应用数学和力学，第9卷第4期（1988年4月）
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

复合材料层合板的分析

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（兰州大学，1987年3月4日收到）

摘　要

在复合材料层合板静力和动力分析方面，本文提出了一个层合板理论．此理论从板的总挠度中分开了由弯曲所产生的挠度W_b与由剪切产生的挠度W_s，因而使得求控制方程的解变得容易了．而且便于讨论和分析横向剪切变形对层合板弯曲、振动和稳定性的影响．

一、引　言

对于纤维层合板，提出了多种适合于复合材料的静力分析及动力分析的层合板理论．首先用来分析复合材料层合板的是经典层合板理论，这种理论建立在Kirchhoff假设的基础上，忽略了横向剪切变形的影响，只适合于用来分析比较薄的层合板．

由于在复合材料层合板中，横向剪切模量相对于平面杨氏模量来说比较小，因而横向剪切变形的影响较大．因此，需建立能考虑这种影响的层合板理论．Yang，Norris和Stavsky[1]把分析各向同性板的Mindlin板理论[2]推广到复合材料层合板中去，建立了能够考虑横向剪切变形影响的YNS层合板理论．而后，Whitney和Pagano[3]对此作了一些发展．

另一类理论称为层合板的高阶理论[4]、[5]．在这些理论中，沿厚度方向位移假设为坐标的高次幂函数，因而能更好地用于层合板的分析．但是，求解高阶理论的控制方程是非常困难的．

无论是YNS理论，还是高阶理论，除了在极其简单的边界条件下能求得解析解外，在其他情况下是相当困难的．针对这一情况，本文对铺层对称的层合板建立了一种四变量的层合板的理论．此理论能考虑横向剪切变形的影响，并由于从板的总挠度W中分开了由弯曲所产生的挠度W_b与由剪切变形所产生的挠度W_s，因此使得求控制方程的精确解变得容易了．特别地，关于W_s的方程是非常简单的，可以容易地求出其精确解．这对于讨论和分析横向剪切变形对板弯曲、振动及稳定性的影响带来了极大的方便．从W_s与W_b的数值大小，就可看出这种影响的大小．为了表明本文理论的精确性，我们给出了数值例子，其结果令人满意．

二、控　制　方　程

首先选取直角坐标系(x,y,z)，把坐标原点取在板的中面上，使$(x\text{-}y)$坐标面与中面重

合，x,y轴与矩形板的二条邻边重合，并假设层合板的每一层都有一个平行于x-y平面的弹性对称面。由于层合板的横向挤压应力σ_z很小，因此可以忽略不计，则每层的应力-应变关系为：

$$\{\sigma^{(i)}\}=\begin{Bmatrix}\sigma_x^{(i)}\\ \sigma_y^{(i)}\\ \sigma_{yz}^{(i)}\\ \sigma_{xz}^{(i)}\\ \sigma_{xy}^{(i)}\end{Bmatrix}=\begin{bmatrix}E_{11}^{(i)} & E_{12}^{(i)} & E_{14}^{(i)} & E_{15}^{(i)} & E_{16}^{(i)}\\ E_{12}^{(i)} & E_{22}^{(i)} & E_{24}^{(i)} & E_{25}^{(i)} & E_{26}^{(i)}\\ E_{14}^{(i)} & E_{24}^{(i)} & E_{44}^{(i)} & E_{45}^{(i)} & E_{46}^{(i)}\\ E_{15}^{(i)} & E_{25}^{(i)} & E_{45}^{(i)} & E_{55}^{(i)} & E_{56}^{(i)}\\ E_{16}^{(i)} & E_{26}^{(i)} & E_{46}^{(i)} & E_{56}^{(i)} & E_{66}^{(i)}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{yz}\\ \varepsilon_{xz}\\ \varepsilon_{xy}\end{Bmatrix} \tag{2.1}$$

