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关于具有大参数a^2/R_0h的$r>0$等厚圆环薄壳轴对称问题的二次渐近解*

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摘 要

在这篇文章中，根据 Love-Kirchhoff 假设的薄壳理论，导出了$r>0$等厚圆环薄壳力矩理论轴对称问题的基本方程．对具有大参数a^2/R_0h的$r>0$等厚圆环薄壳，给出了二次渐近解．本文也给出了当边缘远离圆环薄壳顶点时的边缘问题的二次渐近解．它们的误差都是在Love-Kirchhoff 假设的薄壳理论的允许误差范围之内．

符号说明

a　圆壳环的经线曲率半径
$\tilde{C}_1$和$\tilde{C}_2$　复常数
E　弹性模量
H,V　径向和轴向内力
h　壁厚
M_φ,M_θ　经向和环向弯矩
N_φ,N_θ　经向和环向内力
Q_φ　横剪力
q_H,q_V　分别是单位中面面积上的径向和轴向载荷
R_0　整个环壳的半径
r_2　圆环壳的环向曲率半径
r　$r_2\sin\varphi$
$\varepsilon_\varphi,\varepsilon_\theta$　经向和环向应变
ν　泊松比
ϑ　经线的角位移
φ　壳面的法线与旋转轴之间的夹角
V^*,r^*,φ^*　分别是V,r,φ在圆环壳的上边界处的该值

一、引 言

钱伟长[1]，F．Tölke[2]，R．A．Clark[3]和B．B．Новожилов[4]，都研究了等厚圆环形薄壳轴对称问题复变量方程．钱伟长[1]指出这些复变量方程都在Love-Kirchhoff薄壳理论允许误差范围之内．钱伟长[1][5]，R．A．Clark[3]，B．B．Новожилов[4]和张维[6]，赵鸿宾[7]等给出了不同形式的解．钱伟长[1][5]给出了一般解．R．A．Clark[3]，B．B．Новожилов[4]和张维[6]给出的渐近解的误差都是$\sqrt{R_0h}/a$阶的．而给出的幂级数解，钱伟长[8]指出它不是到处都是一致收敛的．由于基于Love-Kirchhoff假定上的薄壳理论，它本身也包含一定误差，那么给出的正确解的实际精确程度也就不可能超过这个误差范围．本文

* 钱伟长推荐．

给出的二次渐近解，它们在实际计算中是简单而方便的。而且，它的误差是在Love-Kirchhoff 薄壳理论允许误差范围之内。而更高阶渐近解，仅具参考价值。

二、基 本 方 程

圆环壳的几何参数如图 1

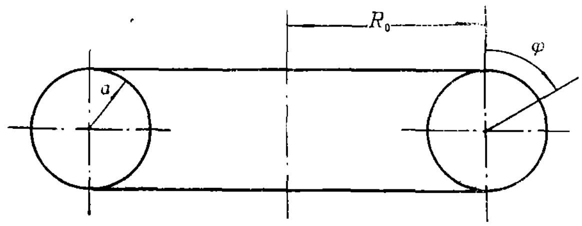

图 1

对等厚圆环薄壳，我们有：

$$a=\text{const},\quad r_2=R_0/\sin\varphi+a \tag{2.1}$$

等厚圆环薄壳轴对称问题的载荷、内力和位移如图 2。它们的定义和正方向标注在图 2 上。

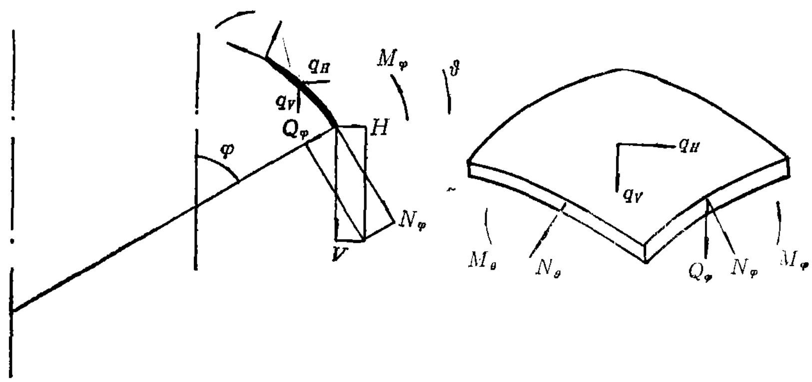

图 2

根据 Love-Kirchhoff 假定的薄壳理论，我们有：

平衡方程：

$$\left.\begin{aligned}&\frac{1}{a}\frac{d(rV)}{d\varphi}+rq_V=0\\&\frac{1}{a}\frac{d(rH)}{d\varphi}-N_\theta+rq_H=0\\&\frac{1}{a}\frac{d(rM_\varphi)}{d\varphi}-M_\theta\cos\varphi-r(V\cos\varphi-H\sin\varphi)=0\end{aligned}\right\} \tag{2.2}$$

内力与应变之间关系：

$$\left.\begin{aligned}&N_\varphi=\frac{Eh}{1-\nu^2}(\varepsilon_\varphi+\nu\varepsilon_\theta),\quad N_\theta=\frac{Eh}{1-\nu^2}(\varepsilon_\theta+\nu\varepsilon_\varphi)\\&M_\varphi=-\frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{a}\frac{d\vartheta}{d\varphi}+\nu\frac{\vartheta}{r_2}\cot\varphi\right),\quad M_\theta=-\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\vartheta}{r_2}\cot\varphi+\frac{\nu}{a}\frac{d\vartheta}{d\varphi}\right)\end{aligned}\right\} \tag{2.3}$$

中面变形连续方程：

$$\frac{1}{a}\frac{d(r\varepsilon_\theta)}{d\varphi}=\varepsilon_\varphi\cos\varphi-\vartheta\sin\varphi \tag{2.4}$$

还有内力之间关系：

$$H=N_\varphi\cos\varphi-Q_\varphi\sin\varphi,\ V=N_\varphi\sin\varphi+Q_\varphi\cos\varphi \tag{2.5}$$

由以上各式，不难求得：

$$L(rH)+\frac{\nu}{a}(rH)=-Eh\vartheta+f(\varphi),\quad L(\vartheta)-\frac{\nu}{a}\vartheta=\frac{12(1-\nu^2)}{Eh^3}(rH-rV\cot\varphi) \tag{2.6}$$

式中：$L(\cdots)$为一线性微分运算子，它是：

$$L(\cdots)=\frac{R_0(1+\alpha\sin\varphi)}{a^2\sin\varphi}\frac{d^2(\cdots)}{d\varphi^2}+\frac{\cot\varphi}{a}\frac{d(\cdots)}{d\varphi}-\frac{\cos^2\varphi}{R_0(1+\alpha\sin\varphi)\sin\varphi}(\cdots)$$

和 $\alpha=\dfrac{a}{R_0}$

$$f(\varphi)=-(2+\nu)\left(\frac{R_0}{\sin\varphi}+a\right)q_H\cos\varphi-\frac{1}{a}\left(\frac{R_0}{\sin\varphi}+a\right)^2\frac{dq_H}{d\varphi}\sin\varphi-\nu\left(\frac{R_0}{\sin\varphi}+a\right)q_V\sin\varphi$$
$$-\left[\frac{\sin\varphi}{R_0(1+\alpha\sin\varphi)}+\frac{\nu}{a}\right]\left[a\int_0^\varphi rq_Vd\varphi-\left(a\int_0^{\varphi^*}rq_Vd\varphi+r^*V^*\right)\right]\cot\varphi$$

