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N-S方程组的通用形式及近似因式分解*

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摘 要

基于张量分析，本文在任意曲线坐标系中导出了用原始变量表达的Navier-Stokes（以下简称N-S）方程组弱守恒型通用形式，其中速度采用了逆变或协变分量；与将复杂的坐标变换嵌入该方程组的流行做法相比，本文所得方程组的形式简捷、直观、更适于在贴体曲线坐标系中直接求解．文中详细讨论了这个方程的因式分解过程即将一个n维流动化为n步一维问题来求解，每一步只需解一个块三对角矩阵，从而避开了大型矩阵求逆，提高了解题速度，进一步推广和发展了Beam-Warming的因式分解法．

一、引 言

随着计算精度的提高，需要较精确处理复杂的物面形状及合理的布置计算网点稠密分布，贴体曲线坐标系便逐渐广泛使用了[1~3]，所以研究在任意曲线坐标系下N-S方程组的通用形式就显得格外必要．目前国内外已发表了各种坐标系下N-S方程组的不同形式[4~7]，但绝大多数都没有使用逆变或协变速度分量而是通过坐标变换将直角坐标系下的速度分量嵌入到曲线坐标下N-S方程组中[7]，因此所得方程在形式上便显得有些复杂．本文则直接使用张量分析得到了N-S方程组的不变形式并进一步把它归并为弱守恒型，表达式简洁并具有较强的通用性．文中还详细讨论了将这组方程进行近似因式分解的过程．

二、高阶张量场的梯度、旋度和散度

高阶张量的梯度、旋度和散度等价于算子$\mathbf{e}^i(\partial/\partial x^i)$，$\mathbf{e}^i\times(\partial/\partial x^i)$，$\mathbf{e}^i\cdot(\partial/\partial x^i)$分别作用于该张量．今考察标量场$f$，矢量场$\mathbf{a}$，二阶张量场$\mathbf{e}^j\mathbf{e}^k\tau_{jk}$其相应的公式是：

$$\nabla f=\mathbf{e}^i\nabla_i f=\mathbf{e}^i\partial f/\partial x^i \tag{2.1}$$

$$\nabla\mathbf{a}=\mathbf{e}^i\mathbf{e}^j\nabla_i a_j \tag{2.2}$$

$$\nabla(\mathbf{e}^j\mathbf{e}^k\tau_{jk})=\mathbf{e}^i\mathbf{e}^j\mathbf{e}^k\nabla_i\tau_{jk} \tag{2.3}$$

$$\nabla\times\mathbf{a}=\epsilon^{ijk}\mathbf{e}_k\nabla_i a_j=\frac{1}{\sqrt{g}}\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \partial/\partial x^1 & \partial/\partial x^2 & \partial/\partial x^3\\ a_1 & a_2 & a_3\end{vmatrix} \tag{2.4}$$

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$$\nabla\times(\mathbf{e}^j\mathbf{e}^k\tau_{jk})=\epsilon^{ijk}\mathbf{e}_k\mathbf{e}^a\nabla_i\tau_{ja} \tag{2.5}$$

$$\nabla\cdot\mathbf{a}=\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}a^i)}{\partial x^i}=\nabla_i a^i \tag{2.6}$$

$$\nabla\cdot(\mathbf{e}^j\mathbf{e}^k\tau_{jk})=\mathbf{e}_i\nabla_j\tau^{ji} \tag{2.7}$$

式中$\{a_i\}$，$\{a^i\}$分别为 **a** 的协变分量和逆变分量；$\{\tau_{ij}\}$，$\{\tau^{ij}\}$分别为相应的二阶张量的协变和逆变分量；$\{\mathbf{e}_i\}$，$\{\mathbf{e}^i\}$分别为曲线坐标系的协变基矢量和逆变基矢量；∇_i 为一阶协变导数；g 为度量张量 g_{ij} 的行列式；$\{\epsilon^{ijk}\}$为 Eddington张量；这些公式的详细推导可参阅文献[8]，[11]。

