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刚性基础上的弹性层在表面垂直集中简谐载荷作用下的动力响应*

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摘　要

本文首次将文献[1]所提出的线载荷积分方程法应用于求解弹性动力学问题．导出了刚性基础上的弹性层在表面垂直集中简谐载荷作用下动力响应问题的一维非奇异积分方程组，并求得了数值解．

一、引　言

参考文献[1]所提出的线载荷积分方程法在求解三维弹性力学轴对称问题时得到了广泛的应用。该方法最大的优点是可将支配这类问题的二维奇异积分方程转化为一维的非奇异积分方程，从而使得计算大为简化。但是，迄今为止尚未见有人将这种方法用于求解弹性动力学问题。

本文首次将文献[1]所提出的线载荷积分方程法应用于弹性动力学领域。在文献[2]的基础上，应用叠加原理导出了刚性基础上的弹性层在表面垂直集中简谐载荷作用下动力响应问题的一维非奇异积分方程组。并通过离散化处理求得了这一问题的数值解。

二、积分方程的推导

图1所示刚性基础上的弹性层在表面垂直集中简谐载荷作用下的主要边界条件可表示为：

$$z=0: \qquad w=0 \tag{2.1}$$

$$z=-h,\ r\neq 0: \qquad \sigma_z=\tau_{rz}=0 \tag{2.2}$$

为了导出求解这一问题的积分方程，设在弹性全空间中沿 z 轴作用有三类轴对称简谐载荷(如图2表示)：

（1）在 $(-\infty,-h]$ 和 $[h,\infty)$ 的区间内对称于 $z=0$ 的平面作用一平行于 z 轴的线分布简谐载荷 $X_1(c)\exp[i\omega t]$。

（2）在 $(-\infty,-h]$ 和 $[h,\infty)$ 的区间内对称于 $z=0$ 的平面作用一线分布的简谐挤压中心

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* 云天铨推荐．

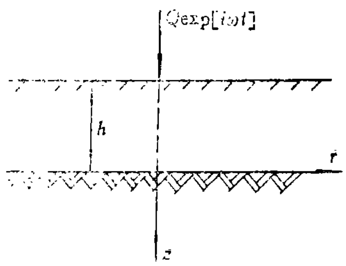

图 1

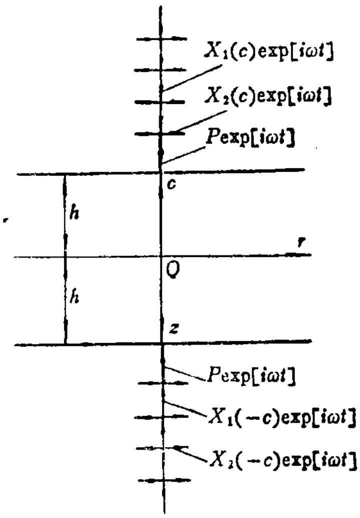

图 2

$X_2(c)\exp[i\omega t]$。

（3）在点 $(r=0, z=h)$ 和 $(r=0, z=-h)$ 处分别作用有一对大小相等、方向相反的垂直集中简谐力 $P\exp[i\omega t]$ 和 $P\exp[-i\omega t]$。

显然，在这一组载荷作用下，由于对称性，边界条件(2.1)可得到满足，而弹性全空间中任一点 (r,z) 的应力和位移可表示为如下形式：

$$
\left.\begin{aligned}
u_r(r,z)&=\int_h^\infty u_r^{(p)}(r,z,c)X_1(c)dc+\int_h^\infty u_r^{(o)}(r,z,c)X_2(c)dc+Pu_r^{(p)}(r,z,h)\\
w(r,z)&=\int_h^\infty w^{(p)}(r,z,c)X_1(c)dc+\int_h^\infty w^{(o)}(r,z,c)X_2(c)dc+Pw^{(p)}(r,z,h)\\
\sigma_r(r,z)&=\int_h^\infty \sigma_r^{(p)}(r,z,c)X_1(c)dc+\int_h^\infty \sigma_r^{(o)}(r,z,c)X_2(c)dc+P\sigma_r^{(p)}(r,z,h)\\
\sigma_\theta(r,z)&=\int_h^\infty \sigma_\theta^{(p)}(r,z,c)X_1(c)dc+\int_h^\infty \sigma_\theta^{(o)}(r,z,c)X_2(c)dc+P\sigma_\theta^{(p)}(r,z,h)\\
\sigma_z(r,z)&=\int_h^\infty \sigma_z^{(p)}(r,z,c)X_1(c)dc+\int_h^\infty \sigma_z^{(o)}(r,z,c)X_2(c)dc+P\sigma_z^{(p)}(r,z,h)\\
\tau_{rz}(r,z)&=\int_h^\infty \tau_{rz}^{(p)}(r,z,c)X_1(c)dc+\int_h^\infty \tau_{rz}^{(o)}(r,z,c)X_2(c)dc+P\tau_{rz}^{(p)}(r,z,h)
\end{aligned}\right\}\quad(2.3)
$$

