$$\begin{cases} u_{b,r}^2(b) \dfrac{\partial^2 v_0^{(b)}(0,b)}{\partial \xi_b^2} + u_{1,r}^2(b) \dfrac{\partial^2 v_0^{(1)}\left(\dfrac{u_1(b)}{\varepsilon}, b\right)}{\partial \xi_1^2} \\ \qquad = -\left(w_{0,rr} + \dfrac{\nu}{\eta} w_{0,r}\right)\Big|_{\eta=b} \qquad (3.19) \\ u_{b,r}^2(1) \dfrac{\partial^2 v_0^{(b)}\left(\dfrac{u_0(1)}{\varepsilon}, 1\right)}{\partial \xi_b^2} + u_{1,r}^2(1) \dfrac{\partial^2 v_0^{(1)}(0,1)}{\partial \xi_1^2} \end{cases}$$

$$= -\left(w_{0,rr} + \frac{\nu}{\eta} w_{0,r}\right)\Big|_{\eta=1} \tag{3.20}$$

为了使方程(3.18)取最简单的形式，可以取待定函数$u_p(r)$ $(p=b,1)$为

$$u_b(r) = \int_b^r \sqrt{\frac{-f_{0,r}(t)}{t}}\, dt, \quad u_1(r) = \int_r^1 \sqrt{\frac{-f_{0,r}(t)}{t}}\, dt \tag{3.21}$$

此时有

$$\frac{\partial^4 v_0^{(p)}}{\partial \xi_p^4} + \frac{\partial^2 v_0^{(p)}}{\partial \xi_p^2} = 0 \qquad (p=b,1) \tag{3.22}$$

它们的解为

$$v_0^{(p)} = A_0^{(p)}(r) \cos \xi_p + \beta_0^{(p)}(r) \sin \xi_p \qquad (p=b,1) \tag{3.23}$$

其中$A_0^{(p)}$、$\beta_0^{(p)}$ $(p=b,1)$ 是 r 的任意函数，以后再确定。将(3.23)式代入边界条件(3.19)和(3.20)，得到$A_0^{(p)}(r)$和$B_0^{(p)}(r)$ $(p=b.1)$所应满足的边界条件

$$\frac{f_{0,r}(b)}{b} A_0^{(b)}(b) + \frac{f_{0,r}(b)}{b}\left[A_0^{(1)}(b) \cos\left(\frac{1}{\varepsilon}\int_b^1 \sqrt{\frac{-f_{0,r}(t)}{t}}\, dt\right)\right.$$

$$\left. + B_0^{(1)}(b) \sin\left(\frac{1}{\varepsilon}\int_b^1 \sqrt{\frac{-f_{0,r}(t)}{t}}\, dt\right)\right] = \varphi(b) \tag{3.24}$$

$$f_{0,r}(1)\left[A_0^{(b)}(1) \cos\left(\frac{1}{\varepsilon}\int_b^1 \sqrt{\frac{-f_{0,r}(t)}{t}}\, dt\right)\right.$$

$$\left. + B_0^{(b)}(1) \sin\left(\frac{1}{\varepsilon}\int_b^1 \sqrt{\frac{-f_{0,r}(t)}{t}}\, dt\right)\right] + f_{0,r}(1) A_0^{(1)}(1) = \varphi(1) \tag{3.25}$$

其中

$$\varphi(\eta) \equiv -\left(w_{0,rr}(\eta) + \frac{\nu}{\eta} w_{0,r}(\eta)\right)$$

将w_0，f_0代入(2.27)，得到关于w_1，f_1的边值问题：

$$\left.\begin{aligned}&\frac{d}{dr}\left(\frac{df_0}{dr}\frac{dw_1}{dr}+\frac{dw_0}{dr}\frac{df_1}{dr}\right)=0\\&\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df_1}{dr}\right)\right]\right\}=\frac{d^2w_0}{dr^2}\frac{dw_1}{dr}+\frac{d^2w_1}{dr^2}\frac{dw_0}{dr}\\&w_1\Big]_{r=b}=0,\qquad w_1\Big]_{r=1}=0\\&\left(\frac{1}{r}f_{1,r}\right)\Big]_{r=b}=0,\qquad\left(\frac{1}{r}f_{1,r}\right)\Big]_{r=1}=0\end{aligned}\right\}\tag{3.26}$$