基于上面基本算子对张量的作用，便易推出 Laplace 算子对 f，**a** 及张量 $\mathbf{e}^j\mathbf{e}^k\tau_{jk}$ 作用的公式：

$$\nabla\cdot\nabla f=\mathbf{e}^i\cdot\frac{\partial}{\partial x^i}\left(\mathbf{e}^j\frac{\partial f}{\partial x^j}\right)=g^{ij}\nabla_i\nabla_j f \tag{2.8}$$

$$\nabla\cdot\nabla\mathbf{a}=\mathbf{e}^i\cdot\frac{\partial}{\partial x^i}\left(\mathbf{e}^j\frac{\partial\mathbf{a}}{\partial x^j}\right)=g^{ij}\mathbf{e}^k\nabla_i\nabla_j a_k \tag{2.9}$$

$$\nabla\cdot\nabla\mathbf{e}^j\mathbf{e}^k\tau_{jk}=g^{ij}\mathbf{e}^k\mathbf{e}^a\nabla_i\nabla_j\tau_{ka} \tag{2.10}$$

其中$\{g^{ij}\}$为度量张量。

三、曲线坐标系中粘性应力张量的表示

设 **V** 为流体速度，变形率张量[9]

$$\varepsilon=\{\nabla\mathbf{V}+(\nabla\mathbf{V})_c\}/2 \tag{3.1}$$

在任意曲线坐标系中为：

$$\varepsilon=\mathbf{e}^i\mathbf{e}^j(\nabla_i v_j+\nabla_j v_i)/2=\mathbf{e}_i\mathbf{e}_j(\nabla^i v^j+\nabla^j v^i)/2=\mathbf{e}^i\mathbf{e}^j\varepsilon_{ij}=\mathbf{e}_i\mathbf{e}_j\varepsilon^{ij} \tag{3.2}$$

式中$\{v_i\}$，$\{v^i\}$分别为 **V** 的协变和逆变分量；$\{\varepsilon_{ij}\}$，$\{\varepsilon^{ij}\}$分别为变形率张量的协变分量和逆变分量；显然 ε_{ij}，ε^{ij} 都是对称的。应力张量是变形率张量的线性函数[10]，可表为[11,8]：

$$\Pi=\mathbf{e}_i\mathbf{e}_j\tau^{ij}=\left[-\left(P+\frac{2}{3}\mu\nabla_i v^i\right)g^{ij}+2\mu\varepsilon^{ij}\right]\mathbf{e}_i\mathbf{e}_j=(\tau'^{ij}-g^{ij}P)\mathbf{e}_i\mathbf{e}_j \tag{3.3}$$

其中$\{\tau^{ij}\}$为应力张量的逆变分量；$\{\tau'^{ij}\}$为粘性应力张量的逆变分量；P 为正压力，μ 为粘性系数；显然 τ^{ij}，τ'^{ij} 也都是对称张量。将(3.2)式代到(3.3)式便得到粘性应力张量逆变分量的表达式：

$$\begin{aligned}\tau'^{ij}&=\mu(\nabla^i v^j+\nabla^j v^i)-(2/3)\mu g^{ij}\nabla_k v^k\\&=\mu(g^{i\alpha}g^{j\beta}+g^{i\beta}g^{j\alpha}-(2/3)g^{ij}g^{\alpha\beta})\nabla_\alpha v_\beta\end{aligned} \tag{3.4}$$

其中∇^i，∇_k 分别为一阶逆变导数和协变导数。

四、弱守恒型N-S方程组的通用形式

连续方程：

$$\frac{\partial\rho}{\partial t}+\nabla_i(\rho v^i)\equiv\frac{\partial\rho}{\partial t}+\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\rho v^i)}{\partial x^i}=0 \tag{4.1}$$

其中 ρ 为密度，t 为时间；v^i 为 **V** 的逆变分量；

动量方程：

$$\mathbf{e}^i\rho\left(\frac{\partial v_i}{\partial t}+v^j\nabla_j v_i\right)\equiv\mathbf{e}^i\left[\rho\frac{\partial v_i}{\partial t}+\rho v^j\nabla_j v_i+v_i\nabla_j(\rho v^j)+v_i\frac{\partial\rho}{\partial t}\right]$$

$$=\mathbf{e}^i\left[\frac{\partial(\rho v_i)}{\partial t}+\nabla_j(\rho v_i v^j)\right]=\mathbf{e}^i\nabla_j\tau^{j}_{\cdot i}$$