式中，$u_r^{(p)}(r,z,c)$ 表示由一对大小相等方向相反分别作用于 $(0,c)$ 和 $(0,-c)$ 处的单位简谐集中力在 (r,z) 处产生的径向位移，$u_r^{(o)}(r,z,c)$ 则表示一对大小和相位均相同的单位简谐挤压中心分别作用于 $(0,c)$ 和 $(0,-c)$ 处时在 (r,z) 处产生的径向位移。其它积分核亦具有与此相似的物理意义。它们均可由弹性动力学的基本奇解[3]导出：

$$
\begin{aligned}
u_r^{(p)}(r,z,c)&=A\frac{\partial^2}{\partial z\partial r}\Big[(\exp[-i\lambda_1R_1]-\exp[-i\lambda_2R_1])/R_1\\
&\qquad-(\exp[-i\lambda_1R_2]-\exp[-i\lambda_2R_2])/R_2\Big]\\
w^{(p)}(r,z,c)&=A\Big[\frac{\partial^2}{\partial z^2}(\exp[-i\lambda_1R_1]/R_1-\exp[-i\lambda_1R_2]/R_2)\\
&\qquad+\Big(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}\Big)(\exp[-i\lambda_2R_1]/R_1-\exp[-i\lambda_2R_2]/R_2)\Big]\\
\sigma_z^{(p)}(r,z,c)&=B\frac{\partial}{\partial z}\Big[\Big(\frac{\partial^2}{\partial z^2}-\frac{\lambda\lambda_1^2}{2\mu}\Big)(\exp[-i\lambda_1R_1]/R_1-\exp[-i\lambda_1R_2]/R_2)\\
&\qquad+\Big(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}\Big)(\exp[-i\lambda_2R_1]/R_1-\exp[-i\lambda_2R_2]/R_2)\Big]
\end{aligned}
$$