显然具有平凡解$w_1=f_1=0$。

w_1，f_1确定后，从(2.28)得到关于$v_1^{(p)}(p=b,1)$的方程：

$$\left(-\frac{f_{0,r}}{r}\right)^2\left(\frac{\partial^4v_1^{(p)}}{\partial\xi_p^4}+\frac{\partial^2v_1^{(p)}}{\partial\xi_p^2}\right)$$

$$=-\left(4u_{p,r}^3\frac{\partial^4v_0^{(p)}}{\partial\xi_p^3\partial\eta}+6u_{p,r}^2u_{p,rr}\frac{\partial^3v_0^{(p)}}{\partial\xi_p^3}+\frac{2}{\eta}u_{p,r}^3\frac{\partial^3v_0^{(p)}}{\partial\xi_p^3}\right.$$

$$\left.-\frac{1}{\eta}f_{0,rr}u_{p,r}\frac{\partial v_0^{(p)}}{\partial\xi_p}+\frac{1}{\eta}f_{0,r}\left(2u_{p,r}\frac{\partial^2v_0^{(p)}}{\partial\xi_p\partial\eta}+u_{p,rr}\frac{\partial v_0^{(p)}}{\partial\xi_p}\right)\right)\qquad(p=b,1)$$

将$v_0^{(p)}$的表达式(3.23)代入上面方程，并令其右端为零，则得到关于$A_0^{(p)}$，$B_0^{(p)}(p=b,1)$的微分方程，它们具有同一形式：

$$\frac{dX}{d\eta}+\frac{1}{4}\left(\frac{3f_{0,rr}}{f_{0,r}}-\frac{1}{\eta}\right)X=0\tag{3.27}$$

根据它们所满足的边界条件(3.24)和(3.25)可以得出无穷组解。

假如取$A_0^{(p)}(\eta)\equiv0$ $(p=b,1)$，有

$$B_0^{(b)}(\eta)=\frac{-\varphi(1)\eta^{1/4}}{(-f_{0,r}(1))^{1/4}(-f_{0,r}(\eta))^{3/4}}\csc\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)$$

$$B_0^{(1)}(\eta)=\frac{-b\varphi(b)}{(-f_{0,r}(b))^{1/4}(-f_{0,r}(\eta))^{3/4}}\left(\frac{\eta}{b}\right)^{1/4}\csc\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)$$

和

$$v_0^{(b)}=B_0^{(b)}(\eta)\sin\xi_0,\qquad v_0^{(1)}=B_0^{(1)}(\eta)\sin\xi_1$$

此外从(2.26)式有

$$h_0^{(b)}=\frac{-w_{0,r}(\eta)}{f_{0,r}(\eta)}B_0^{(b)}(\eta)\sin\xi_0,\quad h_0^{(1)}=\frac{-w_{0,r}(\eta)}{f_{0,r}(\eta)}B_0^{(1)}(\eta)\sin\xi_1\tag{3.28}$$

所以

$$w=\frac{1}{4a_0}\left[C_0(r^2-b^2)+\frac{1}{2}\left(\frac{-a_1}{a_0}C_0-\frac{q}{2}\right)(r^4-b^4)+\cdots\right]+O(\varepsilon^2)$$

$$+\left\{\varepsilon^2\left[\frac{-\varphi(1)}{(-f_{0,r}(1))^{1/4}(-f_{0,r}(r))^{3/4}}r^{1/4}\csc\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)\sin\left(\frac{1}{\varepsilon}\int_b^r\sqrt{\frac{-f_{0,r}}{r}}dr\right)\right.\right.$$

$$+\frac{-b\varphi(b)}{(-f_{0,r}(b))^{1/4}(-f_{0,r}(r))^{3/4}}\left(\frac{r}{b}\right)^{1/4}\csc\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)$$

$$\cdot\sin\left(\frac{1}{\varepsilon}\int_r^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)\Big]+O(\varepsilon^2)\Big\} \qquad (3.29)$$

$$F=\left(a_0\ln r^2+a_1r^2+\frac{1}{2}a_2r^4+\cdots\right)+O(\varepsilon^2)$$

$$+\Big\{\varepsilon^4\Big[\frac{-w_{0,r}(r)}{f_{0,r}(r)}\frac{-\varphi(1)r^{1/4}}{(-f_{0,r}(1))^{1/4}(-f_{0,r}(r))^{3/4}}$$

$$\cdot\csc\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)\sin\left(\frac{1}{\varepsilon}\int_b^r\sqrt{\frac{-f_{0,r}}{r}}dr\right)$$

$$+\frac{-w_{0,r}(r)}{f_{0,r}(r)}\frac{-b\varphi(b)}{(-f_{0,r}(b))^{1/4}(-f_{0,r}(r))^{3/4}}$$

$$\cdot\csc\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)\sin\left(\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr\right)\Big]+O(\varepsilon^5)\Big\} \qquad (3.30)$$