其中$\{\tau^{j}_{\cdot i}\}$为应力张量的混合分量；将上式两边同时点乘 $\mathbf{e}_k$ 后得

$$\partial(\rho v_k)/\partial t+\nabla_j(\rho v^j v_k-\tau^{j}_{\cdot k})=0$$

上式两边同乘 g^{ik} 后得到：

$$\partial(\rho v^i)/\partial t+\nabla_j(\rho v^j v^i-\tau^{ji})=0 \tag{4.2}$$

能量方程：

令 e，$\tilde{e}$ 分别为单位体积和单位质量流体内所具有的广义内能[1,10]，能量方程为：

$$\rho\frac{d\tilde{e}}{dt}=\nabla\cdot(\Pi\cdot\mathbf{V})+\nabla\cdot(\lambda\nabla T)=\rho\frac{\partial\tilde{e}}{\partial t}+\rho v^i\nabla_i\tilde{e}+\tilde{e}\left[\frac{\partial\rho}{\partial t}+\nabla_i(\rho v^i)\right]$$

$$=\partial e/\partial t+\nabla_i(ev^i) \tag{4.3}$$

其中 λ 为流体的热传导系数；T 为温度；又因

$$\nabla\cdot(\Pi\cdot\mathbf{V})=\nabla_i(\tau^{ij}v_j) \tag{4.4}$$

$$\nabla\cdot(\lambda\nabla T)=\nabla_i(\lambda g^{ij}\partial T/\partial x^j) \tag{4.5}$$

于是(4.3)式可表为：

$$\partial e/\partial t+\nabla_i(ev^i-\tau^{ij}v_j-\lambda g^{ij}\partial T/\partial x^j)=0 \tag{4.6}$$

将(4.1)、(4.2)、(4.6)式合并，便得出弱守恒 N-S 方程组的通用形式[8]：

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho v^1\\ \rho v^2\\ \rho v^3\\ e\end{bmatrix}+\frac{\partial}{\partial x^i}\begin{bmatrix}\rho v^i\\ \rho v^1v^i+g^{1i}P\\ \rho v^2v^i+g^{2i}P\\ \rho v^3v^i+g^{3i}P\\ (e+P)v^i\end{bmatrix}=\nabla_i\begin{bmatrix}0\\ \tau'^{1i}\\ \tau'^{2i}\\ \tau'^{3i}\\ \tau'^{ij}v_j+\lambda g^{ij}\partial T/\partial x^j\end{bmatrix}+\begin{bmatrix}-\Gamma^{i}_{ji}\rho v^j\\ -\rho v^iv^j\Gamma^{1}_{ij}-\rho v^1v^j\Gamma^{i}_{ij}\\ -\rho v^iv^j\Gamma^{2}_{ij}-\rho v^2v^j\Gamma^{i}_{ij}\\ -\rho v^iv^j\Gamma^{3}_{ij}-\rho v^3v^j\Gamma^{i}_{ij}\\ -\Gamma^{i}_{ji}(e+P)v^j\end{bmatrix} \tag{4.7}$$

其中 P 为正压力；v^i，g^{ij}，τ'^{ij} 分别为速度、度量张量和粘性应力张量的逆变分量；Γ^i_{jk}为第二类 Christoffel 记号；$i,j,k=1\sim3$。

五、N-S方程的近似因式分解

在本节讨论中引进下面的专用符号：

$$\left.\begin{aligned}&U\equiv\begin{bmatrix}\rho\\ \rho v^1\\ \rho v^2\\ \rho v^3\\ e\end{bmatrix},\qquad E(U)\equiv\begin{bmatrix}\rho v^1\\ \rho v^1v^1+g^{11}P\\ \rho v^2v^1+g^{21}P\\ \rho v^3v^1+g^{31}P\\ (e+P)v^1\end{bmatrix}\\ &F(U)\equiv\begin{bmatrix}\rho v^2\\ \rho v^1v^2+g^{12}P\\ \rho v^2v^2+g^{22}P\\ \rho v^3v^2+g^{32}P\\ (e+P)v^2\end{bmatrix},\quad G(U)\equiv\begin{bmatrix}\rho v^3\\ \rho v^1v^3+g^{13}P\\ \rho v^2v^3+g^{23}P\\ \rho v^3v^3+g^{33}P\\ (e+P)v^3\end{bmatrix}\\ &m\equiv\rho v^1,\ n\equiv\rho v^2,\ b\equiv\rho v^3,\ (V)^2\equiv\mathbf{V}\cdot\mathbf{V}\end{aligned}\right\} \tag{5.1}$$