我们作第一次变换：

$$\left.\begin{aligned}&\Pi_1=\sqrt[4]{\frac{r_2\sin^2\varphi}{h^3}}\left(rH+a\cot\varphi\int_0^\varphi rq_Vd\varphi\right)\\&\Theta_1=\sqrt[4]{r_2h^5\sin^2\varphi}\ \vartheta,\quad \frac{dy}{d\varphi}=\sqrt{\frac{\sin\varphi}{1+\alpha\sin\varphi}}\end{aligned}\right\} \tag{2.7}$$

由式(2.6)得：

$$\left.\begin{aligned}&\frac{d^2\Pi_1}{dy^2}+[\Omega(\varphi)+\nu]\Pi_1=-\frac{Ea^2}{R_0h}(\Theta_1-\Theta_{1m})\\&\frac{d^2\Theta_1}{dy^2}+[\Omega(\varphi)-\nu]\Theta_1=\frac{4\beta^4a^2}{ER_0h}\left[\Pi_1-\sqrt[4]{\frac{r_2\sin^2\varphi}{h^3}}\left(r^*V^*-a\int_0^{\varphi^*}rq_Vd\varphi\right)\cot\varphi\right]\end{aligned}\right\} \tag{2.8}$$

式中： $\beta=\sqrt[4]{3(1-\nu^2)}$

$$\Omega(\varphi)=\frac{a^2}{R_0r_2}\left(-\frac{15}{16}-\frac{r_2}{8a}+\frac{5r_2^2}{16a^2}\right)\cot^2\varphi+\frac{a}{4R_0}+\frac{r_2}{4R_0}$$

$$\Theta_{1m}=\frac{\sqrt[4]{r_2h\sin^2\varphi}}{E}\left\{-(2+\nu)\left(\frac{R_0}{\sin\varphi}+a\right)q_H\cos\varphi-\frac{1}{a}\left(\frac{R_0}{\sin\varphi}+a\right)^2\left(\frac{dq_H}{d\varphi}\sin\varphi\right.\right.$$
$$\left.-\frac{dq_V}{d\varphi}\cos\varphi\right)-\left(2\sin\varphi+\nu\sin\varphi+\frac{2}{\alpha\sin^2\varphi}\right)\left(\frac{R_0}{\sin\varphi}+a\right)q_V-\frac{\cos\varphi}{\sin^3\varphi}\left(\frac{\alpha\sin\varphi}{1+\alpha\sin\varphi}\right.$$
$$\left.\left.-1-\frac{2}{\alpha\sin^2\varphi}\right)\int_0^\varphi rq_Vd\varphi+\left(\frac{\alpha\sin\varphi}{1+\alpha\sin\varphi}+\nu\right)\left(\int_0^{\varphi^*}rq_Vd\varphi+\frac{r^*V^*}{a}\right)\cot\varphi\right\}$$

再作一次变换：

$$\Pi=\sqrt{\frac{d\xi}{dy}}\Pi_1,\quad \Theta=\sqrt{\frac{d\xi}{dy}}\Theta_1,\quad \xi=\left(\frac{3}{2}y\right)^{\frac{2}{3}} \tag{2.9}$$

由式(2.8)得：

$$\left.\begin{aligned}&\frac{d^2\Pi}{d\xi^2}+g_1(\xi)\Pi=-E\lambda_0^2(\Theta\xi-\Theta_m)\\&\frac{d^2\Theta}{d\xi^2}+g_2(\xi)\Theta=\frac{4\beta^4}{E}\lambda_0^2(\Pi\xi-\Pi_m)\end{aligned}\right\}\tag{2.10}$$

式中： $\lambda_0^2=\dfrac{a^2}{R_0h}$， $g_1(\xi)=\left[\Omega(\varphi)+\nu-\dfrac{5}{36y^2}\right]\xi$， $g_2(\xi)=\left[\Omega(\varphi)-\nu-\dfrac{5}{36y^2}\right]\xi$

$$\Theta_m=\Theta_{1m}\xi^{\frac{3}{4}},\quad \Pi_m=\sqrt[4]{\frac{r_2\sin^2\varphi}{h^3}}\left(r^*V^*-a\int_0^{\varphi^*}rq_Vd\varphi\right)\xi^{\frac{3}{4}}\cot\varphi$$

对 $R_0/a=1/\alpha=O(1)$ 的等厚圆环薄壳来讲，λ_0 是一个大参数，而且还有 $g_1(\xi)=O(1)$ 和 $g_2(\xi)=O(1)$。那末，式(2.10)是一个包含大参数的二阶变系数常微分方程组。式(2.10)是 $r>0$ 等厚圆环薄壳力矩理论轴对称问题的基本方程。必须指出，它的误差是在 Love-Kirchhoff 薄壳理论允许误差范围之内。

三、齐次解

式(2.10)的齐次方程是：

$$\frac{d^2\overline{\Pi}}{d\xi^2}+g_1(\xi)\overline{\Pi}=-E\lambda_0^2\overline{\Theta}\xi,\quad\frac{d^2\overline{\Theta}}{d\xi^2}+g_2(\xi)\overline{\Theta}=\frac{4\beta^4}{E}\lambda_0^2\overline{\Pi}\xi\tag{3.1}$$

因为在式(3.1)中，λ_0 是一个大参数且有 $g_1(\xi)=O(1)$ 和 $g_2(\xi)=O(1)$，故有比较方程：

$$\frac{d^2I}{d\xi^2}=-E\lambda_0^2\theta\xi,\quad\frac{d^2\theta}{d\xi^2}=\frac{4\beta^4}{E}\lambda_0^2I\xi\tag{3.2}$$

方程组(3.2)可以合併成一个方程，它是：

$$\frac{d^2\tilde{U}}{d\xi^2}+\mu^2\xi\tilde{U}=0\tag{3.3}$$

式中： $\tilde{U}=I+i\dfrac{E}{2\beta^2}\theta$； $\mu^2=-2i\beta^2\lambda_0^2$， $i=\sqrt{-1}$

上式有解：

$$\tilde{U}=\sqrt{\xi}\ Z_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\tag{3.4}$$

式中 $Z_{\frac{1}{3}}\left(\dfrac{2\mu}{3}\xi^{\frac{3}{2}}\right)$

是 1/3 阶 Bessel 函数。那末，以下式作为式(3.1)的一次渐近解，这同其它作者是一致的[3][4][6]：

$$\overline{\Pi}^{\mathrm{I}}+i\frac{E}{2\beta^2}\overline{\Theta}^{\mathrm{I}}=\sqrt{\xi}\left[\widetilde{C}_1H_{\frac{1}{3}}^{(1)}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\widetilde{C}_2H_{\frac{1}{3}}^{(2)}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\tag{3.5}$$

式中： $H_{\frac{1}{3}}^{(1)}\left(\dfrac{2\mu}{3}\xi^{\frac{3}{2}}\right)$和 $H_{\frac{1}{3}}^{(2)}\left(\dfrac{2\mu}{3}\xi^{\frac{3}{2}}\right)$