其中a_0，a_1，a_2和(3.16)和(3.17)式给出，C_0是方程(3.17)的根。

从(3.29)和(3.30)式我们看到，如果

$$\frac{1}{\varepsilon}\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr=\pi,\qquad 即\frac{\sqrt{12(1-\nu^2)}}{h}r_1\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr=\pi \qquad (3.31)$$

则薄板破裂。由于

$$I\equiv\int_b^1\sqrt{\frac{-f_{0,r}}{r}}dr=\int_b^1\frac{1}{r}\sqrt{2(-a_0-a_1r^2-a_2r^4+\cdots)}\,dr$$

$$=\sqrt{2(-a_0)}\Big\{\ln\frac{1}{b}+\frac{1}{4}\frac{a_1}{a_0}(1-b^2)$$

$$+\frac{1}{3}\left[\frac{1}{2}\frac{a_2}{a_0}-\frac{1}{8}\left(\frac{a_1}{a_0}\right)^2\right](1-b^4)+\cdots\Big\} \qquad (3.32)$$

将(3.16)和(3.17)式代入(3.32)式，可知I是T_0的函数，再从(3.31)式解出T_0，即得到薄板的极限荷载。

特例地，若横向荷载$q=0$，和在(3.13)式中略去高阶项，$O(y^3)$，并在I中只取主要项，则方程(3.17)化为

$$1-\frac{1}{2}(b^2+1)\frac{a_1}{a_0}=0 \qquad (3.33)$$

其中a_0，a_1是由(3.16)式定义的C_0的函数。解此方程得

$$C_0^{2/3}=2^{4/3}\frac{b^{4/3}(b^2+1)}{2b^2+(b^2+1)^2}T_0$$

破裂的条件(3.31)化为

$$\frac{\sqrt{12(1-\nu^2)}r_1}{h}\sqrt{\frac{b^2(b^2+1)}{2b^2+(b^2+1)^2}}T_0^{1/2}\ln\frac{1}{b}=\pi \qquad (3.34)$$

考虑到$b=r_0/r_1$，从(3.34)式得

$$T_0=\frac{(r_0^4+4r_0^2r_1^2+r_1^4)h^2\pi^2}{12(1-\nu^2)r_0^2r_1^2(r_0^2+r_1^2)(\ln r_1-\ln r_2)} \tag{3.35}$$

或用有量纲表示：

$$(T_0)_d=EhT_0=\frac{(r_0^4+r_0^2r_1^2+r_1^4)D\pi^2}{r_0^2r_1^2(r_0^2+r_1^2)(\ln r_1-\ln r_2)} \tag{3.36}$$

四、均匀压力作用下的圆板的屈曲后性态

此时$b=0$，控制w和F的边值问题化为

$$\left.\begin{aligned}&\varepsilon^2\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=\frac{1}{r}\frac{dw}{dr}\frac{dF}{dr}+\frac{qr}{2}\\&\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)\right]=\frac{-1}{2}\frac{1}{r}\left(\frac{dw}{dr}\right)^2\end{aligned}\right\} \tag{4.1}$$

$$w\Big|_{r=1}=0,\quad \frac{d^2w}{dr^2}\Big|_{r=1}=0,\quad \left(\frac{1}{r}\frac{dF}{dr}\right)\Big|_{r=1}=-T_0 \tag{4.2}$$

$$\frac{dw}{dr},\quad \frac{1}{r}\frac{dF}{dr}\text{ 在 } r=0 \text{ 取有限值。} \tag{4.3}$$

假设

$$\left.\begin{aligned}&w=\sum_{n=0}^{\infty}\varepsilon^n w_n(r)+\varepsilon^2\sum_{n=0}^{\infty}\varepsilon^n v_n\left(\frac{u(r)}{\varepsilon},\ r\right)\\&F=\sum_{n=0}^{\infty}\varepsilon^n f_n(r)+\varepsilon^4\sum_{n=0}^{\infty}\varepsilon^n h_n\left(\frac{u(r)}{\varepsilon},\ r\right)\end{aligned}\right\} \tag{4.4}$$

