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集中载荷作用下开顶扁球壳的非线性稳定问题*

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摘　　要

本文使用修正迭代法研究了具有硬中心的边缘固定的开顶扁球壳在中心集中载荷作用下的轴对称非线性稳定问题，得到了决定上、下临界载荷的二次近似解析公式。

一、引　言

在建筑和精密仪器的弹性元件等工程中，常使用具有硬中心的边缘固定的开顶扁球壳。这种壳体在中心集中载荷作用时，在一定条件下会丧失稳定性。对于建筑工程，需要防止这种现象发生；对于精密仪器，则应利用失稳所产生的跳跃来作为自动控制的信号。由于本问题涉及了非线性的数学问题，结构亦较复杂，所以给研究带来了巨大的困难，因而至今尚无人讨论过。

关于开顶扁球壳的非线性稳定问题，以往仅有刘人怀[1~4]、Tillman[5]等人研究过，他们涉及的是无硬中心的开顶扁球壳。

本文使用修正迭代法来求解本问题的非线性微分方程的边值问题，获得了较精确的解析解。这一方法是叶开沅和刘人怀于1965年在研究扁球壳的非线性稳定问题中提出的[1,6,7]，随后应用于板壳的一系列非线性问题，均获得十分满意的结果。

本文所获得的结果对于工程设计说来有现实意义。

二、基 本 方 程

考虑一个在中心集中载荷P作用下，厚度为h，中曲面半径为R，内、外缘半径分别为b,a的具有硬中心的开顶扁球壳(图1)。

按照在均布载荷q作用下的扁球壳的非线性弯曲理论[1,2]，我们有下列关于挠度w和径向薄膜内力N_r所满足的基本方程：

* 本文曾在1986年九华山第三届华东固体力学学术讨论会上宣读。

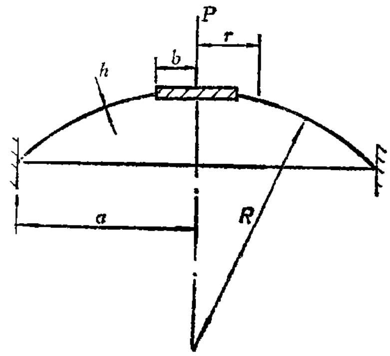

图1 具有硬中心的开顶扁球壳

$$\left.\begin{aligned}&\frac{D}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r\vartheta)-\frac{1}{r}\frac{d}{dr}[rN_r(\theta+\vartheta)]=q\\&\frac{r}{Eh}\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)+\vartheta\left(\theta+\frac{1}{2}\vartheta\right)=0\end{aligned}\right\}\tag{2.1a,b}$$

其中r是扁球中曲面点至对称中心轴的距离，ϑ和θ分别是壳体经线方向弧的旋转角和倾斜角，E是弹性模量，ν是泊松比，D是抗弯刚度，

$$\vartheta=\frac{dw}{dr},\qquad \theta=\frac{r}{R},\qquad D=\frac{Eh^3}{12(1-\nu^2)}\tag{2.2}$$

求得w和N_r后，便可按下列公式计算径向薄膜位移u，径向弯矩M_r和径向剪力Q_r

$$\left.\begin{aligned}&u=\frac{r}{Eh}\left[(1-\nu)N_r+r\frac{dN_r}{dr}\right],\qquad M_r=-D\left(\frac{d\vartheta}{dr}+\frac{\nu}{r}\vartheta\right)\\&Q_r=-D\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r\vartheta)\end{aligned}\right\}\tag{2.3a,b,c}$$

先将方程(2.1a)两端乘以rdr，然后积分一次，得

$$Dr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r\vartheta)-rN_r(\theta+\vartheta)=F(r)\tag{2.4}$$

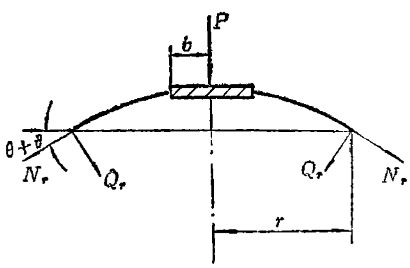

图2 载荷函数$F(r)$的确定

其中

$$F(r)=\int qrdr+C\tag{2.5}$$

应用式(2.3c)，方程(2.4)成为

$$r[Q_r+N_r(\theta+\vartheta)]=-F(r)\tag{2.6}$$

现在，我们需要确定载荷函数$F(r)$。为此，用圆锥曲面沿半径为r的圆周从壳体中切割出如图2所示的部分壳体，将作用在这部分壳体上的全部力投影到对称轴上，便有

$$2\pi r[Q_r\cos(\theta+\vartheta)+N_r\sin(\theta+\vartheta)]+P=0\tag{2.7}$$

根据角$(\theta+\vartheta)$微小的条件，式(2.7)可简化为

$$r[Q_r+N_r(\theta+\vartheta)]=-\frac{P}{2\pi}\tag{2.8}$$

将式(2.8)同式(2.6)比较，就得

$$F(r)=\frac{P}{2\pi}\tag{2.9}$$

应用这一结果，方程(2.4)成为

$$Dr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r\vartheta)-rN_r(\theta+\vartheta)=\frac{P}{2\pi}\tag{2.10}$$