其中正压力 P 可表为：

$$P=(\gamma-1)[e-(1/2\rho)(g_{11}m^2+g_{22}n^2+g_{33}b^2+2g_{12}mn+2g_{13}mb+2g_{23}nb)] \quad (5.2)$$

这里γ为流体的比热比；由于E,F,G都是U的一次齐次函数，故有关系式

$$E=AU,\quad F=BU,\quad G=CU \quad (5.3)$$

其中A，B，C为Jacobian矩阵：

$$A=\frac{\partial E}{\partial U}\equiv\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}g^{11}(V)^2-(v^1)^2 & 2v^1-(\gamma-1)g^{11}a_1 & -(\gamma-1)g^{11}a_2 & -(\gamma-1)g^{11}a_3 & (\gamma-1)g^{11} \\ \frac{\gamma-1}{2}g^{12}(V)^2-v^1v^2 & v^2-(\gamma-1)g^{12}a_1 & v^1-(\gamma-1)g^{12}a_2 & -(\gamma-1)g^{12}a_3 & (\gamma-1)g^{12} \\ \frac{\gamma-1}{2}g^{13}(V)^2-v^1v^3 & v^3-(\gamma-1)g^{13}a_1 & -(\gamma-1)g^{13}a_2 & v^1-(\gamma-1)g^{13}a_3 & (\gamma-1)g^{13} \\ a_4 & a_5 & -(\gamma-1)v^1a_2 & -(\gamma-1)v^1a_3 & \gamma v^1 \end{bmatrix}$$

$$B=\frac{\partial F}{\partial U}\equiv\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{\gamma-1}{2}g^{12}(V)^2-v^1v^2 & v^2-(\gamma-1)g^{12}a_1 & v^1-(\gamma-1)g^{12}a_2 & -(\gamma-1)g^{12}a_3 & (\gamma-1)g^{12} \\ \frac{\gamma-1}{2}g^{22}(V)^2-(v^2)^2 & -(\gamma-1)g^{22}a_1 & 2v^2-(\gamma-1)g^{22}a_2 & -(\gamma-1)g^{22}a_3 & (\gamma-1)g^{22} \\ \frac{\gamma-1}{2}g^{32}(V)^2-v^3v^2 & -(\gamma-1)g^{32}a_1 & v^3-(\gamma-1)g^{32}a_2 & v^2-(\gamma-1)g^{32}a_3 & (\gamma-1)g^{32} \\ a_6 & -(\gamma-1)v^2a_1 & a_7 & -(\gamma-1)v^2a_3 & \gamma v^2 \end{bmatrix}$$

$$C=\frac{\partial G}{\partial U}\equiv\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ \frac{\gamma-1}{2}g^{13}(V)^2-v^1v^3 & v^3-(\gamma-1)g^{13}a_1 & -(\gamma-1)g^{13}a_2 & v^1-(\gamma-1)g^{13}a_3 & (\gamma-1)g^{13} \\ \frac{\gamma-1}{2}g^{23}(V)^2-v^2v^3 & -(\gamma-1)g^{23}a_1 & v^3-(\gamma-1)g^{23}a_2 & v^2-(\gamma-1)g^{23}a_3 & (\gamma-1)g^{23} \\ \frac{\gamma-1}{2}g^{33}(V)^2-(v^3)^2 & -(\gamma-1)g^{33}a_1 & -(\gamma-1)g^{33}a_2 & 2v^3-(\gamma-1)g^{33}a_3 & (\gamma-1)g^{33} \\ a_8 & -(\gamma-1)v^3a_1 & -(\gamma-1)v^3a_2 & a_9 & \gamma v^3 \end{bmatrix}$$