$$
\left.\begin{aligned}
\sigma_r^{(p)}(r,z,c)&=B\frac{\partial}{\partial z}\Big[\Big(\frac{\partial^2}{\partial r^2}-\frac{\lambda\lambda_1^2}{2\mu}\Big)(\exp[-i\lambda_1R_1]/R_1-\exp[-i\lambda_1R_2]/R_2)\\
&\quad-\frac{\partial^2}{\partial r^2}(\exp[-i\lambda_2R_1]/R_1-\exp[-i\lambda_2R_2]/R_2)\Big]\\
\sigma_\theta^{(p)}(r,z,c)&=B\frac{\partial}{\partial z}\Big[\Big(\frac{\partial}{r\partial r}-\frac{\lambda\lambda_1^2}{2\mu}\Big)(\exp[-i\lambda_1R_1]/R_1-\exp[-i\lambda_1R_2]/R_2)\\
&\quad-\frac{\partial}{r\partial r}(\exp[-i\lambda_2R_1]/R_1-\exp[-i\lambda_2R_2]/R_2)\Big]\\
\tau_{rz}^{(p)}(r,z,c)&=B\frac{\partial}{\partial r}\Big[\frac{\partial^2}{\partial z^2}(\exp[-i\lambda_1R_1]/R_1-\exp[-i\lambda_1R_2]/R_2)\\
&\quad-\Big(\frac{\lambda_2^2}{2}+\frac{\partial^2}{\partial z^2}\Big)(\exp[-i\lambda_2R_1]/R_1-\exp[-i\lambda_2R_2]/R_2)\Big]\\
u_r^{(c)}(r,z,c)&=C\frac{\partial}{\partial r}(\exp[-i\lambda_1R_1]/R_1+\exp[-i\lambda_1R_2]/R_2)\\
w^{(c)}(r,z,c)&=C\frac{\partial}{\partial z}(\exp[-i\lambda_1R_1]/R_1+\exp[-i\lambda_1R_2]/R_2)\\
\sigma_z^{(c)}(r,z,c)&=D\Big(\frac{\partial^2}{\partial z^2}-\frac{\lambda\lambda_1^2}{2\mu}\Big)(\exp[-i\lambda_1R_1]/R_1+\exp[-i\lambda_1R_2]/R_2)\\
\sigma_r^{(c)}(r,z,c)&=D\Big(\frac{\partial^2}{\partial r^2}-\frac{\lambda\lambda_1^2}{2\mu}\Big)(\exp[-i\lambda_1R_1]/R_1+\exp[-i\lambda_1R_2]/R_2)\\
\sigma_\theta^{(c)}(r,z,c)&=D\Big(\frac{\partial}{r\partial r}-\frac{\lambda\lambda_1^2}{2\mu}\Big)(\exp[-i\lambda_1R_1]/R_1+\exp[-i\lambda_1R_2]/R_2)\\
\tau_{rz}^{(c)}(r,z,c)&=D\frac{\partial^2}{\partial r\partial z}(\exp[-i\lambda_1R_1]/R_1+\exp[-i\lambda_1R_2]/R_2)
\end{aligned}\right\}\tag{2.4}
$$

$$
\left.\begin{aligned}
&R_1=[r^2+(z-c)^2]^{\frac{1}{2}},\qquad R_2=[r^2+(z+c)^2]^{\frac{1}{2}}\\
&\lambda_1=[\rho\omega^2/(\lambda+2\mu)]^{\frac{1}{2}},\qquad \lambda_2=(\rho\omega^2/\mu)^{\frac{1}{2}}\\
&A=1/4\pi\mu\lambda_2^2,\ B=2\mu A,\ C=1/4\pi(\lambda+2\mu),\ D=2\mu C
\end{aligned}\right\}\tag{2.5}
$$

(2.4)和（2.5）式中，μ为剪切模量，λ 为拉梅系数，ω 表示简谐载荷的角频率。

如果令：

$$R_3=[r^2+(z-h)^2]^{\frac{1}{2}},\qquad R_4=[r^2+(z+h)^2]^{\frac{1}{2}}\tag{2.6}$$

则只要在（2.4）的前六个表达式中用R_3代替R_1，R_4代替R_2，便可直接写出另外的六个积分核。

为了使得图 2 所示的问题等价于图 1 所示的问题，我们考虑一个由平面($z=-h$)，($z=0$)和圆柱面（$r=a$）所界的半径足够小的圆柱体的平衡（参见图 2），则该圆柱体在 z 方向的平衡条件可以写为：

$$Q=-2\pi\Big[a\int_{-h}^{0}\tau_{rz}(a,z)dz+\int_0^a r\sigma_z(r,0)dr\Big]\tag{2.7}$$

将（2.3）的第五式和第六式代入（2.7）式并进行适当的数学处理则可导出：

$$P=-\Big[\frac{Q}{2\pi}+\int_h^\infty T_1(a,h,c)X_1(c)dc+\int_h^\infty T_2(a,h,c)X_2(c)dc\Big]\Big/T_3(a,h)\tag{2.8}$$

式中：

$$\left.\begin{aligned}T_1&=a\int_{-h}^{0}\tau_{rz}^{(p)}(a,z,c)dz+\int_0^a r\ \sigma_z^{(p)}(r,0,c)dr\\T_2&=a\int_{-h}^{0}\tau_{rz}^{(c)}(a,z,c)dz+\int_0^a r\ \sigma_z^{(c)}(r,0,c)dr\\T_3&=a\int_{-h}^{0}\tau_{rz}^{(p)}(a,z,h)dz+\int_0^a r\ \sigma_z^{(p)}(r,0,h)dr\end{aligned}\right\}\tag{2.9}$$