于是，方程(2.10)和(2.1b)就组成具有硬中心的开顶扁球壳在中心集中载荷作用下的轴对称大挠度弯曲方程。关于相应的边界条件为

当$r=a$时，外边缘夹紧固定：

$$w=0,\qquad \vartheta=0,\qquad u=0\tag{2.11}$$

当$r=b$时，内边缘被固定在可上下移动的不变形的硬中心上：

$$\vartheta=0,\quad u=0 \tag{2.12}$$

为了简化计算，我们引入下列无量纲量：

$$\left.\begin{aligned}&\rho=\frac{r}{a},\quad \alpha=\frac{b}{a},\quad y=\sqrt{12\lambda_1\lambda_2}\,\frac{w}{h},\quad \varphi=-\frac{dy}{d\rho}\\&S=\frac{a^2}{D}\rho N_r,\quad k=\sqrt{12\lambda_1\lambda_2}\,\frac{a^2}{Rh},\quad Q=\sqrt{12\lambda_1\lambda_2}\,\frac{a^2P}{2\pi Dh}\end{aligned}\right\} \tag{2.13}$$

其中

$$\lambda_1=1-\nu,\quad \lambda_2=1+\nu \tag{2.14}$$

利用这些无量纲量，方程(2.10)，(2.1b)和边界条件(2.11)，(2.12)简化为

$$L(\rho\varphi)=S(\varphi-k\rho)-Q,\quad L(\rho S)=\varphi\left(k\rho-\frac{1}{2}\varphi\right) \tag{2.15a,b}$$

当$\rho=1$时， $y=0$，$\varphi=0$，$\rho\dfrac{dS}{d\rho}-\nu S=0$ (2.16)

当$\rho=\alpha$时， $\varphi=0$，$\rho\dfrac{dS}{d\rho}-\nu S=0$ (2.17)

其中

$$L=\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho} \tag{2.18}$$

这样，我们的问题便化为在边界条件(2.16)和(2.17)下求解非线性微分方程组(2.15)。

三、边值问题的求解

我们用修正迭代法解边值问题(2.15)～(2.17)。在一次近似中，我们略去方程(2.15a)中含S的项，便得到如下的线性边值问题：

$$L(\rho\varphi_1)=-Q,\quad L(\rho S_1)=\varphi_1\left(k\rho-\frac{1}{2}\varphi_1\right) \tag{3.1a,b}$$

当$\rho=1$时， $y_1=0$，$\varphi_1=0$，$\rho\dfrac{dS_1}{d\rho}-\nu S_1=0$ (3.2a,b,c)

当$\rho=\alpha$时， $\varphi_1=0$， $\rho\dfrac{dS_1}{d\rho}-\nu S_1=0$ (3.3a,b)

将方程(3.1a)积分，利用边界条件(3.2b)和(3.3a)，得小挠度理论的解

$$\varphi_1=\frac{Q}{2(1-\alpha^2)}\left[(\alpha^2-1)\rho\ln\rho-\alpha^2\ln\alpha\left(\rho-\frac{1}{\rho}\right)\right] \tag{3.4}$$

我们以扁球壳的无量纲内边缘挠度Y_m为迭代参数：

$$Y_m=\sqrt{12\lambda_1\lambda_2}\,\frac{w}{h}\bigg|_{r=b} \tag{3.5}$$

再应用式(2.13)和(3.2a)，可得

$$Y_m=\int_\alpha^1\varphi d\rho \tag{3.6}$$

将解(3.4)代入上式，得

$$Q=2(1-\alpha^2)\beta Y_m \tag{3.7}$$

其中

$$\beta=\left(\frac{1}{4}\alpha^4-\alpha^2\ln^2\alpha-\frac{1}{2}\alpha^2+\frac{1}{4}\right)^{-1} \tag{3.8}$$

将式(3.7)代入式(3.4)，有

$$\varphi_1=\beta Y_m\left[(\alpha^2-1)\rho\ln\rho-\alpha^2\ln\alpha\left(\rho-\frac{1}{\rho}\right)\right] \tag{3.9}$$

利用式(3.9)以及边界条件(3.2c)和(3.3b)，我们积分方程(3.1b)，得S的一次近似解为

$$\begin{aligned}S_1=&k\beta Y_m\left[\frac{1}{8}(\alpha^2-1)\left(\rho^3\ln\rho-\frac{3}{4}\rho^3\right)-\frac{1}{2}\alpha^2\ln\alpha\left(\frac{1}{4}\rho^3-\rho\ln\rho\right)+A_1\rho+B_1\frac{1}{\rho}\right]\\&-\beta^2Y_m^2\left[\frac{1}{16}(\alpha^2-1)^2\left(\rho^3\ln^2\rho-\frac{3}{2}\rho^3\ln\rho+\frac{7}{8}\rho^3\right)\right.\\&-\frac{1}{4}(\alpha^2-1)\alpha^2\ln\alpha\left(\frac{1}{2}\rho^3\ln\rho-\frac{3}{8}\rho^3-\rho\ln^2\rho+\rho\ln\rho\right)\\&\left.+\frac{1}{2}\alpha^4\ln^2\alpha\left(\frac{1}{8}\rho^3-\rho\ln\rho-\frac{1}{2}\frac{1}{\rho}\ln\rho\right)-A_2\rho-B_2\frac{1}{\rho}\right]\end{aligned} \tag{3.10}$$