每单位长度的平面力、弯矩以通常的方法定义为：

$$(N_x,N_y,N_{xy})=\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x,\sigma_y,\sigma_{xy})dz,\quad (M_x,M_y,M_{xy})=\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x,\sigma_y,\sigma_{xy})zdz \tag{2.2a,b}$$

每单位长度的剪力定义为：

$$(Q_x,\ Q_y)=K\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_{xz},\sigma_{yz})dz \tag{2.3}$$

式中K是剪切修正系数。

板的上、下表面的边界条件取为：

$$\sigma_{xz}(x,y,\pm h/2,t)=0,\quad \sigma_{yz}(x,y,\pm h/2,t)=0 \tag{2.4a}$$

$$\sigma_z(x,y,-h/2,t)=P,\quad \sigma_z(x,y,h/2,t)=0 \tag{2.4b}$$

式中P是表面载荷，t是时间，前面虽然假设了$\sigma_z=0$，但是它是维持平衡所必须的。

对于对称正交铺设的纤维层合板来说，每层都可以看成是正交各向异性的。因此只有9个独立的弹性常数，故应力-应变关系(2.1)可简化为：

$$\{\sigma^{(i)}\}=\begin{Bmatrix}\sigma_x^{(i)}\\ \sigma_y^{(i)}\\ \sigma_{yz}^{(i)}\\ \sigma_{xz}^{(i)}\\ \sigma_{xy}^{(i)}\end{Bmatrix}=\begin{bmatrix}E_{11}^{(i)} & E_{12}^{(i)} & 0 & 0 & E_{16}^{(i)}\\ E_{12}^{(i)} & E_{22}^{(i)} & 0 & 0 & E_{26}^{(i)}\\ 0 & 0 & E_{44}^{(i)} & E_{45}^{(i)} & 0\\ 0 & 0 & E_{45}^{(i)} & E_{55}^{(i)} & 0\\ E_{16}^{(i)} & E_{26}^{(i)} & 0 & 0 & E_{66}^{(i)}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{yz}\\ \varepsilon_{xz}\\ \varepsilon_{xy}\end{Bmatrix} \tag{2.5}$$