分别是第一种和第二种的 1/3 阶 Hankel 函数。

由于在式(3.1)中包含一个大参数，我们将式(3.1)的解展为 μ 的负次幂的级数。在这篇文章中如下式表示：

$$\bar{\Pi}=\mathrm{Re}\left(\delta_1\tilde{U}+\gamma_1\frac{d\tilde{U}}{d\xi}\right),\quad \bar{\Theta}=\frac{2\beta^2}{E}\mathrm{Im}\left(\delta_2\tilde{U}+\gamma_2\frac{d\tilde{U}}{d\xi}\right) \tag{3.6}$$

式中：

$$\left.\begin{aligned}\delta_1=\sum_{n=0}^{\infty}\delta_{1,n}(\xi)\mu^{-n},\quad \delta_2=\sum_{n=0}^{\infty}\delta_{2,n}(\xi)\mu^{-n}\\ \gamma_1=\sum_{n=0}^{\infty}\gamma_{1,n}(\xi)\mu^{-n},\quad \gamma_2=\sum_{n=0}^{\infty}\gamma_{2,n}(\xi)\mu^{-n}\end{aligned}\right\} \tag{3.7}$$

将式(3.3)和(3.6)代入式(3.1)，得：

$$\left.\begin{aligned}&\left[\frac{d^2\delta_1}{d\xi^2}-\mu^2\xi(\delta_1-\delta_2)-\mu^2\gamma_1-2\mu^2\xi\frac{d\gamma_1}{d\xi}+g_1(\xi)\delta_1\right]\tilde{U}\\&\quad+\left[\frac{d^2\gamma_1}{d\xi^2}-\mu^2\xi(\gamma_1-\gamma_2)+2\frac{d\delta_1}{d\xi}+g_1(\xi)\gamma_1\right]\frac{d\tilde{U}}{d\xi}=0\\&\left[\frac{d^2\delta_2}{d\xi^2}-\mu^2\xi(\delta_2-\delta_1)-\mu^2\gamma_2-2\mu^2\xi\frac{d\gamma_2}{d\xi}+g_2(\xi)\delta_2\right]\tilde{U}\\&\quad+\left[\frac{d^2\gamma_2}{d\xi^2}-\mu^2\xi(\gamma_2-\gamma_1)+2\frac{d\delta_2}{d\xi}+g_2(\xi)\gamma_2\right]\frac{d\tilde{U}}{d\xi}=0\end{aligned}\right\} \tag{3.8}$$

将式(3.7)代入式(3.8)中，并令 $\tilde{U}$ 和 $d\tilde{U}/d\xi$ 的系数分别为零，得：

$$\left.\begin{aligned}&\sum_{n=0}^{\infty}\left\{\frac{d^2\delta_{1,n}(\xi)}{d\xi^2}-\xi[\delta_{1,n+2}(\xi)-\delta_{2,n+2}(\xi)]-\gamma_{1,n+2}(\xi)\right.\\&\quad\left.-2\xi\frac{d\gamma_{1,n+2}(\xi)}{d\xi}+g_1(\xi)\delta_{1,n}(\xi)\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{\frac{d^2\delta_{2,n}(\xi)}{d\xi^2}-\xi[\delta_{2,n+2}(\xi)-\delta_{1,n+2}(\xi)]-\gamma_{2,n+2}(\xi)\right.\\&\quad\left.-2\xi\frac{d\gamma_{2,n+2}(\xi)}{d\xi}+g_2(\xi)\delta_{2,n}(\xi)\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{\frac{d^2\gamma_{1,n}(\xi)}{d\xi^2}-\xi[\gamma_{1,n+2}(\xi)-\gamma_{2,n+2}(\xi)]+2\frac{d\delta_{1,n}(\xi)}{d\xi}+g_1(\xi)\gamma_{1,n}(\xi)\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{\frac{d^2\gamma_{2,n}(\xi)}{d\xi^2}-\xi[\gamma_{2,n+2}(\xi)-\gamma_{1,n+2}(\xi)]+2\frac{d\delta_{2,n}(\xi)}{d\xi}+g_2(\xi)\gamma_{2,n}(\xi)\right\}\mu^{-n}=0\end{aligned}\right\} \tag{3.9}$$

由上式得：

$$\left.\begin{aligned}&\sum_{n=0}^{\infty}\left\{\frac{d^2}{d\xi^2}[\delta_{1,n}(\xi)+\delta_{2,n}(\xi)]-[\gamma_{1,n+2}(\xi)+\gamma_{2,n+2}(\xi)]\right.\\&\quad\left.-2\xi\frac{d}{d\xi}[\gamma_{1,n+2}(\xi)+\gamma_{2,n+2}(\xi)]+g_1(\xi)\delta_{1,n}(\xi)+g_2(\xi)\delta_{2,n}(\xi)\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{\frac{d^2}{d\xi^2}[\gamma_{1,n}(\xi)+\gamma_{2,n}(\xi)]+2\frac{d}{d\xi}[\delta_{1,n}(\xi)+\delta_{2,n}(\xi)]+g_1(\xi)\gamma_{1,n}(\xi)\right.\end{aligned}\right\} \tag{3.10}$$