其中

$$\begin{aligned}A_1=&\frac{1}{2\lambda_1(\alpha^2-1)}\left(\frac{5-3\nu}{16}\alpha^6-\lambda_1\alpha^4\ln^2\alpha-\frac{1}{4}\lambda_2\alpha^4\ln\alpha-\frac{5-3\nu}{16}\alpha^4\right.\\&\left.+\frac{1}{4}\lambda_2\alpha^2\ln\alpha-\frac{5-3\nu}{16}\alpha^2+\frac{5-3\nu}{16}\right)\end{aligned} \tag{3.11a}$$

$$B_1=\frac{1}{2\lambda_2(\alpha^2-1)}\left(\frac{5-3\nu}{16}\alpha^6-\lambda_1\alpha^4\ln^2\alpha-\frac{5-3\nu}{8}\alpha^4+\frac{5-3\nu}{16}\alpha^2\right) \tag{3.11b}$$

$$\begin{aligned}A_2=&\frac{1}{2\lambda_1(\alpha^2-1)}\left(\frac{9-7\nu}{64}\alpha^8-\frac{1}{2}\lambda_1\alpha^6\ln^3\alpha-\frac{1}{2}\lambda_1\alpha^6\ln^2\alpha\right.\\&-\frac{3}{16}\lambda_2\alpha^6\ln\alpha-\frac{9-7\nu}{32}\alpha^6+\nu\alpha^4\ln^3\alpha+\frac{1}{2}\lambda_1\alpha^4\ln^2\alpha+\frac{3}{8}\lambda_2\alpha^4\ln\alpha\\&\left.-\frac{3}{16}\lambda_2\alpha^2\ln\alpha+\frac{9-7\nu}{32}\alpha^2-\frac{9-7\nu}{64}\right)\end{aligned} \tag{3.11c}$$

$$\begin{aligned}B_2=&\frac{1}{2\lambda_2(\alpha^2-1)}\left(\frac{9-7\nu}{64}\alpha^8-\frac{1}{2}\lambda_1\alpha^6\ln^3\alpha+\frac{5}{8}\lambda_2\alpha^6\ln^2\alpha-\frac{27-21\nu}{64}\alpha^6\right.\\&\left.+\nu\alpha^4\ln^3\alpha-\frac{5}{8}\lambda_2\alpha^4\ln^2\alpha+\frac{27-21\nu}{64}\alpha^4-\frac{9-7\nu}{64}\alpha^2\right)\end{aligned} \tag{3.11d}$$

在二次近似中，我们有下述线性边值问题：

$$L(\rho\varphi_2)=S_1(\varphi_1-k\rho)-Q \tag{3.12}$$

当$\rho=1$时，$y_2=0$，$\varphi_2=0$ (3.13a,b)

当$\rho=\alpha$时，$\varphi_2=0$ (3.14)

将方程(3.12)积分，并利用边界条件(3.13b)和(3.14)，得φ的二次近似解为

$$\varphi_2=\frac{Q}{2(1-\alpha^2)}\left[(\alpha^2-1)\rho\ln\rho-\alpha^2\ln\alpha\left(\rho-\frac{1}{\rho}\right)\right]$$

$$-k^2\beta Y_m\Big[\frac{1}{192}(\alpha^2-1)\Big(\rho^5\ln\rho-\frac{1}{6}\rho^5\Big)$$

$$-\frac{1}{16}\alpha^2\ln\alpha\Big(\frac{1}{12}\rho^5-\rho^3\ln\rho+\frac{3}{4}\rho^3\Big)+\frac{1}{8}A_1\rho^3+\frac{1}{2}B_1\rho\ln\rho-C_2\rho-D_2\frac{1}{\rho}\Big]$$

$$+k\beta^2Y_m^2\Big\{\frac{1}{128}(\alpha^2-1)^2\Big(\rho^5\ln^2\rho-\frac{11}{6}\rho^5\ln\rho+\frac{35}{36}\rho^5\Big)$$

$$-\frac{1}{32}(\alpha^2-1)\alpha^2\ln\alpha\Big(\frac{1}{2}\rho^5\ln\rho-\frac{11}{24}\rho^5-3\rho^3\ln^2\rho+5\rho^3\ln\rho-\frac{21}{8}\rho^3\Big)$$

$$+\frac{1}{8}\alpha^4\ln^2\alpha\Big(\frac{1}{16}\rho^5-\rho^3\ln\rho+\frac{5}{8}\rho^3+\frac{1}{2}\rho\ln^2\rho-\frac{1}{2}\rho\ln\rho\Big)$$

$$+\frac{1}{2}A_1\Big[\frac{1}{4}(\alpha^2-1)\Big(\rho^3\ln\rho-\frac{3}{4}\rho^3\Big)-\alpha^2\ln\alpha\Big(\frac{1}{4}\rho^3-\rho\ln\rho\Big)\Big]$$

$$+\frac{1}{2}B_1\Big[\frac{1}{2}(\alpha^2-1)(\rho\ln^2\rho-\rho\ln\rho)-\alpha^2\ln\alpha\Big(\rho\ln\rho+\frac{1}{\rho}\ln\rho\Big)\Big]$$

$$-\frac{1}{8}A_2\rho^3-\frac{1}{2}B_2\rho\ln\rho+C_3\rho+D_3\frac{1}{\rho}\Big\}$$

$$-\beta^3Y_m^3\Big\{\frac{1}{384}(\alpha^2-1)^3\Big(\rho^5\ln^3\rho-\frac{11}{4}\rho^5\ln^2\rho+\frac{35}{12}\rho^5\ln\rho-\frac{71}{72}\rho^5\Big)$$