式中 $$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \equiv \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{Bmatrix} v^1 \\ v^2 \\ v^3 \end{Bmatrix}$$

$$a_4 \equiv (\gamma-1)(V)^2 v^1 - \frac{\gamma e v^1}{\rho}, \quad a_5 \equiv \frac{\gamma e}{\rho} - \frac{\gamma-1}{2}(V)^2 - (\gamma-1)v^1 a_1$$

$$a_6 \equiv (\gamma-1)(V)^2 v^2 - \frac{\gamma e v^2}{\rho}, \quad a_7 \equiv \frac{\gamma e}{\rho} - \frac{\gamma-1}{2}(V)^2 - (\gamma-1)v^2 a_2$$

$$a_8 \equiv (\gamma-1)(V)^2 v^3 - \frac{\gamma e v^3}{\rho}, \quad a_9 \equiv \frac{\gamma e}{\rho} - \frac{\gamma-1}{2}(V)^2 - (\gamma-1)v^3 a_3$$

为了研究粘性应力张量的协变导数，将(3.4)式改写为：

$$\tau'^{ij} = b^{(ij)}_{k\alpha} \partial v^k / \partial x^\alpha - a^{(ij)}_{\alpha\beta} g_{k\sigma} \Gamma^k_{\beta\alpha} v^\sigma \tag{5.4}$$

$$a^{(ij)}_{\alpha\beta} \equiv \mu(g^{i\alpha}g^{j\beta} + g^{i\beta}g^{j\alpha} - (2/3)g^{ij}g^{\alpha\beta}), \quad b^{(ij)}_{k\alpha} \equiv g_{\alpha\beta} a^{(ij)}_{k\beta}$$

式中 k，α，β，σ 为求和指标；显然有

$$a^{(ij)}_{\alpha\beta} \equiv a^{(ij)}_{\beta\alpha}, \qquad b^{(ij)}_{\alpha\beta} \equiv b^{(ij)}_{\beta\alpha} \tag{5.5}$$

于是(4.7)式可整理为：

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v^1 \\ \rho v^2 \\ \rho v^3 \\ e \end{bmatrix} + \frac{\partial}{\partial x^i}\begin{bmatrix} \rho v^i \\ \rho v^1 v^i + g^{1i}P \\ \rho v^2 v^i + g^{2i}P \\ \rho v^3 v^i + g^{3i}P \\ (e+P)v^i \end{bmatrix} = \frac{\partial H(U, U_{x^1})}{\partial x^1} + \frac{\partial M(U, U_{x^2})}{\partial x^2}$$
$$+ \frac{\partial N(U, U_{x^3})}{\partial x^3} + \frac{\partial S_1(U, U_{x^2}, U_{x^3})}{\partial x^1} + \frac{\partial S_2(U, U_{x^1}, U_{x^3})}{\partial x^2}$$
$$+ \frac{\partial S_3(U, U_{x^1}, U_{x^2})}{\partial x^3} + \frac{\partial Q_i(U)}{\partial x^i} + R(U, U_{x^1}, U_{x^2}, U_{x^3}) \tag{5.6}$$

式中

$$H(U, U_{x^1}) \equiv [0, b^{(11)}_{k1}\partial v^k/\partial x^1, \ b^{(12)}_{k1}\partial v^k/\partial x^1, \ b^{(13)}_{k1}\partial v^k/\partial x^1, \ f_1]^T$$

$$M(U, U_{x^2}) \equiv [0, b^{(21)}_{k2}\partial v^k/\partial x^2, \ b^{(22)}_{k2}\partial v^k/\partial x^2, \ b^{(23)}_{k2}\partial v^k/\partial x^2, \ f_2]^T$$

$$N(U, U_{x^3}) \equiv [0, b^{(31)}_{k3}\partial v^k/\partial x^3, \ b^{(32)}_{k3}\partial v^k/\partial x^3, \ b^{(33)}_{k3}\partial v^k/\partial x^3, \ f_3]^T$$

$$Q_i(U) \equiv -[0, \ g_{jk}a^{(i1)}_{\alpha\beta}\Gamma^k_{\alpha\beta}v^j, \ g_{jk}a^{(i2)}_{\alpha\beta}\Gamma^k_{\alpha\beta}v^j, \ g_{jk}a^{(i3)}_{\alpha\beta}\Gamma^k_{\alpha\beta}v^j, f_7]^T$$