$$+g_2(\xi)\gamma_{2,n}(\xi)\Big\}\mu^{-n}=0$$

由式(3.9)和(3.10)，得：

$$\left.\begin{aligned}
&\gamma_{1,2n+1}(\xi)=\gamma_{2,2n+1}(\xi)=\delta_{1,2n+1}(\xi)=\delta_{2,2n+1}(\xi)=0\\
&\gamma_{1,0}(\xi)=\gamma_{2,0}(\xi)=0\\
&\delta_{1,0}(\xi)=\delta_{2,0}(\xi)=1\\
&\gamma_{1,2}(\xi)=\gamma_{2,2}(\xi)=\frac{1}{4\sqrt{\xi}}\int_0^\xi\frac{1}{\sqrt{\xi}}[g_1(\xi)+g_2(\xi)]d\xi\\
&\delta_{1,2}(\xi)=-\frac{1}{4}\frac{d}{d\xi}[\gamma_{1,2}(\xi)+\gamma_{2,2}(\xi)]-\frac{1}{4}\int_0^\xi[g_1(\xi)\gamma_{1,2}(\xi)+g_2(\xi)\gamma_{2,2}(\xi)]d\xi\\
&\qquad+\frac{1}{2\xi}\left[g_1(\xi)-2\xi\frac{d\gamma_{1,2}(\xi)}{d\xi}-2\gamma_{1,2}(\xi)\right]\\
&\delta_{2,2}(\xi)=\delta_{1,2}(\xi)-\frac{1}{\xi}\left[g_1(\xi)-2\xi\frac{d\gamma_{1,2}(\xi)}{d\xi}-2\gamma_{1,2}(\xi)\right]\\
&\gamma_{1,2n+2}(\xi)=\frac{1}{4\sqrt{\xi}}\int_0^\xi\frac{1}{\sqrt{\xi}}\Big\{g_1(\xi)\delta_{1,2n}(\xi)+g_2(\xi)\delta_{2,2n}(\xi)+\frac{d^2}{d\xi^2}[\delta_{1,2n}(\xi)\\
&\qquad+\delta_{2,2n}(\xi)]\Big\}d\xi+\frac{1}{2\xi}\left[\frac{d^2\gamma_{1,2n}(\xi)}{d\xi^2}+g_1(\xi)\gamma_{1,2n}(\xi)+2\frac{d\delta_{1,2n}(\xi)}{d\xi}\right]\\
&\gamma_{2,2n+2}(\xi)=\gamma_{1,2n+2}(\xi)-\frac{1}{\xi}\left[\frac{d^2\gamma_{1,2n}(\xi)}{d\xi^2}+g_1(\xi)\gamma_{1,2n}(\xi)+2\frac{d\delta_{1,2n}(\xi)}{d\xi}\right]\\
&\delta_{1,2n+2}(\xi)=-\frac{1}{4}\frac{d}{d\xi}[\gamma_{1,2n+2}(\xi)+\gamma_{2,2n+2}(\xi)]-\frac{1}{4}\int_0^\xi[g_1(\xi)\gamma_{1,2n+2}(\xi)\\
&\qquad+g_2(\xi)\gamma_{2,2n+2}(\xi)]d\xi+\frac{1}{2\xi}\left[\frac{d^2\delta_{1,2n}(\xi)}{d\xi^2}+g_1(\xi)\delta_{1,2n}(\xi)\right.\\
&\qquad\left.-2\xi\frac{d\gamma_{1,2n+2}(\xi)}{d\xi}-\gamma_{1,2n+2}(\xi)\right]\\
&\delta_{2,2n+2}(\xi)=\delta_{1,2n+2}(\xi)-\frac{1}{\xi}\left[\frac{d^2\delta_{1,2n}(\xi)}{d\xi^2}+g_1(\xi)\delta_{1,2n}(\xi)\right.\\
&\qquad\left.-2\xi\frac{d\gamma_{1,2n+2}(\xi)}{d\xi}-\gamma_{1,2n+2}(\xi)\right]
\end{aligned}\right\}\quad(3.11)$$

利用以下关系：

$$\frac{d}{d\xi}\left[Z_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]=\mu\sqrt{\xi}\left[Z_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)-\frac{1}{2\mu\xi^{\frac{3}{2}}}Z_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\qquad(3.12)$$

式中 $Z_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)$

是(−2/3)阶 Bessel 函数．将式(3.12)代入式(3.6)，得式(3.1)的一致有效渐近解是：

$$\left.\begin{aligned}\overline{\Pi}=&\mathrm{Re}\left\{\widetilde{C}_1\sqrt{\xi}\left[\delta_1H^{(1)}_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\mu\sqrt{\xi}\gamma_1H^{(1)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\right.\\&\left.+\widetilde{C}_2\sqrt{\xi}\left[\delta_1H^{(2)}_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\mu\sqrt{\xi}\gamma_1H^{(2)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\right\}\\\overline{\Theta}=&\frac{2\beta^2}{E}\mathrm{Im}\left\{\widetilde{C}_1\sqrt{\xi}\left[\delta_2H^{(1)}_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\mu\sqrt{\xi}\gamma_2H^{(1)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\right.\\&\left.+\widetilde{C}_2\sqrt{\xi}\left[\delta_2H^{(2)}_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\mu\sqrt{\xi}\gamma_2H^{(2)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\right\}\end{aligned}\right\}\tag{3.13}$$

式中 $H^{(1)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)$ 和$H^{(2)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)$

分别是第一种和第二种的(−2/3)阶 Hankel 函数．

四、特　　解

在工程上，最常见的载荷是均布法向载荷 p，轴向外加集中力 P 和离心力，即：

$$q_H=p\sin\varphi+\rho r\omega^2,\quad q_V=-p\cos\varphi,\quad V^*=\frac{P}{2\pi r^*}\tag{4.1}$$

式中：ρ 是单位圆环壳的中面面积的壳体质量；ω 是圆环壳的旋转角速度．

在以上载荷作用下，我们有：

$$\left.\begin{aligned}\Theta_m=&\frac{\sqrt[4]{\xi^3r_2h\sin^2\varphi}}{E}\left\{\frac{\cot\varphi}{2a}\left[\frac{pa^2}{1+\alpha\sin\varphi}-\nu p(r^{*2}-R_0^2)+\frac{\nu P}{\pi}\right]\right.\\&\left.+\frac{\cos\varphi}{2R_0(1+\alpha\sin\varphi)}\left[\frac{P}{\pi}-p(r^{*2}-R_0^2)\right]-(3+\nu)\left(\frac{R_0}{\sin\varphi}+a\right)^2\rho\omega^2\sin\varphi\cos\varphi\right\}\\\Pi_m=&\frac{1}{2}\sqrt[4]{\frac{\xi^3r_2\sin^2\varphi}{h^3}}\left[\frac{P}{\pi}-p(r^{*2}-R_0^2)\right]\cot\varphi\end{aligned}\right\}\tag{4.2}$$

首先，对下式求特解：

$$\left.\begin{aligned}&\frac{d^2\Pi_{\mathrm{I}}}{d\xi^2}+g_1(\xi)\Pi_{\mathrm{I}}=-E\lambda_0^2(\Theta_{\mathrm{I}}\xi-\Theta_m)\\&\frac{d^2\Theta_{\mathrm{I}}}{d\xi^2}+g_2(\xi)\Theta_{\mathrm{I}}=\frac{4\beta^4}{E}\lambda_0^2\Pi_{\mathrm{I}}\xi\end{aligned}\right\}\tag{4.3}$$

将 Θ_m 展为 ξ 的幂级数：

$$i\frac{E}{2\beta^2}\Theta_m=e_{1,0}+\sum_{n=1}^{\infty}e_{1,n}\xi^n\tag{4.4}$$

式中： $e_{1,0}=\dfrac{i}{2\beta^2}(Rh)^{\frac{1}{4}}\left[\dfrac{pa}{2}-\dfrac{\nu p}{2a}(r^{*2}-R^2)+\nu\dfrac{P}{2a\pi}-(3+\nu)\rho\omega^2R\right]$

由于 λ_0 是一个大参数且有 $g_1(\xi)=O(1)$ 和 $g_2(\xi)=O(1)$，那末式(4.3)的比较方程是：

$$\frac{d^2I_{\mathrm{I}}}{d\xi^2}=-E\lambda_0^2(\theta_{\mathrm{I}}\xi-\Theta_m),\quad\frac{d^2\theta_{\mathrm{I}}}{d\xi^2}=\frac{4\beta^4}{E}\lambda_0^2I_{\mathrm{I}}\xi\tag{4.5}$$

式(4.5)可以合併成一个方程，即

$$\frac{d^2\widetilde{W}_{\mathrm{I}}}{d\xi^2}+\mu^2\xi\widetilde{W}_{\mathrm{I}}=\frac{i}{2\beta^2}E\mu^2\Theta_m \tag{4.6}$$

式中： $\widetilde{W}_{\mathrm{I}}=I_{\mathrm{I}}+i\frac{E}{2\beta^2}\theta_{\mathrm{I}}$， $i=\sqrt{-1}$