下面，我们来假设位移场。垂直挠度W假设为：

$$W(x,y,t)=W_b(x,y,t)+W_s(x,y,t) \tag{2.6}$$

式中W_b是由弯曲所产生的挠度，W_s是由剪切变形所产生的挠度。在这里，我们假设了挠度沿厚度方向不变化。

沿x,y轴的平面位移u，v假设为：

$$u(x,y,z,t)=-\frac{\partial W_b}{\partial x}z+\phi_u(x,y,t)z \tag{2.7a}$$

$$v(x,y,z,t)=-\frac{\partial W_b}{\partial y}z+\phi_v(x,y,t)z \tag{2.7b}$$

将(2.6)、(2.7)式代入弹性理论中的应变-位移关系中，然后代入(2.5)中，便得：

$$\left.\begin{aligned}
\sigma_x^{(i)}&=E_{11}^{(i)}\left(-\frac{\partial^2 W_b}{\partial x^2}z+\frac{\partial\phi_u}{\partial x}z\right)+E_{12}^{(i)}\left(-\frac{\partial^2 W_b}{\partial y^2}z+\frac{\partial\phi_v}{\partial y}z\right)\\
&\quad+E_{16}^{(i)}\left(-2\frac{\partial^2 W_b}{\partial x\partial y}z+\left(\frac{\partial\phi_u}{\partial y}+\frac{\partial\phi_v}{\partial x}\right)z\right)\\
\sigma_y^{(i)}&=E_{12}^{(i)}\left(-\frac{\partial^2 W_b}{\partial x^2}z+\frac{\partial\phi_u}{\partial x}z\right)+E_{22}^{(i)}\left(-\frac{\partial^2 W_b}{\partial y^2}z+\frac{\partial\phi_v}{\partial y}z\right)\\
&\quad+E_{26}^{(i)}\left(-2\frac{\partial^2 W_b}{\partial x\partial y}z+\left(\frac{\partial\phi_u}{\partial y}+\frac{\partial\phi_v}{\partial x}\right)z\right)\\
\sigma_{yz}^{(i)}&=E_{44}^{(i)}\left(\frac{\partial W_s}{\partial y}+\phi_v\right)+E_{45}^{(i)}\left(\frac{\partial W_s}{\partial x}+\phi_u\right)\\
\sigma_{xz}^{(i)}&=E_{45}^{(i)}\left(\frac{\partial W_s}{\partial y}+\phi_v\right)+E_{55}^{(i)}\left(\frac{\partial W_s}{\partial x}+\phi_u\right)\\
\sigma_{xy}^{(i)}&=E_{16}^{(i)}\left(-\frac{\partial^2 W_b}{\partial x^2}z+\frac{\partial\phi_u}{\partial x}z\right)+E_{26}^{(i)}\left(-\frac{\partial^2 W_b}{\partial y^2}z+\frac{\partial\phi_v}{\partial y}z\right)\\
&\quad+E_{66}^{(i)}\left(-2\frac{\partial^2 W}{\partial x\partial y}z+\left(\frac{\partial\phi_u}{\partial y}+\frac{\partial\phi_v}{\partial x}\right)z\right)
\end{aligned}\right\}\tag{2.8}$$

将(2.8)式代入(2.2)、(2.3)式中，可得剪力、弯矩与位移的关系：

$$\left.\begin{aligned}
M_x&=-D_{11}\frac{\partial^2 W_b}{\partial x^2}-D_{12}\frac{\partial^2 W_b}{\partial y^2}-2D_{16}\frac{\partial^2 W_b}{\partial x\partial y}+D_{11}\frac{\partial\phi_u}{\partial x}+D_{16}\frac{\partial\phi_u}{\partial y}+D_{12}\frac{\partial\phi_v}{\partial y}\\
&\quad+D_{16}\frac{\partial\phi_v}{\partial x}\\
M_y&=-D_{12}\frac{\partial^2 W_b}{\partial x^2}-D_{22}\frac{\partial^2 W_b}{\partial y^2}-2D_{26}\frac{\partial^2 W_b}{\partial x\partial y}+D_{12}\frac{\partial\phi_u}{\partial x}+D_{26}\frac{\partial\phi_u}{\partial y}+D_{22}\frac{\partial\phi_v}{\partial y}\\
&\quad+D_{26}\frac{\partial\phi_v}{\partial x}\\
M_{xy}&=-D_{16}\frac{\partial^2 W_b}{\partial x^2}-D_{26}\frac{\partial^2 W_b}{\partial y^2}-2D_{66}\frac{\partial^2 W_b}{\partial x\partial y}+D_{16}\frac{\partial\phi_u}{\partial x}+D_{66}\frac{\partial\phi_u}{\partial y}\\
&\quad+D_{26}\frac{\partial\phi_v}{\partial y}+D_{66}\frac{\partial\phi_v}{\partial x}\\
Q_{bx}&=KA_{45}\phi_v+KA_{55}\phi_u,\quad Q_{by}=KA_{44}\phi_v+KA_{45}\phi_u\\
Q_{sx}&=KA_{45}\frac{\partial W_s}{\partial y}+KA_{55}\frac{\partial W_s}{\partial x},\quad Q_{sy}=KA_{44}\frac{\partial W_s}{\partial y}+KA_{45}\frac{\partial W_s}{\partial x}
\end{aligned}\right\}\tag{2.9}$$

式中

$$A_{ij}=\sum_{k=1}^{n}E_{ij}^{(k)}(z_k-z_{k-1}),\quad D_{ij}=\frac{1}{3}\sum_{k=1}^{n}E_{ij}^{(k)}(z_k^3-z_{k-1}^3)\tag{2.10a,b}$$

(2.9)式中Q_{bx}，Q_{by}是由弯曲所产生的剪力；Q_{sx}，Q_{sy}是由剪切所产生的剪力。为了简化方程，我们引进一个剪力势函数ψ。因此，剪力Q_{bx}，Q_{by}可以表示为：

$$Q_{bx}=\partial\psi/\partial y,\quad Q_{by}=-\partial\psi/\partial x\tag{2.11}$$