$$-\frac{1}{32}(\alpha^2-1)^2\alpha^2\ln\alpha\Big(\frac{1}{4}\rho^5\ln^2\rho-\frac{11}{24}\rho^5\ln\rho+\frac{35}{144}\rho^5-\rho^3\ln^3\rho$$

$$+3\rho^3\ln^2\rho-\frac{27}{8}\rho^3\ln\rho+\frac{25}{16}\rho^3\Big)$$

$$+\frac{1}{16}(\alpha^2-1)\alpha^4\ln^2\alpha\Big(\frac{1}{8}\rho^5\ln\rho-\frac{11}{96}\rho^5-\frac{3}{2}\rho^3\ln^2\rho+\frac{5}{2}\rho^3\ln\rho$$

$$-\frac{21}{16}\rho^3-\rho\ln^2\rho+\rho\ln\rho\Big)$$

$$-\frac{1}{16}\alpha^6\ln^3\alpha\Big(\frac{1}{24}\rho^5-\rho^3\ln\rho+\frac{5}{8}\rho^3+\rho\ln^2\rho-\rho\ln\rho-\frac{1}{\rho}\ln^2\rho-\frac{1}{\rho}\ln\rho\Big)$$

$$-\frac{1}{2}A_2\Big[\frac{1}{4}(\alpha^2-1)\Big(\rho^3\ln\rho-\frac{3}{4}\rho^3\Big)-\alpha^2\ln\alpha\Big(\frac{1}{4}\rho^3-\rho\ln\rho\Big)\Big]$$

$$-\frac{1}{2}B_2\Big[\frac{1}{2}(\alpha^2-1)(\rho\ln^2\rho-\rho\ln\rho)-\alpha^2\ln\alpha\Big(\rho\ln\rho+\frac{1}{\rho}\ln\rho\Big)\Big]$$

$$-C_4\rho-D_4\frac{1}{\rho}\Big\} \qquad (3.15)$$

其中

$$C_2=\frac{1}{4\lambda_1\lambda_2(\alpha^2-1)^2}\Big(\frac{31-13\nu}{576}\lambda_2\alpha^{10}+\frac{1-15\nu+8\nu^2}{24}\alpha^8\ln\alpha-\frac{17+\nu}{576}\lambda_2\alpha^8$$

$$-\lambda_1^2\alpha^8\ln^3\alpha-\frac{1}{4}\lambda_1\lambda_2\alpha^6\ln^2\alpha-\frac{17-54\nu+25\nu^2}{48}\alpha^6\ln\alpha-\frac{26-17\nu}{144}\lambda_2\alpha^6$$

$$+\frac{1}{4}\lambda_1\lambda_2\alpha^4\ln^2\alpha+\frac{14-9\nu+\nu^2}{24}\alpha^4\ln\alpha+\frac{26-17\nu}{144}\lambda_2\alpha^4$$

$$-\frac{13-7\nu}{48}\lambda_2\alpha^2\ln\alpha+\frac{17+\nu}{576}\lambda_2\alpha^2-\frac{31-13\nu}{576}\lambda_2\Big) \tag{3.16a}$$

$$D_2=-\frac{1}{4\lambda_1\lambda_2(\alpha^2-1)^2}\Big(\frac{31-13\nu}{576}\lambda_2\alpha^{10}+\frac{1-15\nu+8\nu^2}{24}\alpha^8\ln\alpha-\frac{31-13\nu}{288}\lambda_2\alpha^8$$

$$-\lambda_1^2\alpha^6\ln^3\alpha-\frac{1-15\nu+8\nu^2}{12}\alpha^6\ln\alpha+\frac{1-15\nu+8\nu^2}{24}\alpha^4\ln\alpha$$

$$+\frac{31-13\nu}{288}\lambda_2\alpha^4-\frac{31-13\nu}{576}\lambda_2\alpha^2\Big) \tag{3.16b}$$

$$C_3=\frac{1}{16\lambda_1\lambda_2(\alpha^2-1)^2}\Big[\frac{73-37\nu}{288}\lambda_2\alpha^{12}-\frac{5+12\nu-9\nu^2}{8}\alpha^{10}\ln^2\alpha$$

$$-\frac{10+63\nu-43\nu^2}{24}\alpha^{10}\ln\alpha-\frac{37-13\nu}{96}\lambda_2\alpha^{10}-(1-7\nu)\lambda_1\alpha^8\ln^4\alpha$$

$$+\frac{5+14\nu-7\nu^2}{2}\alpha^8\ln^3\alpha-(1+2\nu)\lambda_1\alpha^8\ln^2\alpha-\frac{17-348\nu+211\nu^2}{48}\alpha^8\ln\alpha$$

$$-\frac{71-59\nu}{96}\lambda_2\alpha^8-(5-11\nu)\lambda_1\alpha^6\ln^4\alpha-\frac{5+12\nu-9\nu^2}{2}\alpha^6\ln^3\alpha$$

$$+\frac{31+52\nu-59\nu^2}{8}\alpha^6\ln^2\alpha+\frac{57-39\nu}{16}\lambda_1\alpha^6\ln\alpha+\frac{251-179\nu}{144}\lambda_2\alpha^6$$

$$-\nu\lambda_2\alpha^4\ln^3\alpha-\frac{9+16\nu-17\nu^2}{4}\alpha^4\ln\alpha-\frac{211-36\nu-55\nu^2}{48}\alpha^4\ln\alpha$$

