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一类二阶微分差分方程边值问题的奇摄动解*

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摘　要

本文利用两变量展开直接构造边界层项的方法，讨论了一类二阶微分差分方程边值问题的奇摄动解，构造了形式渐近解，作出了余项估计，从而证明了解的存在性。

一、前　　言

对于下列微分差分方程的边值问题

$$\begin{cases} \varepsilon^2 y''(x;\ \varepsilon)-\eta^2(x)y(x;\ \varepsilon)+\alpha(x)y'(x-1;\ \varepsilon) \\ \qquad\qquad +\beta(x)y(x-1;\ \varepsilon)=\psi(x) \qquad (0<x<l) \\ y(x;\ \varepsilon)=\varphi(x) \qquad (-1\leqslant x\leqslant 0) \\ y(l;\ \varepsilon)=\gamma \end{cases}$$

文[1]利用逐步求解法求解，且由匹配方法[2]确定边界层内解，给出了渐近解的首项。

我们利用[3]、[4]提出的两变量展开直接构造边界层的方法讨论了这类问题，得到了求外解和边界层校正函数的 n 阶递推方程，给出了形式渐近解的通式，改进了渐近解的形式，并且利用极值原理[5],[6]作出了余项估计，从而证明了一致有效渐近解的存在。

二、形 式 渐 近 解

对于方程

$$\varepsilon^2 y''(x;\ \varepsilon)-\eta^2(x)y(x;\ \varepsilon)+\alpha(x)y'(x-1;\ \varepsilon) +\beta(x)y(x-1;\ \varepsilon)=\psi(x) \qquad (0<x<l) \tag{2.1}$$

边值条件

$$\left.\begin{aligned} &y(x;\ \varepsilon)=\varphi(x) \qquad (-1\leqslant x\leqslant 0) \\ &y(l;\ \varepsilon)=\gamma \end{aligned}\right\} \tag{2.2}$$

设其中 $\eta(x)$，$\alpha(x)$，$\beta(x)$，$\psi(x)$，$\varphi(x)$ 是 x 的足够光滑函数，l 与 γ 为常数（不妨设 $1<l<2$）。另设

$$[\mathrm{H}_1] \qquad \eta(x)\geqslant\delta>0 \qquad (x\in[0,l])$$

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* 江福汝推荐。

$[H_2]$ $\eta(0)\neq\eta(1)$

我们要求给出在$[0,l]$上连续，$(0,l)$上一阶导数连续，在$(0,l)$上除$x=1$以外二阶导数存在的解，使它在二阶导数存在处满足方程(2.1)，且满足边值条件(2.2)。

为求解，我们把区间$[0,l]$分成如图1所示六个区域[1]，其中Ⅰ，Ⅲ，Ⅳ，Ⅵ为边界层区域

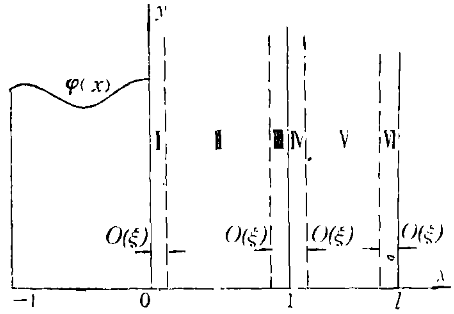

图 1

(1) 区域Ⅱ

设解为

$$Y_2(x;\ \varepsilon)\sim\sum_{n=0}^{\infty}Y_2^{(n)}\varepsilon^n$$

代入方程(2.1)，注意到边值条件(2.2)有

$$\varepsilon^2Y_2''(x;\ \varepsilon)-\eta^2Y_2(x;\ \varepsilon)=\psi(x)-\alpha(x)\varphi'(x-1)-\beta(x)\varphi(x-1)$$

设 $g(x)=\psi(x)-\alpha(x)\varphi'(x-1)-\beta(x)\varphi(x-1)$

便有 $$\varepsilon^2Y_2''(x;\ \varepsilon)-\eta^2Y_2(x;\ \varepsilon)=g(x) \tag{2.3}$$

ε^0: $-\eta^2(x)Y_2^{(0)}(x)=g(x)$, $Y_2^{(0)}(x)=-\dfrac{1}{\eta^2(x)}g(x)$

ε^1: $-\eta^2(x)Y_2^{(1)}(x)=0$, $Y_2^{(1)}(x)\equiv0$

ε^2: $-\eta^2(x)Y_2^{(2)}(x)=-Y_2^{(0)''}$, $Y_2^{(2)}(x)=\dfrac{-1}{\eta^2(x)}\left[\dfrac{g(x)}{\eta^2(x)}\right]''$