由(2.9)式得：

$$A_{45}\phi_v+A_{55}\phi_u=K^{-1}\partial\psi/\partial y,\quad A_{44}\phi_v+A_{45}\phi_u=-K^{-1}\partial\psi/\partial x \tag{2.12a,b}$$

因而

$$\phi_u=\frac{1}{KG}\left(A_{44}\frac{\partial\psi}{\partial y}+A_{45}\frac{\partial\psi}{\partial x}\right),\quad \phi_v=-\frac{1}{KG}\left(A_{45}\frac{\partial\psi}{\partial y}+A_{55}\frac{\partial\psi}{\partial x}\right) \tag{2.13a,b}$$

式中　$G=A_{44}A_{55}-A_{45}^2$

而板的转角以及板的平面位移u，v可表示为：

$$\left.\begin{aligned}&u=-\frac{\partial W_b}{\partial x}z+\frac{1}{KG}\left(A_{44}\frac{\partial\psi}{\partial y}+A_{45}\frac{\partial\psi}{\partial x}\right)z,\quad v=-\frac{\partial W_b}{\partial y}z-\frac{1}{KG}\left(A_{45}\frac{\partial\psi}{\partial y}+A_{55}\frac{\partial\psi}{\partial x}\right)z\\&\phi_x=-\frac{\partial W_b}{\partial x}+\frac{1}{KG}\left(A_{44}\frac{\partial\psi}{\partial y}+A_{45}\frac{\partial\psi}{\partial x}\right),\quad \phi_y=-\frac{\partial W_b}{\partial y}-\frac{1}{KG}\left(A_{45}\frac{\partial\psi}{\partial y}+A_{55}\frac{\partial\psi}{\partial x}\right)\end{aligned}\right\} \tag{2.14}$$

将(2.13)式代入(2.9)式，可用W_b，W_s，ψ三个变量把各个力表示出来：

$$\left.\begin{aligned}M_x=&-D_{11}\frac{\partial^2W_b}{\partial x^2}-D_{12}\frac{\partial^2W_b}{\partial y^2}-2D_{16}\frac{\partial^2W_b}{\partial x\partial y}+\frac{1}{KG}(D_{11}A_{45}-D_{16}A_{55})\frac{\partial^2\psi}{\partial x^2}\\&+\frac{1}{KG}(D_{11}A_{44}-D_{12}A_{55})\frac{\partial^2\psi}{\partial x\partial y}+\frac{1}{KG}(D_{16}A_{44}-D_{12}A_{45})\frac{\partial^2\psi}{\partial y^2}\\M_y=&-D_{12}\frac{\partial^2W_b}{\partial x^2}-D_{22}\frac{\partial^2W_b}{\partial y^2}-2D_{26}\frac{\partial^2W_b}{\partial x\partial y}+\frac{1}{KG}(D_{12}A_{45}-D_{26}A_{55})\frac{\partial^2\psi}{\partial x^2}\\&+\frac{1}{KG}(D_{12}A_{44}-D_{22}A_{55})\frac{\partial^2\psi}{\partial x\partial y}+\frac{1}{KG}(D_{26}A_{44}-D_{22}A_{45})\frac{\partial^2\psi}{\partial y^2}\\M_{xy}=&-D_{16}\frac{\partial^2W_b}{\partial x^2}-D_{26}\frac{\partial^2W_b}{\partial y^2}-2D_{66}\frac{\partial^2W_b}{\partial x\partial y}+\frac{1}{KG}(D_{16}A_{45}-D_{66}A_{55})\frac{\partial^2\psi}{\partial x^2}\\&+\frac{1}{KG}(D_{16}A_{44}-D_{26}A_{55})\frac{\partial^2\psi}{\partial x\partial y}+\frac{1}{KG}(D_{66}A_{44}-D_{26}A_{45})\frac{\partial^2\psi}{\partial y^2}\\Q_{bx}=&\frac{\partial\psi}{\partial y},\quad Q_{by}=-\frac{\partial\psi}{\partial x}\\Q_{sx}=&KA_{45}\frac{\partial W_s}{\partial y}+KA_{55}\frac{\partial W_s}{\partial x},\quad Q_{sy}=KA_{44}\frac{\partial W_s}{\partial y}+KA_{45}\frac{\partial W_s}{\partial x}\end{aligned}\right\} \tag{2.15}$$