$$-\frac{71-59\nu}{96}\lambda_2\alpha^4+\frac{77-47\nu}{48}\lambda_2\alpha^2\ln\alpha-\frac{37-13\nu}{96}\lambda_2\alpha^2+\frac{73-37\nu}{288}\lambda_2\Big] \tag{3.16c}$$

$$D_3=-\frac{1}{16\lambda_1\lambda_2(\alpha^2-1)^2}\Big[\frac{73-37\nu}{288}\lambda_2\alpha^{12}-\frac{5+12\nu-9\nu^2}{8}\alpha^{10}\ln^2\alpha$$

$$-\frac{35+132\nu-95\nu^2}{48}\alpha^{10}\ln\alpha-\frac{73-37\nu}{96}\lambda_2\alpha^{10}-(1-7\nu)\lambda_1\alpha^8\ln^4\alpha$$

$$+(4+7\nu-5\nu^2)\alpha^8\ln^3\alpha+\frac{15-4\nu-3\nu^2}{8}\alpha^8\ln^2\alpha+\frac{35+132\nu-95\nu^2}{16}\alpha^8\ln\alpha$$

$$+\frac{73-37\nu}{144}\lambda_2\alpha^8-(5-11\nu)\lambda_1\alpha^6\ln^4\alpha-(4+7\nu-5\nu^2)\alpha^6\ln^3\alpha$$

$$-\frac{15-44\nu+21\nu^2}{8}\alpha^6\ln^2\alpha-\frac{35+132\nu-95\nu^2}{16}\alpha^6\ln\alpha+\frac{73-37\nu}{144}\lambda_2\alpha^6$$

$$+\frac{5-28\nu+15\nu^2}{8}\alpha^4\ln^2\alpha+\frac{35+132\nu-95\nu^2}{48}\alpha^4\ln\alpha-\frac{73-37\nu}{96}\lambda_2\alpha^4$$

$$+\frac{73-37\nu}{288}\lambda_2\alpha^2\Big] \tag{3.16d}$$

$$C_4=\frac{1}{8\lambda_1\lambda_2(\alpha^2-1)^2}\left(\frac{445-283\nu}{13824}\lambda_2\alpha^{14}-\frac{9+20\nu-21\nu^2}{64}\alpha^{12}\ln^2\alpha\right.$$

$$-\frac{179+216\nu-251\nu^2}{576}\alpha^{12}\ln\alpha-\frac{1051-565\nu}{13824}\lambda_2\alpha^{12}+\nu\lambda_1\alpha^{10}\ln^5\alpha$$

$$+\frac{9+17\nu}{8}\lambda_1\alpha^{10}\ln^4\alpha+\frac{12-3\nu}{8}\lambda_2\alpha^{10}\ln^3\alpha-\frac{83-71\nu}{192}\lambda_2\alpha^{10}\ln^2\alpha$$

$$+\frac{19+33\nu-34\nu^2}{24}\alpha^{10}\ln\alpha-\frac{325-379\nu}{4608}\lambda_2\alpha^{10}-\frac{1-4\nu+11\nu^2}{2}\alpha^8\ln^5\alpha$$

$$+\frac{4-3\nu}{2}\lambda_2\alpha^8\ln^4\alpha-\frac{24-9\nu}{8}\lambda_2\alpha^8\ln^3\alpha+\frac{137+132\nu-197\nu^2}{64}\alpha^8\ln^2\alpha$$

$$-\frac{17+504\nu-377\nu^2}{288}\alpha^8\ln\alpha+\frac{5065-4255\nu}{13824}\lambda_2\alpha^8-\frac{1-5\nu}{2}\lambda_1\alpha^6\ln^5\alpha$$

$$-\frac{25+12\nu-29\nu^2}{8}\alpha^6\ln^4\alpha+\frac{12-9\nu}{8}\lambda_2\alpha^6\ln^3\alpha-\frac{155+172\nu-239\nu^2}{64}\alpha^6\ln^2\alpha$$

$$-\frac{211-108\nu-31\nu^2}{144}\alpha^6\ln\alpha-\frac{5065-4255\nu}{13824}\lambda_2\alpha^6+\frac{3}{8}\nu\lambda_2\alpha^4\ln^3\alpha$$

$$+\frac{41+48\nu-65\nu^2}{48}\alpha^4\ln^2\alpha+\frac{287+24\nu-167\nu^2}{192}\alpha^4\ln\alpha$$

$$+\frac{325-379\nu}{4608}\lambda_2\alpha^4-\frac{65-47\nu}{144}\lambda_2\alpha^2\ln\alpha+\frac{1051-565\nu}{13824}\lambda_2\alpha^2$$

$$\left.-\frac{445-283\nu}{13824}\lambda_2\right) \tag{3.16e}$$

$$D_4=-\frac{1}{8\lambda_1\lambda_2(\alpha^2-1)^2}\left(\frac{445-283\nu}{13824}\lambda_2\alpha^{14}-\frac{9+20\nu-21\nu^2}{64}\alpha^{12}\ln^2\alpha\right.$$

$$-\frac{439+450\nu-565\nu^2}{1152}\alpha^{12}\ln\alpha-\frac{445-283\nu}{3456}\lambda_2\alpha^{12}+\nu\lambda_1\alpha^{10}\ln^5\alpha$$