$$S_1(U, U_{x^2}, U_{x^3}) \equiv \begin{bmatrix} 0 \\ b^{(11)}_{k2}\partial v^k/\partial x^2 + b^{(11)}_{k3}\partial v^k/\partial x^3 \\ b^{(12)}_{k2}\partial v^k/\partial x^2 + b^{(12)}_{k3}\partial v^k/\partial x^3 \\ b^{(13)}_{k2}\partial v^k/\partial x^2 + b^{(13)}_{k3}\partial v^k/\partial x^3 \\ f_4 \end{bmatrix}$$

$$S_2(U, U_{x^1}, U_{x^3}) \equiv \begin{bmatrix} 0 \\ b^{(21)}_{k1}\partial v^k/\partial x^1 + b^{(21)}_{k3}\partial v^k/\partial x^3 \\ b^{(22)}_{k1}\partial v^k/\partial x^1 + b^{(22)}_{k3}\partial v^k/\partial x^3 \\ b^{(23)}_{k1}\partial v^k/\partial x^1 + b^{(23)}_{k3}\partial v^k/\partial x^3 \\ f_5 \end{bmatrix}$$

$$S_3(U,U_{x^1},U_{x^2})\equiv\begin{bmatrix}0\\ b_{k1}^{(31)}\partial v^k/\partial x^1+b_{k2}^{(31)}\partial v^k/\partial x^2\\ b_{k1}^{(32)}\partial v^k/\partial x^1+b_{k2}^{(32)}\partial v^k/\partial x^2\\ b_{k1}^{(33)}\partial v^k/\partial x^1+b_{k2}^{(33)}\partial v^k/\partial x^2\\ f_6\end{bmatrix}$$

$$R(U,U_{x^1},U_{x^2},U_{x^3})\equiv-\begin{bmatrix}\rho\Gamma_{ij}^i v^j\\ \rho\Gamma_{ij}^i v^j v^1+\rho\Gamma_{ij}^1 v^i v^j-\Gamma_{ij}^i\tau'^{j1}-\Gamma_{ij}^1\tau'^{ij}\\ \rho\Gamma_{ij}^i v^j v^2+\rho\Gamma_{ij}^2 v^i v^j-\Gamma_{ij}^i\tau'^{j2}-\Gamma_{ij}^2\tau'^{ij}\\ \rho\Gamma_{ij}^i v^j v^3+\rho\Gamma_{ij}^3 v^i v^j-\Gamma_{ij}^i\tau'^{j3}-\Gamma_{ij}^3\tau'^{ij}\\ (e+P)\Gamma_{ij}^i v^j-\Gamma_{ij}^i\tau'^{jk}v_k-\lambda\Gamma_{ij}^i g^{jk}\partial T/\partial x^k\end{bmatrix}$$

其中i,j,k,α,β为求和指标；上注脚T表示矩阵转置；符号$f_1\sim f_7$的定义如下：

$$f_1\equiv g_{j\beta}b_{k1}^{(1j)}v^\beta\partial v^k/\partial x^1+\lambda g^{11}\partial T/\partial x^1$$

$$f_2\equiv g_{j\beta}b_{k2}^{(2j)}v^\beta\partial v^k/\partial x^2+\lambda g^{22}\partial T/\partial x^2$$

$$f_3\equiv g_{j\beta}b_{k3}^{(3j)}v^\beta\partial v^k/\partial x^3+\lambda g^{33}\partial T/\partial x^3$$

$$f_4\equiv g_{j\beta}b_{k2}^{(1j)}v^\beta\frac{\partial v^k}{\partial x^2}+g_{j\beta}b_{k3}^{(1j)}v^\beta\frac{\partial v^k}{\partial x^3}+\lambda g^{12}\frac{\partial T}{\partial x^2}+\lambda g^{13}\frac{\partial T}{\partial x^3}$$

$$f_5\equiv g_{j\beta}b_{k1}^{(2j)}v^\beta\frac{\partial v^k}{\partial x^1}+g_{j\beta}b_{k3}^{(2j)}v^\beta\frac{\partial v^k}{\partial x^3}+\lambda g^{21}\frac{\partial T}{\partial x^1}+\lambda g^{23}\frac{\partial T}{\partial x^3}$$

$$f_6\equiv g_{j\beta}b_{k1}^{(3j)}v^\beta\frac{\partial v^k}{\partial x^1}+g_{j\beta}b_{k2}^{(3j)}v^\beta\frac{\partial v^k}{\partial x^2}+\lambda g^{31}\frac{\partial T}{\partial x^1}+\lambda g^{32}\frac{\partial T}{\partial x^2}$$

$$f_7\equiv a_{\alpha\beta}^{(ij)}\Gamma_{\alpha\beta}^k g_{k\sigma}v_j v^\sigma$$