在忽略板的体力后，层合板的平衡方程可表示为：

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=-R_1(W_b)_{,xtt}+\frac{R_1}{KG}(A_{44}\psi_{,ytt}+A_{45}\psi_{,xtt}) \tag{2.16a}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=-R_1(W_b)_{,ytt}-\frac{R_1}{KG}(A_{45}\psi_{,ytt}+A_{55}\psi_{,xtt}) \tag{2.16b}$$

$$Q_{x,x}+Q_{y,y}+P=R_0W_{b,tt}+R_0W_{s,tt} \tag{2.16c}$$

式中

$$R_0=\sum_{k=1}^{n}\rho^{(k)}(z_k-z_{k-1}),\quad R_1=\frac{1}{3}\sum_{k=1}^{n}\rho^{(k)}(z_k^3-z_{k-1}^3) \tag{2.17a,b}$$

其中$\rho^{(k)}$是第k层的质量密度。在(2.16)式中，平面位移u，v已用(2.14a,b)代入。

将(2.15)式代入(2.16)式，便得层合板理论的控制方程：

$$L_{11}W_b+L_{12}W_s+L_{13}\psi=0,\ L_{21}W_b+L_{22}W_s+L_{23}\psi=0,\ L_{31}W_b+L_{32}W_s+L_{33}\psi+P=0 \tag{2.18}$$

式中

$$\left.\begin{aligned}
&L_{11}=-D_{11}(\ )_{,xxx}-(D_{12}+2D_{66})(\ )_{,xyy}-3D_{16}(\ )_{,xxy}-D_{26}(\ )_{,yyy}\\
&\quad+R_1(\ )_{,xtt}\\
&L_{12}=-KA_{45}(\ )_{,y}-KA_{55}(\ )_{,x}\\
&L_{13}=(KG)^{-1}(D_{11}A_{45}-D_{16}A_{55})(\ )_{,xxx}+[(KG)^{-1}(D_{11}A_{44}-D_{12}A_{55})\\
&\quad+(KG)^{-1}(D_{16}A_{45}-D_{66}A_{55})](\ )_{,xxy}+[(KG)^{-1}(D_{16}A_{44}-D_{22}A_{45})\\
&\quad+(KG)^{-1}(D_{16}A_{44}-D_{26}A_{55})](\ )_{,xyy}+(KG)^{-1}(D_{66}A_{44}\\
&\quad-D_{22}A_{45})(\ )_{,yyy}-(\ )_{,y}-(R_1/KG)A_{45}(\ )_{,xtt}\\
&\quad-(R_1/KG)A_{44}(\ )_{,ytt}\\
&L_{21}=-D_{16}(\ )_{,xxx}-(D_{12}+2D_{66})(\ )_{,xxy}-3D_{26}(\ )_{,xyy}-D_{22}(\ )_{,yyy}\\
&\quad+R_1(\ )_{,ytt}\\
&L_{22}=-KA_{44}(\ )_{,y}-KA_{45}(\ )_{,x}\\
&L_{23}=(KG)^{-1}(D_{16}A_{45}-D_{66}A_{55})(\ )_{,xxx}+[(KG)^{-1}(D_{12}A_{45}-D_{26}A_{55})\\
&\quad+(KG)^{-1}(D_{16}A_{44}-D_{26}A_{55})](\ )_{,xxy}+[(KG)^{-1}(D_{66}A_{44}\\
&\quad-D_{26}A_{45})+(KG)^{-1}(D_{12}A_{44}-D_{22}A_{55})](\ )_{,xyy}+(KG)^{-1}(D_{26}A_{44}\\
&\quad-D_{22}A_{45})(\ )_{,yyy}+(\ )_{,x}+(R_1/KG)A_{55}(\ )_{,xtt}\\
&\quad+(R_1/KG)A_{45}(\ )_{,ytt}\\
&L_{31}=-R_0(\ )_{,tt}\\
&L_{32}=KA_{55}(\ )_{,xx}+2KA_{45}(\ )_{,xy}+KA_{44}(\ )_{,yy}-R_0(\ )_{,tt}\\
&L_{33}=0
\end{aligned}\right\} \tag{2.19}$$