$$+\frac{11+19\nu}{8}\lambda_1\alpha^{10}\ln^4\alpha+\frac{109-55\nu}{48}\lambda_2\alpha^{10}\ln^3\alpha+\frac{9-7\nu}{16}\lambda_2\alpha^{10}\ln^2\alpha$$

$$+\frac{439+450\nu-565\nu^2}{288}\alpha^{10}\ln\alpha+\frac{2225-1415\nu}{13824}\lambda_2\alpha^{10}$$

$$-\frac{1-4\nu+11\nu^2}{2}\alpha^8\ln^5\alpha+\frac{7}{4}\lambda_1\lambda_2\alpha^8\ln^4\alpha-\frac{109-55\nu}{24}\lambda_2\alpha^8\ln^3\alpha$$

$$-\frac{27-21\nu}{32}\lambda_1\alpha^8\ln^2\alpha-\frac{439+450\nu-565\nu^2}{192}\alpha^8\ln\alpha-\frac{1-5\nu}{2}\lambda_1\alpha^6\ln^5\alpha$$

$$-\frac{25+33\nu}{8}\lambda_1\alpha^6\ln^4\alpha+\frac{109-55\nu}{48}\lambda_2\alpha^6\ln^3\alpha+\frac{9-34\nu+21\nu^2}{16}\alpha^6\ln^2\alpha$$

$$+\frac{439+450\nu-565\nu^2}{288}\alpha^6\ln\alpha-\frac{2225-1415\nu}{13824}\lambda_2\alpha^6$$

$$-\frac{9-52\nu+35\nu^2}{64}\alpha^4\ln^2\alpha-\frac{439+450\nu-565\nu^2}{1152}\alpha^4\ln\alpha$$

$$+\frac{445-283\nu}{3456}\lambda_2\alpha^4-\frac{445-283\nu}{13824}\lambda_2\alpha^2\Big) \tag{3.16f}$$

将式(3.15)代入式(3.6)，得此壳体的二次近似特征关系式:

$$Q=(\alpha_1+\alpha_2k^2)Y_m+\alpha_3kY_m^2+\alpha_4Y_m^3 \tag{3.17}$$

其中

$$\alpha_1=2(1-\alpha^2)\beta \tag{3.18a}$$

$$\alpha_2=\frac{\beta^2}{2\lambda_1\lambda_2(1-\alpha^2)}\Big[\frac{83-29\nu}{6912}\lambda_2\alpha^{12}-\frac{31-13\nu}{288}\lambda_2\alpha^{10}\ln\alpha+\frac{71-162\nu+55\nu^2}{1152}\alpha^{10}$$

$$-\frac{17-54\nu+25\nu^2}{48}\alpha^8\ln^2\alpha+\frac{31-13\nu}{144}\lambda_2\alpha^8\ln\alpha-\frac{817-1134\nu+353\nu^2}{2304}\alpha^8$$

$$+\lambda_1^2\alpha^6\ln^4\alpha+\frac{17-54\nu+25\nu^2}{24}\alpha^6\ln^2\alpha+\frac{971-1242\nu+379\nu^2}{1728}\alpha^6$$

$$-\frac{17-54\nu+25\nu^2}{48}\alpha^4\ln^2\alpha-\frac{31-13\nu}{144}\lambda_2\alpha^4\ln\alpha$$

$$-\frac{817-1134\nu+353\nu^2}{2304}\alpha^4+\frac{31-13\nu}{288}\lambda_2\alpha^2\ln\alpha+\frac{71-162\nu+55\nu^2}{1152}\alpha^2$$

$$+\frac{83-29\nu}{6912}\lambda_2\Big] \tag{3.18b}$$

$$\alpha_3=\frac{\beta^3}{8\lambda_1\lambda_2(1-\alpha^2)}\Big[\frac{95-41\nu}{2304}\lambda_2\alpha^{14}-\frac{73-37\nu}{96}\lambda_2\alpha^{12}\ln\alpha$$

$$+\frac{835-1890\nu+731\nu^2}{2304}\alpha^{12}+\frac{15-9\nu}{4}\nu\alpha^{10}\ln^3\alpha+\frac{19+66\nu-49\nu^2}{16}\alpha^{10}\ln^2\alpha$$

$$+\frac{73-37\nu}{32}\lambda_2\alpha^{10}\ln\alpha-\frac{1835-2898\nu+1027\nu^2}{768}\alpha^{10}+(3-9\nu)\lambda_1\alpha^8\ln^5\alpha$$

$$-(6-3\nu)\lambda_2\alpha^8\ln^4\alpha-\frac{15-9\nu}{4}\nu\alpha^8\ln^3\alpha-\frac{57+198\nu-147\nu^2}{16}\alpha^8\ln^2\alpha$$

$$-\frac{73-37\nu}{48}\lambda_2\alpha^8\ln\alpha+\frac{11675-17010\nu+5875\nu^2}{2304}\alpha^8+(3-9\nu)\lambda_1\alpha^6\ln^5\alpha$$

$$+(6-3\nu)\lambda_2\alpha^6\ln^4\alpha-\frac{15-9\nu}{4}\nu\alpha^6\ln^3\alpha+\frac{57+198\nu-147\nu^2}{16}\alpha^6\ln^2\alpha$$

$$-\frac{73-37\nu}{48}\lambda_2\alpha^6\ln\alpha-\frac{11675-17010\nu+5875\nu^2}{2304}\alpha^6$$