为了满足边界条件（2.2）式，我们令：

$$\sigma_z(r,-h)=0,\qquad \tau_{rz}(r,-h)=0\tag{2.10}$$

将（2.8）代入(2.3)，然后将（2.3）中关于σ_z和τ_{rz}的表达式代入（2.10）则可得这一问题的控制方程：

$$\left.\begin{aligned}\int_h^\infty S_{11}(r,c)X_1(c)dc+\int_h^\infty S_{12}(r,c)X_2(c)dc=Q_1(r)\\\int_h^\infty S_{21}(r,c)X_1(c)dc+\int_h^\infty S_{22}(r,c)X_2(c)dc=Q_2(r)\end{aligned}\right\}\tag{2.11}$$

式中：

$$\left.\begin{aligned}S_{11}(r,c)&=\sigma_z^{(p)}(r,-h,c)-\sigma_z^{(p)}(r,-h,h)T_1(a,h,c)/T_3(a,h)\\S_{12}(r,c)&=\sigma_z^{(c)}(r,-h,c)-\sigma_z^{(p)}(r,-h,h)T_2(a,h,c)/T_3(a,h)\\S_{21}(r,c)&=\tau_{rz}^{(p)}(r,-h,c)-\tau_{rz}^{(p)}(r,-h,h)T_1(a,h,c)/T_3(a,h)\\S_{22}(r,c)&=\tau_{rz}^{(c)}(r,-h,c)-\tau_{rz}^{(p)}(r,-h,h)T_2(a,h,c)/T_3(a,h)\\Q_1&=Q\sigma_z^{(p)}(r,-h,h)/2\pi T_3(a,h)\\Q_2&=Q\tau_{rz}^{(p)}(r,-h,h)/2\pi T_3(a,h)\end{aligned}\right\}\tag{2.12}$$

(2.11)是一个定义在复平面上的第一种Fredholm积分方程组，其中，S_{ij}为积分核，$X_1(c)$和$X_2(c)$为未知函数。如果由（2.11）解得$X_1(c)$和$X_2(c)$，则可由（2.3）式求得刚性基础上的弹性层在表面垂直集中简谐载荷作用下的应力场和 多场，且可全部满足边界条件（2.1）和(2.2)。

三、积分方程的求解

如果令：

$$\left.\begin{aligned}&x=r/h,\ y=z/h,\ \xi=c/h,\ \beta=a/h,\ \alpha_1=\lambda_1 h\\&\alpha_2=\lambda_2 h,\ Y_1(\xi)=hX_1(c)/Q\alpha_2^2,\ Y_2(\xi)=\mu X_2(c)/Q(\lambda+2\mu)\end{aligned}\right\}\tag{3.1}$$

则可将（2.11）式写成如下无量纲形式：

$$\left.\begin{aligned}\int_1^\infty K_{11}(x,\xi)Y_1(\xi)d\xi+\int_1^\infty K_{12}(x,\xi)Y_2(\xi)d\xi=F_1(x)\\\int_1^\infty K_{21}(x,\xi)Y_1(\xi)d\xi+\int_1^\infty K_{22}(x,\xi)Y_2(\xi)d\xi=F_2(x)\end{aligned}\right\}\tag{3.2}$$

式中：

$$
\left.
\begin{aligned}
&K_{11}(x,\xi)=\sigma_z^{(p)}(x,\ \xi)-\sigma_z^{(p)}(x)t_1(\xi)/t_3\\
&K_{12}(x,\xi)=\sigma_z^{(o)}(x,\ \xi)-\sigma_z^{(p)}(x)t_2(\xi)/t_3\\
&K_{21}(x,\xi)=\tau_{rz}^{(p)}(x,\xi)-\tau_{rz}^{(p)}(x)t_1(\xi)/t_3\\
&K_{22}(x,\xi)=\tau_{rz}^{(o)}(x,\xi)-\tau_{rz}^{(p)}(x)t_2(\xi)/t_3\\
&F_1(x)=\sigma_z^{(p)}(x)/t_3,\qquad F_2(x)=\tau_{rz}^{(p)}(x)/t_3
\end{aligned}
\right\} \tag{3.3}
$$