三、对称正交铺设层合板的静力问题

对称正交铺设层合板是由[0°/90°/0°/…]的奇数层铺设而成，它具有比较好的弹性性质，也是在工程上应用最为广泛的一种层合板。由于这种层合板的特殊性，它的一些弹性常数为零：

$$E_{16}^{(i)}=E_{26}^{(i)}=E_{45}^{(i)}=0$$

因此，

$$A_{16}=A_{26}=A_{45}=D_{16}=D_{26}=0$$

因而，(2.19)式的微分算子简化为：

$$\left.\begin{aligned}
&L_{11}=-D_{11}(\ )_{,xxx}-D_3(\ )_{,xyy},\quad L_{12}=-KA_{55}(\ )_{,x}\\
&L_{13}=-\frac{D_2}{KG}(\ )_{,xxy}+\frac{1}{KG}D_{66}A_{44}(\ )_{,yyy}-(\ )_{,y}\\
&L_{21}=-D_3(\ )_{,xxy}-D_{22}(\ )_{,yyy},\quad L_{22}=-KA_{44}(\ )_{,y}\\
&L_{23}=-\frac{1}{KG}D_{66}A_{55}(\ )_{,xxx}+\frac{D_1}{GK}(\ )_{,xyy}+(\ )_{,x}\\
&L_{31}=L_{33}=0,\quad L_{32}=KA_{55}(\ )_{,xx}+KA_{44}(\ )_{,yy}
\end{aligned}\right\} \tag{3.1}$$

式中 $D_1=(D_{12}+D_{66})A_{44}-D_{22}A_{55}$，$D_2=(D_{12}+D_{66})A_{55}-D_{11}A_{44}$，$D_3=D_{12}+2D_{66}$

控制方程为：

$$\left.\begin{aligned}&L_{11}W_b+L_{12}W_s+L_{13}\psi=0,\ L_{21}W_b+L_{22}W_s+L_{23}\psi=0\\&KA_{55}\frac{\partial^2W_s}{\partial x^2}+KA_{44}\frac{\partial^2W_s}{\partial y^2}+P=0\end{aligned}\right\}\tag{3.2}$$

从(3.2)式中可以看到，第三个方程与前二个方程是解藕的。这个关于W_s的微分方程是一个简单的二阶偏微分方程，可以很容易地获得它的精确解。

层合板的转角、挠度、弯矩、剪力简化为：

$$\left.\begin{aligned}&\phi_x=-\frac{\partial W_b}{\partial x}+\frac{1}{KA_{55}}\ \frac{\partial\psi}{\partial y},\ \phi_y=-\frac{\partial W_b}{\partial y}-\frac{1}{KA_{44}}\ \frac{\partial\psi}{\partial x}\\&M_x=-D_{11}\frac{\partial^2W_b}{\partial x^2}-D_{12}\frac{\partial^2W_b}{\partial y^2}+\frac{1}{KG}(D_{11}A_{44}-D_{12}A_{55})\frac{\partial^2\psi}{\partial x\partial y}\\&M_y=-D_{12}\frac{\partial^2W_b}{\partial x^2}-D_{22}\frac{\partial^2W_b}{\partial y^2}+\frac{1}{KG}(D_{12}A_{44}-D_{22}A_{55})\frac{\partial^2\psi}{\partial x\partial y}\\&M_{xy}=-2D_{66}\frac{\partial^2W_b}{\partial x\partial y}-\frac{1}{KG}D_{66}A_{55}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{KG}D_{66}A_{44}\frac{\partial^2\psi}{\partial y^2}\\&Q_{bx}=\frac{\partial\psi}{\partial y},\qquad Q_{by}=-\frac{\partial\psi}{\partial x}\\&Q_{sx}=KA_{55}\frac{\partial W_s}{\partial x},\quad Q_{sy}=KA_{44}\frac{\partial W_s}{\partial y}\end{aligned}\right\}\tag{3.3}$$