$$+\frac{15-9\nu}{4}\nu\alpha^4\ln^3\alpha-\frac{19+66\nu-49\nu^2}{16}\alpha^4\ln^2\alpha+\frac{73-37\nu}{32}\lambda_2\alpha^4\ln\alpha$$

$$+\frac{1835-2898\nu+1027\nu^2}{768}\alpha^4-\frac{73-37\nu}{96}\lambda_2\alpha^2\ln\alpha-\frac{835-1890\nu+731\nu^2}{2304}\alpha^2$$

$$-\frac{95-41\nu}{2304}\lambda_2\Big] \tag{3.18c}$$

$$
\begin{aligned}
\alpha_4=&\frac{\beta^4}{4\lambda_1\lambda_2(1-\alpha^2)}\Big[\frac{353-191\nu}{82944}\lambda_2\alpha^{16}\frac{445-283\nu}{3456}\lambda_2\alpha^{14}\ln\alpha\\
&+\frac{1199-2754\nu+1231\nu^2}{20736}\alpha^{14}+\frac{9-7\nu}{8}\nu\alpha^{12}\ln^3\alpha+\frac{601-439\nu}{576}\lambda_2\alpha^{12}\ln^2\alpha\\
&+\frac{445-283\nu}{864}\lambda_2\alpha^{12}\ln\alpha-\frac{8959-15714\nu+6431\nu^2}{20736}\alpha^{12}\\
&+\frac{1-5\nu}{3}\lambda_1\alpha^{10}\ln^6\alpha-\frac{5}{2}\lambda_1\lambda_2\alpha^{10}\ln^5\alpha-\frac{8+\nu}{6}\lambda_2\alpha^{10}\ln^4\alpha\\
&-\frac{9-7\nu}{4}\nu\alpha^{10}\ln^3\alpha-\frac{601-439\nu}{144}\lambda_2\alpha^{10}\ln^2\alpha-\frac{2225-1415\nu}{3456}\lambda_2\alpha^{10}\ln\alpha\\
&+\frac{23633-38718\nu+15409\nu^2}{20736}\alpha^{10}+\frac{1-6\nu+17\nu^2}{3}\alpha^8\ln^6\alpha\\
&+\frac{8+\nu}{3}\lambda_2\alpha^8\ln^4\alpha+\frac{601-439\nu}{96}\lambda_2\alpha^8\ln^2\alpha-\frac{63845-102870\nu+40645\nu^2}{41472}\alpha^8\\
&+\frac{1-5\nu}{3}\lambda_1\alpha^6\ln^6\alpha+\frac{5}{2}\lambda_1\lambda_2\alpha^6\ln^5\alpha-\frac{8+\nu}{6}\lambda_2\alpha^6\ln^4\alpha\\
&+\frac{9-7\nu}{4}\nu\alpha^6\ln^3\alpha-\frac{601-439\nu}{144}\lambda_2\alpha^6\ln^2\alpha+\frac{2225-1415\nu}{3456}\lambda_2\alpha^6\ln\alpha\\
&+\frac{23633-38718\nu+15409\nu^2}{20736}\alpha^6-\frac{9-7\nu}{8}\nu\alpha^4\ln^3\alpha\\
&+\frac{601-439\nu}{576}\lambda_2\alpha^4\ln^2\alpha-\frac{445-283\nu}{864}\lambda_2\alpha^4\ln\alpha\\
&-\frac{8959-15714\nu+6431\nu^2}{20736}\alpha^4+\frac{445-283\nu}{3456}\lambda_2\alpha^2\ln\alpha\\
&+\frac{1199-2754\nu+1231\nu^2}{20736}\alpha^2+\frac{353-191\nu}{82944}\lambda_2\Big]
\end{aligned}
\tag{3.18d}
$$

我们以 $\alpha=0.3$，$\nu=0.3$ 的情况为例，按照公式(3.17)在图3上绘出了不同几何参数 k 值下的特征曲线。由图看出，当 k 很小时，$Q\sim Y_m$ 曲线单调上升，说明壳体具有平板性质，无跳跃现象产生。当 k 较大时，$Q\sim Y_m$ 曲线出现了迴形线状态，这时壳体便产生了跳跃现象。

对式(3.17)应用极值条件

$$\frac{dQ}{dY_m}=0 \tag{3.19}$$

我们得到产生跳跃现象时的无量纲内边缘临界挠度的公式：

$$Y_m^*=\frac{-\alpha_3k\pm\sqrt{(\alpha_3^2-3\alpha_2\alpha_4)k^2-3\alpha_1\alpha_4}}{3\alpha_4} \tag{3.20}$$

将此 Y_m^* 值代入式(3.17)，得二次近似的临界载荷公式：

$$Q^*=(\alpha_1+\alpha_2k^2)Y_m^*+\alpha_3kY_m^{*2}+\alpha_4Y_m^{*3} \tag{3.21}$$

其中对应于式(3.20)负号的Q^*是上临界载荷，正号的Q^*是下临界载荷。它们分别是$Q \sim Y_m$曲线上的极大值点和极小值点。

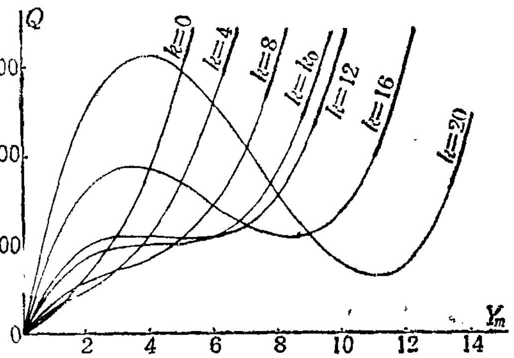

图3　各种k值下的特征曲线($\alpha=0.3$, $\nu=0.3$)