在上面的符号定义式中要注意对指标j求和；符号U_{x^1}，U_{x^2}，U_{x^3}分别表示U对x^1，x^2，x^3的普通偏导数．如果将U的时间微分用下列差分近似[5]

$$\Delta U^n=\frac{\theta\Delta t}{1+\xi}\frac{\partial}{\partial t}\Delta U^n+\frac{\Delta t}{1+\xi}\frac{\partial}{\partial t}U^n+\frac{\xi}{1+\xi}\Delta U^{n-1}+O\left[\left(\theta-\frac{1}{2}-\xi\right)\Delta t^2+\Delta t^3\right]\tag{5.7}$$

式中上标n表示时间步数；θ和ξ为格式选择参数，例如取$\theta=0$，$\xi=-1/2$时便得到显式的跳步格式；注意在上式及以后章节的讨论中，符号Δ不再表示Laplace算子而表示增量，例如$\Delta U^n\equiv U^{n+1}-U^n$；如果将$H,M,N,Q_i$作Taylor级数展开整理后可得下面形式：

$$\Delta H^n=\left(D_1^n-\frac{\partial P_1^n}{\partial x^1}\right)\Delta U^n+\frac{\partial}{\partial x^1}(P_1\Delta U)^n+O(\Delta t^2)$$

$$\Delta M^n=\left(D_2^n-\frac{\partial P_2^n}{\partial x^2}\right)\Delta U^n+\frac{\partial}{\partial x^2}(P_2\Delta U)^n+O(\Delta t^2)$$

$$\Delta N^n=\left(D_3^n-\frac{\partial P_3^n}{\partial x^3}\right)\Delta U^n+\frac{\partial}{\partial x^3}(P_3\Delta U)^n+O(\Delta t^2)$$

$$\Delta Q_i^n=W_i^n\Delta U^n+O(\Delta t^2)$$

其中 $D_1\equiv\partial H/\partial U$，$D_2\equiv\partial M/\partial U$，$D_3\equiv\partial N/\partial U$

$P_1 \equiv \partial H/\partial U_{x^1}$, $P_2 \equiv \partial M/\partial U_{x^2}$, $P_3 \equiv \partial N/\partial U_{x^3}$

$W_i \equiv \partial Q_i/\partial U \qquad (i=1,2,3)$

将(5.6)代入(5.7)并注意关系式(5.3)，整理后可得

$$\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^1}\left(A-D_1-W_1+\frac{\partial P_1}{\partial x^1}\right)^n-\frac{\partial^2}{(\partial x^1)^2}P_1^n+\frac{\partial}{\partial x^2}\left(B-D_2-W_2+\frac{\partial P_2}{\partial x^2}\right)^n\right.\right.$$
$$\left.\left.-\frac{\partial^2}{(\partial x^2)^2}P_2^n+\frac{\partial}{\partial x^3}\left(C-D_3-W_3+\frac{\partial P_3}{\partial x^3}\right)^n-\frac{\partial^2}{(\partial x^3)^2}P_3^n\right]\right\}\Delta U^n=\tilde{R} \tag{5.8}$$

其中 I 为么矩阵；上标 n 仍表示时间步数；$\tilde{R}$ 的定义是：

$$\tilde{R}\equiv\frac{\bar{\theta}\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^1}(\Delta S_1)^{n-1}+\frac{\partial}{\partial x^2}(\Delta S_2)^{n-1}+\frac{\partial}{\partial x^3}(\Delta S_3)^{n-1}+(\Delta R)^{n-1}\right]$$
$$+\frac{\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^1}(-E+H+S_1)^n+\frac{\partial}{\partial x^2}(-F+M+S_2)^n+\frac{\partial}{\partial x^3}(-G+N+S_3)\right.$$
$$\left.+\frac{\partial Q_i^n}{\partial x^i}+R^n\right]+\frac{\xi}{1+\xi}\Delta U^{n-1}+O\left[\left(\theta-\frac{1}{2}-\xi\right)\Delta t^2+\Delta t^3\right]$$