$$
\left.
\begin{aligned}
\sigma_z^{(p)}(x,\xi)=&\frac{\partial}{\partial y}\Big[\Big(\frac{\partial^2}{\partial y^2}-\frac{\lambda\alpha_1^2}{2\mu}\Big)(\exp[-i\alpha_1 r_1]/r_1-\exp[-i\alpha_1 r_2]/r_2)\\
&+\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial}{x\partial x}\Big)(\exp[-i\alpha_2 r_1]/r_1-\exp[-i\alpha_2 r_2]/r_2)\Big]_{y=-1}\\
\sigma_z^{(p)}(x)=&\frac{\partial}{\partial y}\Big[\Big(\frac{\partial^2}{\partial y^2}-\frac{\lambda\alpha_1^2}{2\mu}\Big)(\exp[-i\alpha_1 r_3]/r_3-\exp[-i\alpha_1 r_4]/r_4)\\
&+\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial}{x\partial x}\Big)(\exp[-i\alpha_2 r_3]/r_3-\exp[-i\alpha_2 r_4]/r_4)\Big]_{y=-1}\\
\tau_{rz}^{(p)}(x,\xi)=&\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial y^2}(\exp[-i\alpha_1 r_1]/r_1-\exp[-i\alpha_1 r_2]/r_2)\\
&-\Big(\frac{\alpha_2^2}{2}+\frac{\partial^2}{\partial y^2}\Big)(\exp[-i\alpha_2 r_1]/r_1-\exp[-i\alpha_2 r_2]/r_2)\Big]_{y=-1}\\
\tau_{rz}^{(p)}(x)=&\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial y^2}(\exp[-i\alpha_1 r_3]/r_3-\exp[-i\alpha_1 r_4]/r_4)\\
&-\Big(\frac{\alpha_2^2}{2}+\frac{\partial^2}{\partial y^2}\Big)(\exp[-i\alpha_2 r_3]/r_3-\exp[-i\alpha_2 r_4]/r_4)\Big]_{y=-1}\\
\sigma_z^{(o)}(x,\xi)=&\Big[\Big(\frac{\partial^2}{\partial y^2}-\frac{\lambda\alpha_1^2}{2\mu}\Big)(\exp[-i\alpha_1 r_1]/r_1+\exp[-i\alpha_2 r_2]/r_2)\Big]_{y=-1}\\
\tau_{rz}^{(o)}(x,\xi)=&\Big[\frac{\partial^2}{\partial x\partial y}(\exp[-i\alpha_1 r_1]/r_1+\exp[-i\alpha_1 r_2]/r_2)\Big]_{y=-1}\\
\tau_{rz}^{(p)}(y,\xi)=&\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial y^2}(\exp[-i\alpha_1 r_1]/r_1-\exp[-i\alpha_1 r_2]/r_2)\\
&-\Big(\frac{\alpha_2^2}{2}+\frac{\partial^2}{\partial y^2}\Big)(\exp[-i\alpha_2 r_1]/r_1-\exp[-i\alpha_2 r_2]/r_2)\Big]_{z=\beta}\\
\tau_{rz}^{(p)}(y)=&\frac{\partial}{\partial x}\Big[\frac{\partial^2}{\partial y^2}(\exp[-i\alpha_1 r_3]/r_3-\exp[-i\alpha_1 r_4]/r_4)\\
&-\Big(\frac{\alpha_2^2}{2}+\frac{\partial^2}{\partial y^2}\Big)(\exp[-i\alpha_2 r_3]/r_3-\exp[-i\alpha_2 r_4]/r_4)\Big]_{z=\beta}\\
\sigma_z^{(p)}(x,\xi)=&\frac{\partial}{\partial y}\Big[\Big(\frac{\partial^2}{\partial y^2}-\frac{\lambda\alpha_1^2}{2\mu}\Big)(\exp[-i\alpha_1 r_1]/r_1-\exp[-i\alpha_1 r_2]/r_2)
\end{aligned}
\right\} \tag{3.4}
$$

$$+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{x\partial x}\right)(\exp[-i\alpha_2r_1]/r_1-\exp[-i\alpha_2r_2]/r_2)\Big]_{y=0}$$