为了说明本文理论的精确性。我们考虑四边简支的对称正交铺设层合板在载荷

$$P=P_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

作用下的弯曲问题。

简支边界条件为：

$$\left.\begin{aligned}&W_b=W_s=0,\ M_x=\phi_y=0\qquad x=0,a\\&W_b=W_s=0,\ M_y=\phi_x=0\qquad y=0,b\end{aligned}\right\}\tag{3.4}$$

假设

$$\left.\begin{aligned}&W_b=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{bmn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&W_s=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{smn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&\psi=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\psi_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\end{aligned}\right\}\tag{3.5}$$

显然，这样假设的W_b，W_s，ψ满足边界条件(3.4)。

将载荷$P=P_0\sin(\pi x/a)\sin(\pi y/b)$和(3.5)式代入控制方程(3.2)中，可以解得：

$$\left.\begin{aligned}&W_{b11}=-KW_{s11}[A_{44}(\pi/b)\alpha_2-A_{55}(\pi/a)\beta_2]/(\alpha_1\beta_2+\beta_1\alpha_2)\\&W_{s11}=P_0/K[A_{55}(\pi/a)^2+A_{44}(\pi/b)^2]\\&\psi_{11}=KW_{s11}[A_{44}(\pi/b)\alpha_1+A_{55}(\pi/a)\beta_1]/(\alpha_1\beta_2+\alpha_2\beta_1)\\&W_{bmn}=W_{smn}=\psi_{mn}=0\qquad(m=2,3,\cdots;\ n=2,3,\cdots)\end{aligned}\right\}\tag{3.6}$$

式中

$$\alpha_1=D_{11}\left(\frac{\pi}{a}\right)^3+D_3\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)^2,\quad \alpha_2=-\frac{D_2}{KG}\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)+\frac{1}{KG}D_{66}A_{44}\left(\frac{\pi}{b}\right)^3+\frac{\pi}{b}$$

$$\beta_1=-D_3\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)+D_{22}\left(\frac{\pi}{b}\right)^3,\quad \beta_2=-\frac{1}{KG}D_{66}A_{55}\left(\frac{\pi}{a}\right)^3+\frac{D_1}{KG}\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)^2-\frac{\pi}{a}$$

挠度和剪力函数的解为：

$$\left.\begin{aligned}&W_b=W_{b11}\sin(\pi x/a)\sin(\pi y/b)\\&W_s=W_{s11}\sin(\pi x/a)\sin(\pi y/b)\\&\psi=\psi_{11}\cos(\pi x/a)\cos(\pi y/b)\end{aligned}\right\}\tag{3.7}$$

层合板的总挠度为：

$$W=W_b+W_s=(W_{b11}+W_{s11})\sin(\pi x/a)\sin(\pi y/b)\tag{3.8}$$

四、数 值 例 子

例1 三层的纤维层合板，四边为简支边($b/a=3$)，其铺层为[0°/90°/0°]，承受载荷为$P=P_0\sin(\pi x/a)\sin(\pi y/b)$。每层的材料弹性常数为：

$$E_1/E_2=25,\ G_{12}/E_2=0.5,\ G_{23}/E_2=0.2,\ \nu_{12}=0.25,\ G_{13}=G_{12},\ \nu_{13}=\nu_{12}$$