…………

ε^n: $-\eta^2(x)Y_2^{(n)}(x)=-Y_2^{(n-2)''}(x)$, $Y_2^{(n)}(x)=\dfrac{1}{\eta^2(x)}Y_2^{(n-2)''}(x)$

(2) 区域Ⅰ

令 $\tilde{x}_1=u(x)/\varepsilon$, $u(x)$为待定函数

设解为 $Y_2(x;\ \varepsilon)+y_1(x,\ \tilde{x}_1;\ \varepsilon)$

其中边界层校正函数为

$$y_1(x,\ \tilde{x}_1;\ \varepsilon)\sim\sum_{n=0}^{\infty}y_1^{(n)}(x,\ \tilde{x}_1)\varepsilon^n$$

当$\tilde{x}_1\to\infty$时，$y_1(x,\ \tilde{x}_1;\ \varepsilon)\to0$

把解代入方程(2.1)，注意到(2.3)有

$$(K_0+\varepsilon K_1+\varepsilon^2K_2)y_1(x,\ \tilde{x}_1;\ \varepsilon)=0$$

其中 $K_0=u'^2(x)\dfrac{\partial^2}{\partial\tilde{x}_1^2}-\eta^2(x)$, $K_1=2u'(x)\dfrac{\partial^2}{\partial x\partial\tilde{x}_1}+u''(x)\dfrac{\partial}{\partial\tilde{x}_1}$, $K_2=\dfrac{\partial^2}{\partial x^2}$

ε^0: $K_0y_1^{(0)}=0$

令 $u'(x)=\eta(x)$，$u(x)=\int_0^x \eta(s)ds>0$

有 $y_1^{(0)}=A_0(x)\exp[-\tilde{x}_1]$

ε^1: $K_0 y_1^{(1)}=-K_1 y_1^{(0)}$

我们选取 $A_0(x)$ 使 $K_1 y_1^{(0)}=0$，即

$$-2\eta(x_1)A_0'(x)-\eta'(x)A_0(x)=0$$

$$A_0(x)=a_0\sqrt{\frac{\eta(0)}{\eta(x)}}$$

这样 $y_1^{(1)}=A_1(x)\exp[-\tilde{x}_1]$

ε^2: $K_0 y_1^{(2)}=-K_1 y_1^{(1)}-K_2 y_1^{(0)}$

我们选取 $A_1(x)$ 使 $K_1 y_1^{(1)}+K_2 y_1^{(0)}=0$，即

$$-2\eta(x)A_1'(x)-\eta'(x)A_1(x)+A_0''(x)=0$$

$$A_1(x)=a_1\sqrt{\frac{\eta(0)}{\eta(x)}}+\frac{1}{\sqrt{\eta(x)}}\int_0^x \frac{A_0''(s)}{2\sqrt{\eta(s)}}ds$$

这样 $y_1^{(2)}=A_2(x)\exp[-\tilde{x}_1]$

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ε^n: $K_0 y_1^{(n)}=-K_1 y_1^{(n-1)}-K_2 y_1^{(n-2)}$

有 $y_1^{(n)}=A_n(x)\exp[-\tilde{x}_1]$

其中 $$A_n(x)=a_n\sqrt{\frac{\eta(0)}{\eta(x)}}+\frac{1}{\sqrt{\eta(x)}}\int_0^x \frac{A_{n-1}''(s)}{2\sqrt{\eta(s)}}ds \tag{2.4}$$

满足 $K_1 y_1^{(n)}+K_2 y_1^{(n-1)}=0$

(3) 区域Ⅲ

令 $\tilde{x}_3=v(x)/\varepsilon$，$v(x)$为待定函数

设解为 $Y_2(x;\varepsilon)+y_3(x,\tilde{x}_3;\varepsilon)$

其中边界层校正函数为

$$y_3(x,\tilde{x}_3;\varepsilon)\sim\frac{1}{\varepsilon}\sum_{n=0}^{\infty}y_3^{(n)}(x,\tilde{x}_3)\varepsilon^n$$

当$\tilde{x}_3\to\infty$时，$y_3(x,\tilde{x}_3;\varepsilon)\to 0$

把解代入方程(2.1)，注意到(2.3)有

$$(\widetilde{K}_0+\varepsilon\widetilde{K}_1+\varepsilon^2\widetilde{K}_2)y_3(x,\ \widetilde{x}_3;\ \varepsilon)=0$$