$$\sigma_z^{(p)}(x)=\frac{\partial}{\partial y}\left[\left(\frac{\partial^2}{\partial y^2}-\frac{\lambda\alpha_1^2}{2\mu}\right)(\exp[-i\alpha_1r_3]/r_3-\exp[-i\alpha_1r_4]/r_4)\right.$$

$$\left.+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{x\partial x}\right)(\exp[-i\alpha_2r_3]/r_3-\exp[-i\alpha_2r_4]/r_4)\right]_{y=0}$$

$$\tau_{rz}^{(o)}(y,\xi)=\left[\frac{\partial^2}{\partial x\partial y}(\exp[-i\alpha_1r_1]/r_1+\exp[-i\alpha_1r_2]/r_2)\right]_{x=\beta}$$

$$\sigma_z^{(o)}(x,\xi)=\left[\left(\frac{\partial^2}{\partial y^2}-\frac{\lambda\alpha_1}{2\mu}\right)(\exp[-i\alpha_1r_1]/r_1+\exp[-i\alpha_1r_2]/r_2)\right]_{y=0}$$

$$r_1=R_1/h=[x^2+(y-\xi)^2]^{\frac{1}{2}},\quad r_2=R_2/h=[x^2+(y+\xi)^2]^{\frac{1}{2}}$$

$$r_3=R_3/h=[x^2+(y-1)^2]^{\frac{1}{2}},\quad r_4=R_4/h=[x^2+(y+1)^2]^{\frac{1}{2}}$$

$$\left.\begin{aligned}t_1(\xi)&=\beta\int_{-1}^{0}\tau_{rz}^{(p)}(y,\ \xi)dy+\int_0^\beta x\sigma_z^{(p)}(x,\xi)dx\\t_2(\xi)&=\beta\int_{-1}^{0}\tau_{rz}^{(o)}(y,\ \xi)dy+\int_0^\beta x\sigma_z^{(o)}(x,\xi)dx\\t_3(\xi)&=\beta\int_{-1}^{0}\tau_{rz}^{(p)}(y)dy-\int_0^\beta x\sigma_z^{(p)}(x)dx\end{aligned}\right\}\tag{3.5}$$

显然，要求得积分方程组（3.2）的解析解是非常困难的，为此，本文选用了数值解法。首先我们将（3.2）式的积分上限近似地取为b，然后将积分区间（1，b）离散为n个长为$\varepsilon_1=(b-1)/n$的子区间，并假设在每个子区间内Y_1和Y_2为一常量，则当b和n足够大的时候，(3.2)式可近似地表为：

$$\left.\begin{aligned}&\sum_{j=1}^{n}Y_1^{(j)}\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{11}(x,\ \xi)d\xi+\sum_{j=1}^{n}Y_2^{(j)}\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{12}(x,\xi)d\xi=F_1(x)\\&\sum_{j=1}^{n}Y_1^{(j)}\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{21}(x,\ \xi)d\xi+\sum_{j=1}^{n}Y_2^{(j)}\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{22}(x,\xi)d\xi=F_2(x)\end{aligned}\right\}\tag{3.6}$$

设$x=x_1$，$x_2,\cdots$，x_n，则由（3.6）式可得一组定义在复平面的线性代数方程组：

$$\begin{bmatrix}A_{11}&A_{12}&\cdots&A_{1n}&B_{11}&B_{12}&\cdots&B_{1n}\\A_{21}&A_{22}&\cdots&A_{2n}&B_{21}&B_{22}&\cdots&B_{2n}\\\vdots&\vdots&&\vdots&\vdots&\vdots&&\vdots\\A_{n1}&A_{n2}&\cdots&A_{nn}&B_{n1}&B_{n2}&\cdots&B_{nn}\\C_{11}&C_{12}&\cdots&C_{1n}&D_{11}&D_{12}&\cdots&D_{1n}\\C_{21}&C_{22}&\cdots&C_{2n}&D_{21}&D_{22}&\cdots&D_{2n}\\\vdots&\vdots&&\vdots&\vdots&\vdots&&\vdots\\C_{n1}&C_{n2}&\cdots&C_{nn}&D_{n1}&D_{n2}&\cdots&D_{nn}\end{bmatrix}\begin{bmatrix}Y_1^{(1)}\\Y_1^{(2)}\\\vdots\\Y_1^{(n)}\\Y_2^{(1)}\\Y_2^{(2)}\\\vdots\\Y_2^{(n)}\end{bmatrix}=\begin{bmatrix}F_1^{(1)}\\F_1^{(2)}\\\vdots\\F_1^{(n)}\\F_2^{(1)}\\F_2^{(2)}\\\vdots\\F_2^{(n)}\end{bmatrix}\tag{3.7}$$