重复以上的运算，得到关于w_0，f_0的边值问题：

$$\left.\begin{aligned}&\frac{dw_0}{dr}\frac{df_0}{dr}=\frac{qr^2}{2}\\&\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df_0}{dr}\right)\right]=\frac{-1}{2r}\left(\frac{dw_0}{dr}\right)^2\end{aligned}\right\} \tag{4.5}$$

$$w_0\Big|_{r=1}=0,\qquad \left(\frac{1}{r}\frac{df_0}{dr}\right)\Big|_{r=1}=-T_0 \tag{4.6}$$

$$\frac{dw_0}{dr},\quad \frac{1}{r}\frac{df_0}{dr}\text{ 在} r=0 \text{ 取有限值。} \tag{4.7}$$

令$z=df_0/dr$，边值问题(4.5)～(4.7)化为

$$\left.\begin{aligned}&r^2\frac{d^2z}{dr^2}+r\frac{dz}{dr}-z=\frac{-q^2r^6}{8z^2}\\&z\frac{dw_0}{dr}=\frac{-qr^2}{2}\end{aligned}\right\} \tag{4.8}$$

$$z\Big|_{r=1}=-T_0,\qquad w_0\Big|_{r=1}=0 \tag{4.9}$$

$$\frac{z}{r},\ \frac{dw_0}{dr}\ 在 r=0\ 取有限值 \tag{4.10}$$

假设

$$z=a_1r+a_2r^2+a_3r^3+\cdots \tag{4.11}$$

将(4.11)式代入(4.8)，比较r的同次幂的系数知$a_2=a_4=\cdots=0$，和

$$a_3=\frac{-q^2}{64a_1^2},\quad a_5=\frac{-2a_3^2}{3a_1}=\frac{-q^4}{6144a_1^5},\ \cdots \tag{4.12}$$

其中a_1是任意常数。从边界条件(4.9)得到关于a_1的代数方程:

$$a_1-\frac{q}{64a_1^2}-\frac{q^4}{6144a_1^5}+\cdots=-T_0 \tag{4.13}$$

从(4.13)解出a_1，再代入(4.12)，(4.11)式，则确定了z。又从(4.8)式有

$$w_0=\int_r^1\frac{qr^2}{2(a_1r+a_3r^3+\cdots)}dr\approx\frac{q}{4a_3}\ln\frac{a_1+a_3}{a_1+a_3r^2} \tag{4.14}$$

如果$q=0$，则得

$$w_0=0,\quad \frac{1}{r}\frac{df_0}{dr}=a_1=-T_0 \tag{4.15}$$

类似地（参看(3.18)，(3.19)）关于v_0的边值问题为

$$u_r^2\frac{\partial^4v_0}{\partial\xi^4}-\frac{f_{0,r}}{\eta}\frac{\partial^2v_0}{\partial\xi^2}=0 \tag{4.16}$$

$$u_r^2\frac{\partial^2v_0}{\partial\xi^2}\bigg]_{\eta=1}=-\left(w_{0,rr}+\frac{\nu}{\eta}w_{0,r}\right)\bigg|_{\eta=1} \tag{4.17}$$

$$v_0\ 在 \eta=0 取有限值 \tag{4.18}$$

其中

$$\xi=\frac{u(r)}{\varepsilon},\quad \eta=r \tag{4.19}$$

今取

$$u(r)=\int_0^r\sqrt{\frac{-f_{0,r}}{r}}dr \tag{4.20}$$

方程(4.16)化为

$$\frac{\partial^4v_0}{\partial\xi^4}+\frac{\partial^2v_0}{\partial\xi^2}=0$$

具有解

$$v_0=A_0(\eta)\sin(\xi+B_0(\eta))$$

其中$A_0(\eta)$，$B_0(\eta)$是待定的任意函数。从边界条件(4.17)有

$$A_0(1)=\frac{\varphi(1)}{f_{0,r}(1)}\csc\left(\frac{1}{\varepsilon}\int_0^1\sqrt{\frac{-f_{0,r}}{r}}dr+B_0(1)\right) \tag{4.21}$$

其中$\varphi(1)=-\left(w_{0,rr}+\frac{\nu}{r}w_{0,r}\right)\Big]_{r=1}$。为简单起见今取$B(\eta)\equiv0$。

类似地可知$w_1=f_1=0$，又关于$A_0(\eta)$的微分方程为

$$\frac{dA}{d\eta}+\frac{1}{4}\left(\frac{3f_{0,rr}}{f_{0,r}}-\frac{1}{\eta}\right)A=0 \tag{4.22}$$

考虑到(4.21)式，从(4.22)得

$$A_0(\eta)=\frac{-\varphi(1)\eta^{1/4}}{(-f_{0,r}(1))^{1/4}(-f_{0,r}(\eta))^{3/4}}\csc\left(\frac{1}{\varepsilon}\int_0^1\sqrt{\frac{-f_{0,r}}{r}}\,dr\right)$$