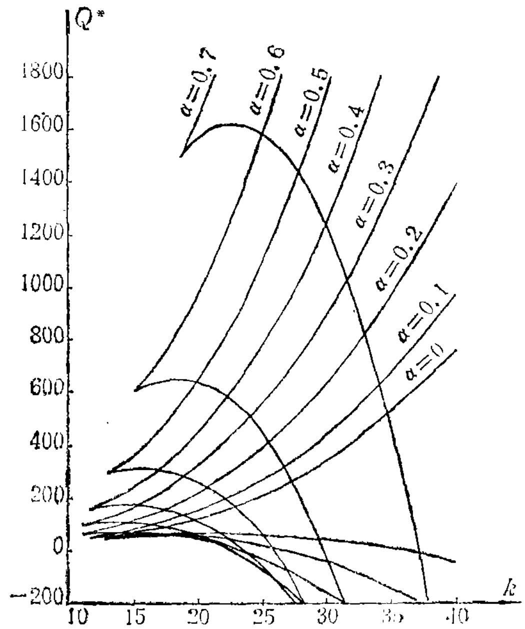

图4　各种α值下的稳定曲线($\nu=0.3$)

按照公式(3.21)进行数值计算，我们得到了不同α值的上、下临界载荷数据，结果给在图4中。由图看到，上临界载荷随k的增加而增加；下临界载荷在很小的范围内随k的增加而增加，之后随k的增加而减少。当k值相当大时，下临界载荷为负值，这表明此时若无反向载荷作用，扁壳本身无能力恢复至原来形状。

下面，我们来讨论上、下临界载荷曲线的相重点，亦即壳体屈曲临界点。记临界点的壳体几何参数为k_0，临界载荷为Q_0^*。众所周知，二次方程有重根的条件应为方程的判别式等于零，于是由式(3.19)得

$$(\alpha_3^2-3\alpha_2\alpha_4)k^2-3\alpha_1\alpha_4=0 \tag{3.22}$$

解此方程，即得

$$k_0=\sqrt{\frac{3\alpha_1\alpha_4}{\alpha_3^2-3\alpha_2\alpha_4}} \tag{3.23}$$

显然，当$k<k_0$时，壳体不会发生失稳现象。当$k\geqslant k_0$时，壳体就会出现失稳现象。因此，几何参数k_0是区分壳体失稳与否的分界点。

图5与图6分别给出了$k_0\sim\alpha$和$Q_0^*\sim\alpha$的关系曲线。由图5可知，此曲线存在k_0的一个极小值。相应于这一极小值的壳体是对失稳反映最灵敏的壳体。为了得到此值，我们令式(3.23)的关于α的导数为零

$$\frac{dk_0}{d\alpha}=0 \tag{3.24}$$

于是得到

$$\alpha=0.2436 \tag{3.25}$$

将此值代入式(3.23)，便得k_0的极小值

$$(k_0)_{\min}=11.05 \tag{3.26}$$

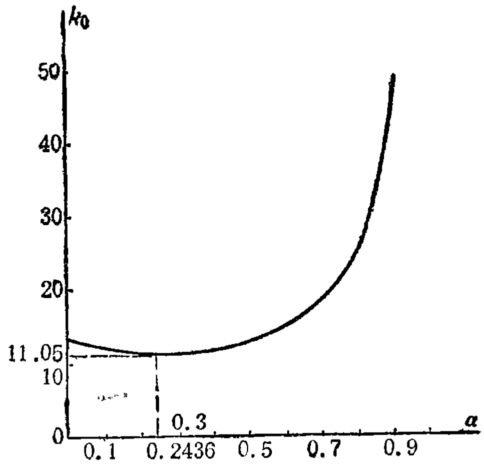

图5 临界几何参数k_0与α的关系曲线($\nu=0.3$)

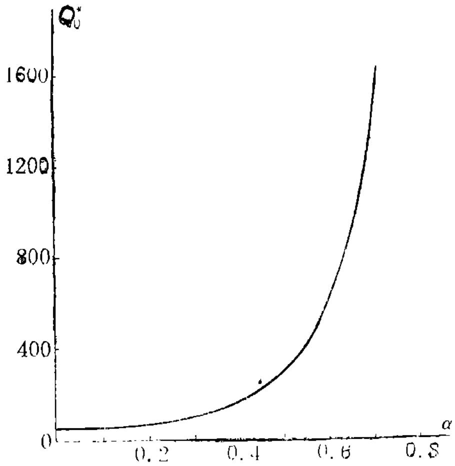

图6 临界点载荷Q_0^*与α的关系曲线($\nu=0.3$)

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On the Nonlinear Stability of a Truncated Shallow Spherical Shell under a Concentrated Load

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Abstract

In this paper, the axisymmetric nonlinear stability of a clamped truncated shallow spherical shell with a nondeformable rigid body at the center under a concentrated load is investigated by use of the modified iteration method. The analytic formulas of second approximation for determining the upper and lower critical buckling loads are obtained.