式中：

$$\left.\begin{aligned}&A_{ij}=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{11}(x_i,\xi)d\xi,&&B_{ij}=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{12}(x_i,\xi)d\xi\\&C_{ij}=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{21}(x_i,\xi)d\xi,&&D_{ij}=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}K_{22}(x_i,\xi)d\xi\\&F_1^{(i)}=F_1(x_i),&&F_2^{(i)}=F_2(x_i)\end{aligned}\right\}\tag{3.8}$$

解代数方程(3.7)，并将由此解得的$Y_1^{(j)}$和$Y_2^{(j)}$代入（2.8）式可得：

$$P=-Qa_2^2\left[1+\sum_{j=1}^{n}\left(Y_1^{(j)}t_1^{(j)}+Y_2^{(j)}t_1^{(j)}\right)\right]\Big/t_3=-Qa_2^2P_0 \tag{3.9}$$

式中：

$$\left.\begin{aligned}&t_1^{(j)}=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}t_1(\xi)d\xi,\qquad t_2^{(j)}=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}t_2(\xi)d\xi\\&P_0=\left[1+\sum_{j=1}^{n}\left(Y_1^{(j)}t_1^{(j)}+Y_2^{(j)}t_2^{(j)}\right)\right]\Big/t_3\end{aligned}\right\} \tag{3.10}$$

将（3.9）代入（2.3）中关于w和u_r的表达式，并令$z=-h$，则可将表面位移表为：

$$\left.\begin{aligned}&w(r,-h)=\frac{Q}{4\pi\mu h}\left[\sum_{j=1}^{n}Y_1^{(j)}w_j^{(p)}(x)+\sum_{j=1}^{n}Y_2^{(j)}w_j^{(c)}(x)-P_0w^{(p)}(x)\right]\\&u_r(r,-h)=\frac{Q}{4\pi\mu h}\left[\sum_{j=1}^{n}Y_1^{(j)}u_j^{(p)}+\sum_{j=1}^{n}Y_2^{(j)}u_j^{(c)}(x)-P_0u_r^{(p)}(x)\right]\end{aligned}\right\} \tag{3.11}$$

式中：

$$\left.\begin{aligned}&w_j^{(p)}(x)=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}w^{(p)}(x,\xi)d\xi,\qquad w_j^{(c)}(x)=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}w^{(c)}(x,\xi)d\xi\\&u_j^{(p)}(x)=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}u_r^{(p)}(x,\xi)d\xi,\qquad u_j^{(c)}(x)=\int_{1+(j-1)\varepsilon_1}^{1+j\varepsilon_1}u_r^{(c)}(x,\xi)d\xi\\&u_r^{(p)}(x,\xi)=\frac{\partial^2}{\partial y\partial x}\left[(\exp[-i\alpha_1r_1]-\exp[-i\alpha_2r_1])/r_1\right.\\&\qquad\left.-(\exp[-i\alpha_1r_2]-\exp[-i\alpha_2r_2])/r_2\right]_{y=-1}\\&u_r^{(p)}(x)=\frac{\partial^2}{\partial y\partial x}\left[(\exp[-i\alpha_1r_3]-\exp[-i\alpha_2r_3])/r_3\right.\\&\qquad\left.-(\exp[-i\alpha_1r_4]-\exp[-i\alpha_2r_4])/r_4\right]_{y=-1}\\&u_r^{(c)}(x,\xi)=\frac{\partial}{\partial x}(\exp[-i\alpha_1r_1]/r_1+\exp[-i\alpha_1r_2]/r_2)_{y=-1}\\&w^{(p)}(x,\xi)=\left[\frac{\partial^2}{\partial y^2}(\exp[-i\alpha_1r_1]/r_1-\exp[-i\alpha_1r_2]/r_2)\right.\\&\qquad\left.+\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}\right)(\exp[-i\alpha_2r_1]/r_1-\exp[-i\alpha_2r_2]/r_2)\right]_{y=-1}\\&w^{(p)}(x)=\left[\frac{\partial^2}{\partial y^2}(\exp[-i\alpha_1r_3]/r_3-\exp[-i\alpha_1r_4]/r_4)\right.\\&\qquad\left.+\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}\right)(\exp[-i\alpha_2r_3]/r_3-\exp[-i\alpha_2r_4]/r_4)\right]_{y=-1}\\&w^{(c)}(x,\xi)=\frac{\partial}{\partial y}(\exp[-i\alpha_1r_1]/r_1+\exp[-i\alpha_1r_2]/r_2)_{y=-1}\end{aligned}\right\} \tag{3.12}$$