将式(4.4)代入上式，得：

$$\frac{d^2\widetilde{W}_{\mathrm{I}}}{d\xi^2}+\mu^2\xi\widetilde{W}_{\mathrm{I}}=\mu^2\left(e_{1,0}+\sum_{n=1}^{\infty}e_{1,n}\xi^n\right) \tag{4.7}$$

式中： $\sum_{n=1}^{\infty}e_{1,n}\xi^n=i\frac{E}{2\beta^2}\Theta_m-e_{1,0}$

分别对式(4.7)的右边每一项 $e_{1,n}\xi^n$ 求特解。因为

$$\sum_{n=1}^{\infty}e_{1,n}\xi^{n-1}$$

在 $\xi=0$ 是可微的，故式(4.7)有以下特解：

$$\widetilde{W}_{\mathrm{I}}^{p}=T_1(\mu^{\frac{2}{3}}\xi)+\frac{1}{\xi}\left(\frac{i}{2\beta^2}E\Theta_m-e_{1,0}\right)+\sum_{n=1}^{\infty}b_n(\xi)\mu^{-n} \tag{4.8}$$

式中：$b_n(\xi)$是 ξ 的待定函数

和 $T_1(\mu^{\frac{2}{3}}\xi)=\mu^{\frac{2}{3}}e_{1,0}T(\eta)$， $\eta=\mu^{\frac{2}{3}}\xi$

$T(\eta)$是由下式决定的 Lommel 函数：

$$\frac{d^2T(\eta)}{d\eta^2}+\eta T(\eta)=1$$

在本文中，以下式作为式(4.3)的一次渐近特解，

即：

$$\left.\begin{aligned}\Pi_{\mathrm{I}}^{p\,\mathrm{I}}&=\mathrm{Re}\left[T_1(\mu^{\frac{2}{3}}\xi)+\frac{1}{\xi}\left(\frac{i}{2\beta^2}E\Theta_m-e_{1,0}\right)\right]\\ \Theta_{\mathrm{I}}^{p\,\mathrm{I}}&=\frac{2\beta^2}{E}\mathrm{Im}\left[T_1(\mu^{\frac{2}{3}}\xi)+\frac{1}{\xi}\left(\frac{i}{2\beta^2}E\Theta_m-e_{1,0}\right)\right]\end{aligned}\right\} \tag{4.9}$$

本文将它展为 μ 的负次幂级数，以下式来表示：

$$\left.\begin{aligned}\Pi_{\mathrm{I}}^{p}&=\mathrm{Re}\left[\delta_1^pT_1(\mu^{\frac{2}{3}}\xi)+\gamma_1^p\frac{dT_1(\mu^{\frac{2}{3}}\xi)}{d\xi}+\Delta_1^p\right]\\ \Theta_{\mathrm{I}}^{p}&=\frac{2\beta^2}{E}\mathrm{Im}\left[\delta_2^pT_1(\mu^{\frac{2}{3}}\xi)+\gamma_2^p\frac{dT_1(\mu^{\frac{2}{3}}\xi)}{d\xi}+\Delta_2^p\right]\end{aligned}\right\} \tag{4.10}$$

式中：δ_1^p，δ_2^p，γ_1^p，γ_2^p 和 Δ_1^p，Δ_2^p 都是待定函数：

$$\left.\begin{aligned}&\delta_1^p=\sum_{n=0}^{\infty}\delta_{1,n}^p(\xi)\mu^{-n},\quad \delta_2^p=\sum_{n=0}^{\infty}\delta_{2,n}^p(\xi)\mu^{-n}\\ &\gamma_1^p=\sum_{n=0}^{\infty}\gamma_{1,n}^p(\xi)\mu^{-n},\quad \gamma_2^p=\sum_{n=0}^{\infty}\gamma_{2,n}^p(\xi)\mu^{-n}\end{aligned}\right\} \tag{4.11}$$

$$\Delta_1^p=\sum_{n=0}^{\infty}\Delta_{1,n}^p(\xi)\mu^{-n},\quad \Delta_2^p=\sum_{n=0}^{\infty}\Delta_{2,n}^p(\xi)\mu^{-n}$$

将式(4.10)代入式(4.3)，得：

$$\left.\begin{aligned}
&\mathrm{Re}\Big\{\Big[\frac{d^2\delta_1^p}{d\xi^2}-\mu^2\xi(\delta_1^p-\delta_2^p)-\mu^2\gamma_1^p-2\mu^2\xi\frac{d\gamma_1^p}{d\xi}\\
&\quad+g_1(\xi)\delta_1^p\Big]T_1(\mu^{\frac{2}{3}}\xi)+\Big[\frac{d^2\gamma_1^p}{d\xi^2}-\mu^2\xi(\gamma_1^p-\gamma_2^p)+2\frac{d\delta_1^p}{d\xi}\\
&\quad+g_1(\xi)\gamma_1^p\Big]\frac{dT_1(\mu^{\frac{2}{3}}\xi)}{d\xi}+\Big[\frac{d^2\Delta_1^p}{d\xi^2}+g_1(\xi)\Delta_1^p+\mu^2\xi\Delta_2^p\\
&\quad+\mu^2e_{1,0}\delta_1^p+2\mu^2e_{1,0}\frac{d\gamma_1^p}{d\xi}-i\frac{E}{2\beta^2}\mu^2\Theta_m\Big]\Big\}=0\\
&\mathrm{Im}\Big\{\Big[\frac{d^2\delta_2^p}{d\xi^2}-\mu^2\xi(\delta_2^p-\delta_1^p)-\mu^2\gamma_2^p-2\mu^2\xi\frac{d\gamma_2^p}{d\xi}\\
&\quad+g_2(\xi)\delta_2^p\Big]T_1(\mu^{\frac{2}{3}}\xi)+\Big[\frac{d^2\gamma_2^p}{d\xi^2}-\mu^2\xi(\gamma_2^p-\gamma_1^p)+2\frac{d\delta_2^p}{d\xi}\\
&\quad+g_2(\xi)\gamma_2^p\Big]\frac{dT_1(\mu^{\frac{2}{3}}\xi)}{d\xi}+\Big[\frac{d^2\Delta_2^p}{d\xi^2}+g_2(\xi)\Delta_2^p+\mu^2\xi\Delta_1^p\\
&\quad+\mu^2e_{1,0}\delta_2^p+2\mu^2e_{1,0}\frac{d\gamma_2^p}{d\xi}\Big]\Big\}=0
\end{aligned}\right\}\tag{4.12}$$

令上式各个方括号中的项为零，得：

$$\left.\begin{aligned}
&\frac{d^2\delta_1^p}{d\xi^2}-\mu^2\xi(\delta_1^p-\delta_2^p)-\mu^2\gamma_1^p-2\mu^2\xi\frac{d\gamma_1^p}{d\xi}+g_1(\xi)\delta_1^p=0\\
&\frac{d^2\delta_2^p}{d\xi^2}-\mu^2\xi(\delta_2^p-\delta_1^p)-\mu^2\gamma_2^p-2\mu^2\xi\frac{d\gamma_2^p}{d\xi}+g_2(\xi)\delta_2^p=0\\
&\frac{d^2\gamma_1^p}{d\xi^2}-\mu^2\xi(\gamma_1^p-\gamma_2^p)+2\frac{d\delta_1^p}{d\xi}+g_1(\xi)\gamma_1^p=0\\
&\frac{d^2\gamma_2^p}{d\xi^2}-\mu^2\xi(\gamma_2^p-\gamma_1^p)+2\frac{d\delta_2^p}{d\xi}+g_2(\xi)\gamma_2^p=0
\end{aligned}\right\}\tag{4.13}$$