每层厚度都相等，剪切修正系数取为$K=5/6$（在一些其他文献中，K有时取为$\pi^2/12$，2/3，见[6]、[7]）。由(3.7)、(3.8)式就可求得层合板的中点（$a/2,b/2$）处的挠度，其数值结果见表1。

例2 四层层合板，四边简支，铺层为[0°/90°/90°/0°]，每层厚度相等，承受与例1相同的载荷，材料弹性常数也与上例一样，取$K=5/6$。这种层合板可以用铺层为[0°/90°/0°]的层合板来进行求解[8]，其中两0°层的厚度均为$h/4$，中间90°层的厚度为$h/2$。同样，由(3.7)、(3.8)二式可求得层合板中点处的挠度，其数值结果见表2。

五、结 论

从前面的推导中显见，本文所建立的层合板理论能方便地应用于对称正交铺设层合板的分析。由于垂直挠度W的假设，使得在控制方程中W_s与其他二个变量(W_b,ψ)解藕，简化了方程的求解。特别，这对于分析和讨论横向剪切变形的影响极为方便。

从数值结果中可以看到，当$a/h<10$时，横向剪切变形的影响很大。当$a/h<30$时，这种影响还不能忽略。因此，在复合材料层合板中，薄板的概念不同于各向同性板中的薄板概念。

表 1 [0°/90°/0°]层合板中点的挠度($b/a=3$)

$\frac{a}{h}$	本文结果				[8]	[9]	[10]	[11]
	$\overline{W}_b$	$\overline{W}_s$	$\overline{W}$	$\frac{\overline{W}_s}{\overline{W}}\times 100\%$	$\overline{W}$	$\overline{W}$	$\overline{W}$	$\overline{W}$
5	0.575	1.122	1.697	66.10	1.695	—	—	—
10	0.522	0.281	0.803	34.99	0.802	0.919	0.752	1.141
20	0.508	0.070	0.578	12.11	0.578	0.610	0.565	0.664
30	0.506	0.031	0.537	5.77	—	—	—	—
50	0.504	0.012	0.516	2.33	0.515	0.520	0.513	0.529
100	0.504	0.003	0.507	0.59	0.506	0.508	0.505	0.510

注：$(\overline{W},\overline{W}_b,\overline{W}_s)=100E_2h^3(W,W_b,W_s)/(P_0a^4)$

表 2 [0°/90°/90°/0°]层合板中点挠度($b/a=1$)

$\frac{a}{h}$	本文结果				[8]	[9]	[10]	[11]
	$\overline{W}_b$	$\overline{W}_s$	$\overline{W}$	$\frac{\overline{W}_s}{\overline{W}}\times 100\%$	$\overline{W}$	$\overline{W}$	$\overline{W}$	$\overline{W}$
5	1.357	1.611	2.968	54.28	2.964	—	—	—
10	1.134	0.403	1.537	26.22	1.534	1.709	1.448	2.034
20	1.038	0.101	1.139	8.87	1.136	1.189	1.114	1.273
30	1.017	0.045	1.062	4.24	—	—	—	—
50	1.006	0.016	1.022	1.57	1.019	1.031	1.016	1.048
100	1.002	0.004	1.006	0.40	1.005	1.008	1.003	1.016

注：$(\overline{W},\overline{W}_b,\overline{W}_s)=100\alpha(W,W_b,W_s)h^3/(P_0a^4)$，$\alpha=(\pi^4/12)\{4G_{12}+[E_1+(1+2\nu_{12})E_2]/(1-\nu_{12}\nu_{21})\}$

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Analysis of Composite Laminated Plates

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Abstract

In the static and dynamic analysis of composite laminates, a theory for the laminated plates is presented in this paper. Because the deflection W_b which is caused by the classical bending deformation and the deflection W_s which is caused by the shear deformation are divided from the total deflection W in the theory, this makes it easy to solve the governing equations. In addition, this theory is convenient for the discussion and analysis of the effects of transverse shear deformations on bendings, vibrations and stabilities of laminated plates.