实际上，由（2.3）式还可进一步求得弹性层内部任意点的应力和位移，只是由于在工程问题中，人们通常只对表面位移感兴趣，故本文仅给出了表面位移的数值结果。

四、数 值 结 果

本文分别就（$\lambda_1=0.5\pi/h$，$\lambda_2=2\lambda_1$）和（$\lambda_1=0.3\pi/h$，$\lambda_2=2\lambda_1$）两种情况在IBM—PC微型计算机上作了具体计算，数值结果如图3和图4所示。其中，图3表示自由表面无量纲径向位移幅（$4\pi\mu h u_r/Q$）随径向坐标r的变化规律，图4表示自由表面无量纲竖向位移幅（$4\pi\mu h w/Q$）随r的变化规律。

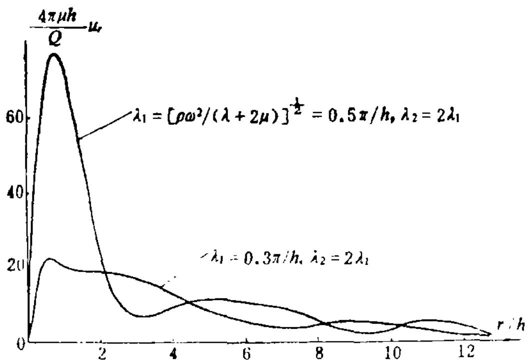

图 3

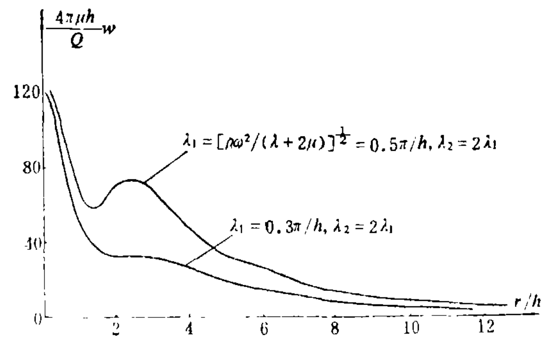

图 4

五、讨 论

（1）由边界条件（2.1）和（2.2）可以看出，在弹性层和刚性基础之间的接触面上，z方向的连续条件是精确满足的，但对r方向的接触条件未加考虑，所以本文的解是近似的。如欲满足r方向的接触条件$u_r=0$（完全接触）或$\sigma_{rz}=0$（光滑接触），则还应增加一种与$z=0$的平面对称的轴对称简谐载荷。在这种情况下，我们需联立求解三个相互偶合的一维非奇异积分方程。

（2）计算过程中应注意β值的选取。由于($z=-h$，$r=0$）处的奇异性，β取值太小会引起计算机上溢；而β取值太大，又将引起计算精度下降。本文参照文献[2]选取$\beta=0.1$。

（3）由图3和图4可以看出，当$r\to\infty$时，u_r和$w\to 0$，除此以外，图3和图4中的位移曲线还具有一定程度的衰减振荡的特征。这种现象可以解释为刚性基础对弹性波的反射在自由表面所产生的影响。

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Dynamic Response of Elastic Layer on Stiff Foundation under Time Harmonic Surface Vertical Concentrated Load

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Abstract

In this paper, the line-load integral equation method proposed in reference [1] is first used for solving the elastodynamic problems. A set of one-dimensional regular integral equations is derived for calculating the dynamic response of elastic layer on stiff foundation under time harmonic surface vertical concentrated load. And the numerical solution of the integral equation is obtained.