和

$$\left.\begin{aligned}
&\mathrm{Re}\Big[\frac{d^2\Delta_1^p}{d\xi^2}+g_1(\xi)\Delta_1^p+\mu^2\xi\Delta_2^p+\mu^2e_{1,0}\delta_1^p+2\mu^2e_{1,0}\frac{d\gamma_1^p}{d\xi}-i\frac{E}{2\beta^2}\mu^2\Theta_m\Big]=0\\
&\mathrm{Im}\Big[\frac{d^2\Delta_2^p}{d\xi^2}+g_2(\xi)\Delta_2^p+\mu^2\xi\Delta_1^p+\mu^2e_{1,0}\delta_2^p+2\mu^2e_{1,0}\frac{d\gamma_2^p}{d\xi}\Big]=0
\end{aligned}\right\}\tag{4.14}$$

将式(4.11)代入式(4.13)中，由 μ 的同次幂系数为零，得：

$$\begin{aligned}
&\Delta_{1,2n+1}^p(\xi)=\Delta_{2,2n+1}^p(\xi)=0,\quad \Delta_{1,0}^p(\xi)=0,\quad \Delta_{2,0}^p(\xi)=\frac{1}{\xi}\Big(\frac{iE}{2\beta^2}\Theta_m-e_{1,0}\Big)\\
&\Delta_{1,2n+2}^p(\xi)=-\frac{1}{\xi}\Big[\frac{d^2\Delta_{2,2n}^p(\xi)}{d\xi^2}+g_2(\xi)\Delta_{2,2n}^p(\xi)\\
&\qquad+e_{1,0}\delta_{2,2n+2}^p(\xi)+2e_{1,0}\frac{d\gamma_{2,2n}^p(\xi)}{d\xi}\Big]
\end{aligned}\qquad(4.15)$$

$$\varDelta^{p}_{2,2n+2}(\xi)=-\frac{1}{\xi}\left[\frac{d^2\varDelta^{p}_{1,2n}(\xi)}{d\xi^2}+g_1(\xi)\varDelta^{p}_{1,2n}(\xi)+e_{1,0}\delta^{p}_{1,2n+2}(\xi)+2e_{1,0}\frac{d\gamma^{p}_{1,2n}(\xi)}{d\xi}\right]$$

和

$$\left.\begin{aligned}&\delta^{p}_{1,0}(\xi)=\delta^{p}_{2,0}(\xi)=1,\ \delta^{p}_{1,2n+1}(\xi)=\delta^{p}_{2,2n+1}(\xi)=\gamma^{p}_{1,2n+1}(\xi)=\gamma^{p}_{2,2n+1}(\xi)=0\\&\gamma^{p}_{1,0}(\xi)=\gamma^{p}_{2,0}(\xi)=0,\ \gamma^{p}_{1}=\gamma_1,\ \gamma^{p}_{2}=\gamma_2,\ \delta^{p}_{1}=\delta_1+c_1,\ \delta^{p}_{2}=\delta_2+c_2\end{aligned}\right\}\quad(4.16)$$

式中 c_1 和 c_2 是待定常数。

为了使 $\varDelta^{p}_{1}$ 和 $\varDelta^{p}_{2}$ 在 $\xi=0$ 有效，我们令：

$$\left.\begin{aligned}&\delta^{p}_{1,2n+2}(0)=-\frac{1}{e_{1,0}}\left[\frac{d^2\varDelta^{p}_{1,2n}(\xi)}{d\xi^2}+g_1(\xi)\varDelta^{p}_{1,2n}(\xi)+2e_{1,0}\frac{d\gamma^{p}_{1,2n}(\xi)}{d\xi}\right]_{\xi=0}\\&\delta^{p}_{2,2n+2}(0)=-\frac{1}{e_{1,0}}\left[\frac{d^2\varDelta^{p}_{2,2n}(\xi)}{d\xi^2}+g_2(\xi)\varDelta^{p}_{2,2n}(\xi)+2e_{1,0}\frac{d\gamma^{p}_{2,2n}(\xi)}{d\xi}\right]_{\xi=0}\end{aligned}\right\}\quad(4.17)$$

由式(4.17)来决定常数 c_1 和 c_2。

现在再对下式求特解：

$$\frac{d^2\Pi_{\rm I}}{d\xi^2}+g_1(\xi)\Pi_{\rm I}=-E\lambda_0^2\Theta_{\rm I}\xi,\quad \frac{d^2\Theta_{\rm I}}{d\xi^2}+g_2(\xi)\Theta_{\rm I}=\frac{4\beta^4}{E}\lambda_0^2(\Pi_{\rm I}\xi-\Pi_m)\quad(4.18)$$

用相似方法，不难求得式(4.18)的特解。那末式(2.10)的特解由以上二个特解叠加而得：

$$\Pi^{p}=\Pi^{p}_{\rm I}+\Pi^{p}_{\rm II},\quad \Theta^{p}=\Theta^{p}_{\rm I}+\Theta^{p}_{\rm II}\quad(4.19)$$

五、边 缘 问 题

当边缘远离圆环壳顶点时，即当 $\Omega(\varphi)=O(1)$时，我们以式(2.8)作为 $R_0/a=O(1)$，$r>0$ 等厚圆环薄壳力矩理论轴对称边缘问题的基本方程。下面给出齐次解。

式(2.8)的齐次方程是：

$$\frac{d^2\overline{\Pi}_1}{dy^2}+[\Omega(\varphi)+\nu]\overline{\Pi}_1=-E\lambda_0^2\overline{\Theta}_1,\quad \frac{d^2\overline{\Theta}_1}{dy^2}+[\Omega(\varphi)-\nu]\overline{\Theta}_1=\frac{4\beta^4}{E}\lambda_0^2\overline{\Pi}_1\quad(5.1)$$

由于式(5.1)也包含大参数 λ_0^2 和 $\Omega(\varphi)=O(1)$，故我们也有比较方程：

$$\frac{d^2I_1}{dy^2}=-E\lambda_0^2\theta_1,\quad \frac{d^2\theta_1}{dy^2}=\frac{4\beta^4}{E}\lambda_0^2I_1\quad(5.2)$$

上式有解：

$$U_1=I_1+i\frac{E}{2\beta^2}\theta_1=\exp[\pm\mu y]\quad(5.3)$$

那末，式(5.1)的一次渐近解是：

$$\left.\begin{aligned}&\overline{\Pi}^{1}_{1}={\rm Re}[\widetilde{C}_1\exp[-(1+i)\beta\bar{y}]+\widetilde{C}_2\exp[-(1+i)\beta\bar{y}_1]]\\&\overline{\Theta}^{1}_{1}=\frac{2\beta^2}{E}{\rm Im}[\widetilde{C}_1\exp[-(1+i)\beta\bar{y}]+\widetilde{C}_2\exp[-(1+i)\beta\bar{y}_1]]\end{aligned}\right\}\quad(5.4)$$