注意(5.8)式采用了文献[5]中(6)式的约定写法例如式中的

$$\left[\frac{\partial}{\partial x^1}(A-D_1)^n\right]\Delta U^n \quad \text{则表示} \frac{\partial}{\partial x^1}[(A-D_1)^n\Delta U^n]$$

的意思。参照文献[5]，将(5.8)式作下面近似因式分解：

$$\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^1}\left(A-D_1-W_1+\frac{\partial P_1}{\partial x^1}\right)^n-\frac{\partial^2}{(\partial x^1)^2}P_1^n\right]\right\}$$
$$\cdot\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^2}\left(B-D_2-W_2+\frac{\partial P_2}{\partial x^2}\right)^n-\frac{\partial^2}{(\partial x^2)^2}P_2^n\right]\right\}$$
$$\cdot\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^3}\left(C-D_3-W_3+\frac{\partial P_3}{\partial x^3}\right)^n-\frac{\partial^2}{(\partial x^3)^2}P_3^n\right]\right\}\Delta U^n$$
$$=\tilde{R}+O(\Delta t^3) \tag{5.9}$$

此式把一个三维流动化为三个一维问题求解，具体计算时分别沿 x^1，x^2 与 x^3 方向进行：

$$\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^1}\left(A-D_1-W_1+\frac{\partial P_1}{\partial x^1}\right)^n-\frac{\partial^2}{(\partial x^1)^2}P_1^n\right]\right\}\Delta\bar{\bar{U}}=\tilde{R} \tag{5.10a}$$

$$\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^2}\left(B-D_2-W_2+\frac{\partial P_2}{\partial x^2}\right)^n-\frac{\partial^2}{(\partial x^2)^2}P_2^n\right]\right\}\Delta\bar{U}=\Delta\bar{\bar{U}} \tag{5.10b}$$

$$\left\{I+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x^3}\left(C-D_3-W_3+\frac{\partial P_3}{\partial x^3}\right)^n-\frac{\partial^2}{(\partial x^3)^2}P_3^n\right]\right\}\Delta U=\Delta\bar{U} \tag{5.10c}$$

$$U^{n+1}=U^n+\Delta U^n \tag{5.10d}$$

求解块三对角矩阵是近似因式分解法解三维 N-S 方程组的主要工作量，它避免了大型矩阵求逆，节省了计算机内存且提高了解题速度。

应指出，这里导出的(5.10)式是文献[5]方法的进一步推广和发展，是在非正交曲线坐标系下用近似因式分解法解 N-S 方程的最一般形式，将此方程应用于二维直角坐标系便得到与文献[5]完全一致的结果。另外在参变量梯度变化较大且易出现数值求解不稳定的区域还要在(5.9)式右边加入数值耗散项，例如可参照文献[5]或[12]加入四阶或适当的数值耗散项。

吴仲华教授一直关心和指导着本文工作的进展，卞荫贵教授也给予了细致的指导并阅读

了全文，在此作者对他们表示深切谢意。

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On General Form of Navier-Stokes Equations and Implicit Factored Scheme

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Abstract

A general weak conservative form of Navier-Stokes equations expressed with respect to non-orthogonal curvilinear coordinates and with primitive variables was obtained by using tensor analysis technique, where the contravariant and covariant velocity components were employed. Compared with the current coordinate transformation method, the established equations are concise and forthright, and they are more convenient to be used for solving problems in body-fitted curvilinear coordinate system. An implicit factored scheme for solving the equations is presented with detailed discussions in this paper. For *n*-dimensional flow the algorithm requires *n* steps and for each step only a block tridiagonal matrix equation needs to be solved. It avoids inverting the matrix for large systems of equations and enhances the speed of arithmetic. In this study, the Beam-Warming's implicit factored scheme is extended and developed in non-orthogonal curvilinear coordinate system.