又（参看(3.28)）

$$h_0(\xi,\eta)=\frac{1}{2}A_0(\eta)\frac{-w_{0,r}(\eta)}{f_{0,r}(\eta)}\sin\xi$$

所以

$$w=\frac{q}{4a_3}\ln\frac{a_1+a_3}{a_1+a_3r}+O(\varepsilon^2)+\varepsilon^2\left[\frac{-\varphi(1)r^{1/4}}{(-f_{0,r}(1))^{1/4}(-f_{0,r}(r))^{3/4}}\right.$$
$$\left.\cdot\csc\left(\frac{1}{\varepsilon}\int_0^1\sqrt{\frac{-f_{0,r}}{r}}\,dr\right)\sin\left(\frac{1}{\varepsilon}\int_0^r\sqrt{\frac{-f_{0,r}}{r}}\,dr\right)+O(\varepsilon)\right] \tag{4.23}$$

$$F=\frac{a_1}{2}r^2-\frac{1}{3}a_2r^3+O(\varepsilon^2)+\varepsilon^4\left[\frac{-\varphi(1)r^{1/4}w_{0,r}(r)}{2(-f_{0,r}(1))^{1/4}(-f_{0,r}(r))^{7/4}}\right.$$
$$\left.\cdot\csc\left(\frac{1}{\varepsilon}\int_0^1\sqrt{\frac{-f_{0,r}}{r}}\,dr\right)\sin\left(\frac{1}{\varepsilon}\int_0^r\sqrt{\frac{-f_{0,r}}{r}}\,dr\right)+O(\varepsilon)\right] \tag{4.24}$$

从(4.23)，(4.24)式我们看到，如果成立

$$\frac{1}{\varepsilon}\int_0^1\sqrt{\frac{-f_{0,r}}{r}}\,dr=\pi,\quad 即\quad \sqrt{\frac{Eh}{D}}\,r_1\int_0^1\sqrt{\frac{-f_{0,r}}{r}}\,dr=\pi \tag{4.25}$$

则薄板破裂．从方程(4.25)中解出的根T_0就表示薄板的极限荷载．

特例地若$q=0$，则

$$\int_0^1\sqrt{\frac{-f_{0,r}}{r}}\,dr\approx\sqrt{-a_1}=\sqrt{T_0}$$

将上式代入(4.25)式，解出T_0得

$$T_0=\frac{D\pi^2}{Ehr_1^2} \tag{4.26}$$

近似地表示薄板的极限荷载．或写成有量纲形式

$$(T_0)_d=\frac{D\pi^2}{r_1^2} \tag{4.27}$$

注 1、为了与已知结果比较，在推导公式(3.36)和(4.27)时，我们忽略了所有低阶项，否则将可得到更精确的结果．但计算较繁．

2、从(4.27)式可以看出一简支圆薄板的极限荷载是其临界荷载的2.74倍．所以一屈曲后的薄板仍具有足够强度以承受较大的轴向压力不致破裂．

3、从(3.29)，(4.23)式可以看到屈曲的薄板可以具有不同的形态．因都含有快变量ξ的正弦或余弦项，所以，其皱纹波长随着薄板刚度的减小而减小。

参 考 文 献

[1] Timoshenko, S. P. and T. M. Gere, *Theory of Elastic Stability*, McGraw Hill (1961).

[2] Вольмир A. C., *Гибкие Пластинки и Оболочка*, Гостехиздат, Москва (1956).

[3] Chia Chuen-Yuan, *Nonlinear Analysis of Plates*, McGraw Hill (1980).

[4] 秦圣立、张爱淑，关于环形薄板的屈曲问题，应用数学和力学，**6**，2 (1985)，175—190.

[5] 江福汝，摄动方法在薄板屈曲问题中的某些应用，应用数学和力学，**1**，1 (1980)，37—53.

[6] 江福汝，环形和薄板在各种支承条件下的非对称屈曲问题(Ⅰ)，应用数学和力学，**3**，5 (1982)，629—640.

[7] 江福汝，环形和圆形薄板在各种支承条件下的非对称弯曲问题(Ⅱ)，应用数学和力学，**5**，2 (1984)，191—203.

Nonlinear Analyses for the Postbuckling Behaviors of Annular and Circular Plates

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Abstract

In this paper we apply the modified method of multiple scales to study the postbuckling behaviors of annular and circular plates. The asymptotic solutions have been constructed, the ultimate loads have been determined, and the relations between length of twisted waves formed by buckling and the flexural rigidity of plates have been discovered.