式中：$\bar{y}=\int_{\varphi_*}^{\varphi}\sqrt{\frac{\sin\varphi}{1+\alpha\sin\varphi}}d\varphi,\quad \bar{y}_1=\int_{\varphi}^{\varphi_*}\sqrt{\frac{\sin\varphi}{1+\alpha\sin\varphi}}d\varphi$

φ_* 是环壳下边界处的 φ 值。

将式(5.1)的解也展为 μ 的负次幂级数：

$$\bar{\Pi}_1=\mathrm{Re}\left(\delta_{1,e}U_1+\gamma_{1,e}\frac{dU_1}{dy}\right),\quad \bar{\Theta}_1=\frac{2\beta^2}{E}\mathrm{Im}\left(\delta_{2,e}U_1+\gamma_{2,e}\frac{dU_1}{dy}\right) \tag{5.5}$$

式中：
$$\left.\begin{aligned}&\delta_{1,e}=\sum_{n=0}^{\infty}\delta_{1,en}(y)\mu^{-n},\quad \delta_{2,e}=\sum_{n=0}^{\infty}\delta_{2,en}(y)\mu^{-n}\\&\gamma_{1,e}=\sum_{n=0}^{\infty}\gamma_{1,en}(y)\mu^{-n},\quad \gamma_{2,e}=\sum_{n=0}^{\infty}\gamma_{2,en}(y)\mu^{-n}\end{aligned}\right\} \tag{5.6}$$

将式(5.5)代入式(5.1)中得：

$$\left.\begin{aligned}&\left\{2\frac{d\delta_{1,e}}{dy}+\frac{d^2\gamma_{1,e}}{dy^2}-\mu^2(\gamma_{1,e}-\gamma_{2,e})+[\Omega(\varphi)+\nu]\gamma_{1,e}\right\}U_1\\&\quad+\left\{\frac{d^2\delta_{1,e}}{dy^2}-\mu^2(\delta_{1,e}-\delta_{2,e})+[\Omega(\varphi)+\nu]\delta_{1,e}-2\mu^2\frac{d\gamma_{1,e}}{dy}\right\}\frac{dU_1}{dy}=0\\&\left\{2\frac{d\delta_{2,e}}{dy}+\frac{d^2\gamma_{2,e}}{dy^2}-\mu^2(\gamma_{2,e}-\gamma_{1,e})+[\Omega(\varphi)-\nu]\gamma_{2,e}\right\}U_1\\&\quad+\left\{\frac{d^2\delta_{2,e}}{dy^2}-\mu^2(\delta_{2,e}-\delta_{1,e})+[\Omega(\varphi)-\nu]\delta_{2,e}-2\mu^2\frac{d\gamma_{2,e}}{dy}\right\}\frac{dU_1}{dy}=0\end{aligned}\right\} \tag{5.7}$$

将式(5.6)代入式(5.7)中，由 U_1 和 dU_1/dy 的系数分别为零，得：

$$\left.\begin{aligned}&\sum_{n=0}^{\infty}\left\{2\frac{d\delta_{1,e,n}(y)}{dy}+\frac{d^2\gamma_{1,e,n}(y)}{dy^2}-[\gamma_{1,e,n+2}(y)-\gamma_{2,e,n+2}(y)]\right.\\&\quad\left.+[\Omega(\varphi)+\nu]\gamma_{1,e,n}(y)\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{2\frac{d\delta_{2,e,n}(y)}{dy}+\frac{d^2\gamma_{2,e,n}(y)}{dy^2}-[\gamma_{2,e,n+2}(y)-\gamma_{1,e,n+2}(y)]\right.\\&\quad\left.+[\Omega(\varphi)-\nu]\gamma_{2,e,n}(y)\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{\frac{d^2\delta_{1,e,n}(y)}{dy^2}-[\delta_{1,e,n+2}(y)-\delta_{2,e,n+2}(y)]+[\Omega(\varphi)\right.\\&\quad\left.+\nu]\delta_{1,e,n}(y)-2\frac{d\gamma_{1,e,n+2}(y)}{dy}\right\}\mu^{-n}=0\\&\sum_{n=0}^{\infty}\left\{\frac{d^2\delta_{2,e,n}(y)}{dy^2}-[\delta_{2,e,n+2}(y)-\delta_{1,e,n+2}(y)]+[\Omega(\varphi)\right.\\&\quad\left.-\nu]\delta_{2,e,n}(y)-2\frac{d\gamma_{2,e,n+2}(y)}{dy}\right\}\mu^{-n}=0\end{aligned}\right\} \tag{5.8}$$

由式(5.8)得：

$$\left.\begin{aligned}&\gamma_{1,e,2n+1}(y)=\gamma_{2,e,2n+1}(y)=\delta_{1,e,2n+1}(y)=\delta_{2,e,2n+1}(y)=0\\&\gamma_{1,e,0}(y)=\gamma_{2,e,0}(y)=0,\quad \delta_{1,e,0}(y)=\delta_{2,e,0}(y)=1\\&\gamma_{1,e,2}(y)=\gamma_{2,e,2}(y)=\frac{1}{2}\int^{y}\Omega(\varphi)dy\\&\cdots\cdots\end{aligned}\right\} \tag{5.9}$$

六、二次渐近解

在 Love-Kirchhoff 薄壳理论假定下，推导求得的等厚圆环薄壳方程(2.8) 和 (2.10)，它们本身就包含着一定误差。作为本文的一次渐近解，张维和 Clark，Новожилов 给出的渐近解，它们的误差都是$|\mu^{-1}|$阶数量级的。现在让我们研究 $\xi\to0$ 的情况，当 $\xi=0$ 时，有

$$g_1(0)=g_2(0)=\frac{1}{14}-\frac{129a^2}{140R_0^2}$$

如果在式(2.10)中略去 $g_1(\xi)$ 和 $g_2(\xi)$ 这二项，而给出的解即上述的近似解，在圆环壳 $\xi=0$ 及其附近，就会有更大误差。

在 Love-Kirchhoff 假定上的薄壳理论，对式 (2.10) 来讲，如果给出具有$|\mu^{-2}|$阶数量级相对误差的渐近解，那末它的误差是在 Love-Kirchhoff 薄壳理论所允许误差范围之内的。本文给出的二次渐近解的误差是$|\mu^{-2}|$阶数量级的。它是：

$$\begin{aligned}\Pi^{\mathrm{I}}+i\frac{E}{2\beta^2}\Theta^{\mathrm{I}}=&\sqrt{\xi}\left\{\widetilde{C}_1\left[H^{(1)}_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\frac{b(\xi)}{\mu}\cdot H^{(1)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\right.\\&\left.+\widetilde{C}_2\left[H^{(2)}_{\frac{1}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)+\frac{b(\xi)}{\mu}H^{(2)}_{-\frac{2}{3}}\left(\frac{2\mu}{3}\xi^{\frac{3}{2}}\right)\right]\right\}+\frac{1}{\xi}\left(i\frac{E}{2\beta^2}\Theta_m-e_{1,0}\right)\\&+T_1(\mu^{\frac{2}{3}}\xi)+\frac{b(\xi)}{\mu^{\frac{4}{3}}\xi^{\frac{1}{2}}}\frac{dT_1(\mu^{\frac{2}{3}}\xi)}{d(\mu^{\frac{2}{3}}\xi)}+\frac{1}{\xi}(\Pi_m-e_{2,0})+T_2(\mu^{\frac{2}{3}}\xi)\\&+\frac{b(\xi)}{\mu^{\frac{4}{3}}\xi^{\frac{1}{2}}}\frac{dT_2(\mu^{\frac{2}{3}}\xi)}{d(\mu^{\frac{2}{3}}\xi)}\end{aligned}\tag{6.1}$$

式中： $b(\xi)=\frac{5}{72y}+\frac{1}{2}\int_0^{\psi}\Omega(\varphi)dy$

和 $e_{2,0}=\frac{1}{2}\left(\frac{R_0}{h^3}\right)^{\frac{1}{4}}\left[\frac{P}{\pi}-p(r^{*2}-R_0^2)\right]$, $T_2(\mu^{\frac{2}{3}}\xi)=\mu^{\frac{2}{3}}e_{2,0}T(\eta)$

对 $\frac{1}{2}\int_0^{\psi}\Omega(\varphi)dy$

我们有：

(1) $\frac{a}{R_0}\leqslant1$ 且 $0\leqslant\varphi\leqslant\pi$:

$$\begin{aligned}\frac{1}{2}\int_0^{\psi}\Omega(\varphi)dy=&\frac{a}{8R_0}y-\frac{5\sqrt{2}}{24\cos\psi\sin^3\psi\sqrt{1+\frac{1}{2}\left(\frac{a}{R_0}-1\right)\sin^2\psi}}\\&+\frac{\sqrt{2}}{4}\left(\frac{5}{4}-\frac{a}{3R_0}\right)\sqrt{\frac{\cot\psi}{1+\frac{1}{2}\left(\frac{a}{R_0}-1\right)\sin^2\psi}}-\frac{\sqrt{2}}{8}\left(\frac{a}{3R_0}+\frac{3}{2}\right)F_1\left(\psi,\sqrt{\frac{1-a/R_0}{2}}\right)\\&+\frac{5\sqrt{2}}{24}F_3\left(\psi,-1,\sqrt{\frac{1-a/R_0}{2}}\right)+\frac{\sqrt{2}}{24}\left(5-11\frac{a}{R_0}-11\frac{a^2}{R_0^2}\right)F_3\left(\psi,\frac{a/R_0-1}{2},\right.\\&\left.\sqrt{\frac{1-a/R_0}{2}}\right)+\frac{a}{\sqrt{2}R_0}F_3\left(\psi,-\frac{1}{2},\sqrt{\frac{1-a/R_0}{2}}\right)\end{aligned}\tag{6.2}$$

式中： $y=\sqrt{2}\left[F_3\left(\psi,\ -\frac{1}{2},\ \sqrt{\frac{1-a/R_0}{2}}\right)-F_1\left(\psi,\ \sqrt{\frac{1-a/R_0}{2}}\right)\right]$

$$\psi=\cos^{-1}\tan\frac{1}{2}\left(\frac{\pi}{2}-\varphi\right)$$

和 $F_1(\cdots)$，$F_3(\cdots)$分别是第一种和第三种椭圆积分：

$$F_1(\psi,k)=\int_0^{\psi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}},\quad F_3(\psi,m,k)=\int_0^{\psi}\frac{d\psi}{(1+m\sin^2\psi)\sqrt{1-k^2\sin^2\psi}}$$

（2） $\frac{a}{R_0}<1$ 且 $\pi\leqslant\varphi\leqslant2\pi$

$$\begin{aligned}\frac{1}{2}\int_0^{y_1}\Omega(\varphi)dy=&\frac{a}{8R_0}y_1+\frac{5\sqrt{2}\,i}{24\cos\psi_1\sin^3\psi_1\sqrt{1-\frac{1}{2}\left(1+\frac{a}{R_0}\right)\sin^2\psi_1}}\\&-i\frac{\sqrt{2}}{4}\left(\frac{5}{4}+\frac{a}{3R_0}\right)\frac{\cot\psi_1}{\sqrt{1-\frac{1}{2}\left(1+\frac{a}{R_0}\right)\sin^2\psi_1}}+i\frac{\sqrt{2}}{8}\left(\frac{3}{2}-\frac{a}{3R_0}\right)F_1\left(\psi_1\sqrt{\frac{1+a/R_0}{2}}\right)\\&-i\frac{5\sqrt{2}}{24}F_3\left(\psi_1,\ -1,\ \sqrt{\frac{1+a/R_0}{2}}\right)-i\frac{\sqrt{2}}{24}\left(5+11\frac{a}{R_0}-11\frac{a^2}{R_0^2}\right)F_3\left(\psi_1,-\frac{1+a/R_0}{2},\right.\\&\left.\sqrt{\frac{1+a/R_0}{2}}\right)+i\frac{a}{\sqrt{2}R_0}F_3\left(\psi_1,-\frac{1}{2},\ \sqrt{\frac{1+a/R_0}{2}}\right)\end{aligned}\tag{6.3}$$

式中：$y_1=i\sqrt{2}\left[F_3\left(\psi_1,-\frac{1}{2},\ \sqrt{\frac{1+a/R_0}{2}}\right)-F_1\left(\psi_1,\ \sqrt{\frac{1+a/R_0}{2}}\right)\right]$

$$\psi_1=\cos^{-1}\left[-\tan\frac{1}{2}\left(\frac{\pi}{2}+\varphi\right)\right]$$

对远离圆环壳的顶点时的边缘问题，式(2.8)的二次渐近齐次解是：

$$\bar{\Pi}_1^{\pi}+i\frac{E}{2\beta^2}\bar{\Theta}_1^{\pi}=\widetilde{C}_1\left[1+i\frac{b_e(y)}{\mu}\right]\exp[i\mu y]+\widetilde{C}_2\left[1-i\frac{b_e(y)}{\mu}\right]\exp[-i\mu y]\tag{6.4}$$

式中：当 $0\leqslant\varphi\leqslant\pi$ 时， $b_e(y)=\frac{1}{2}\int_0^{y}\Omega(\varphi)dy$

当 $\pi\leqslant\varphi\leqslant2\pi$ 时， $b_e(y)=\frac{1}{2}\int_0^{y_1}\Omega(\varphi)dy$

二次渐近解式(6.4)的误差也是$|\mu^{-2}|$阶数量级的，它的误差也是在 Love-Kirchhoff 假定的薄壳理论允许误差范围之内。

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On Second Order Asymptotic Solutions of Axial Symmetrical Problems of $r>0$ Thin Uniform Circular Toroidal Shells with a Large Parameter a^2/R_0h

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Abstract

According to the classical shell theory based on the Love-Kirchhoff assumptions, the basic differential equations for the axial symmetrical problems of $r>0$ thin uniform circular toroidal shells in bending are derived, and the second order asymptotic solutions are given for $r>0$ thin uniform circular toroidal shells with a large parameter a^2/R_0h. In the present paper, the second order asymptotic solutions of the edge problems far from the apex of toroidal shells are given, too. Their errors are within the margins allowed in the classical theory based on the Love-Kirchhoff assumptions.