其中 $\widetilde{K}_0=v'^2(x)\dfrac{\partial^2}{\partial\widetilde{x}_3^2}-\eta^2(x)$, $\widetilde{K}_1=2v'(x)\dfrac{\partial^2}{\partial x\partial\widetilde{x}_3}+v''(x)\dfrac{\partial}{\partial\widetilde{x}_3}$, $\widetilde{K}_2=\dfrac{\partial^2}{\partial x^2}$

ε^{-1}:
$$\widetilde{K}_0y_3^{(0)}=0$$

$$\left[v'^2(x)\frac{\partial^2}{\partial\widetilde{x}_3^2}-\eta^2(x)\right]y_3^{(0)}=0$$

令
$$v'(x)=-\eta(x),\quad v(x)=\int_x^1\eta(s)ds>0$$

有
$$y_3^{(0)}=B_0(x)\exp[-\widetilde{x}_3]$$

相仿区域Ⅰ的做法，有

$$B_0(x)=b_0\sqrt{\frac{\eta(1)}{\eta(x)}}$$

$$y_3^{(1)}=B_1(x)\ \exp[-\widetilde{x}_3],\quad B_1(x)=b_1\sqrt{\frac{\eta(1)}{\eta(x)}}-\frac{1}{\sqrt{\eta(x)}}\int_1^x\frac{B_0''(s)}{2\sqrt{\eta(s)}}ds$$

…………

$$y_3^{(n)}=B_n(x)\exp[-\widetilde{x}_3],\quad B_n(x)=b_n\sqrt{\frac{\eta(1)}{\eta(x)}}-\frac{1}{\sqrt{\eta(x)}}\int_1^x\frac{B_{n-1}''(s)}{2\sqrt{\eta(s)}}ds \tag{2.5}$$

其中$B_n(x)$满足 $\widetilde{K}_1y_3^{(n)}+\widetilde{K}_2y_3^{(n-1)}=0$

（4） 区域Ⅴ

设解为

$$Y_5(x;\ \varepsilon)\sim\sum_{n=0}^{\infty}Y_5^{(n)}(x)\varepsilon^n$$

代入方程(2.1)有

$$\varepsilon^2Y_5''(x;\ \varepsilon)-\eta^2(x)Y_5(x;\ \varepsilon)+\alpha(x)\ Y_2'(x-1;\ \varepsilon)+\beta(x)Y_2(x-1;\ \varepsilon)=\psi(x)$$

从而有

$$Y_5^{(0)}=\frac{1}{\eta^2(x)}[-\psi(x)+\alpha(x)Y_2^{(0)\prime}(x-1)+\beta(x)Y_2^{(0)}(x-1)],\quad Y_5^{(1)}\equiv 0$$

$$Y_5^{(2)}=\frac{1}{\eta^2(x)}[Y_5^{(0)\prime\prime}(x)+\alpha(x)Y_2^{(2)\prime}(x-1)+\beta(x)Y_2^{(2)}(x-1)]$$

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$$Y_5^{(n)}=\frac{1}{\eta^2(x)}[Y_5^{(n-2)\prime\prime}(x)+\alpha(x)Y_2^{(n)\prime}\ (x-1)+\beta(x)Y_2^{(n)}\ (x-1)]$$

（5） **区域Ⅳ**

令 $\tilde{x}_4=\omega(x)/\varepsilon$，$\omega(x)$为待定函数

设解为 $Y_5(x;\ \varepsilon)+y_4(x,\ \tilde{x}_4;\ \varepsilon)$

其中边界层校正函数为

$$y_4(x,\ \tilde{x}_4;\ \varepsilon)\sim\frac{1}{\varepsilon}\sum_{n=0}^{\infty}y_4^{(n)}(x,\tilde{x}_4)\varepsilon^n$$

当$\tilde{x}_4\to\infty$时，$y_4(x,\tilde{x}_4;\ \varepsilon)\to 0$

把解代入方程（2.1）有

$$(\bar{K}_0+\varepsilon\bar{K}_1+\varepsilon^2\bar{K}_2)y_4(x,\tilde{x}_4;\ \varepsilon)+\alpha(x)y_1'\left(x-1,\ \frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds;\ \varepsilon\right)$$

$$+\beta(x)y_1\left(x-1,\ \frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds,\ \varepsilon\right)=0$$

其中
$$\bar{K}_0=\omega'^2(x)\frac{\partial^2}{\partial\tilde{x}_4^2}-\eta^2(x)$$

$$\bar{K}_1=2\omega'(x)\frac{\partial^2}{\partial x\partial\tilde{x}_4}+\omega''(x)\frac{\partial}{\partial\tilde{x}_4}$$

$$\bar{K}_2=\frac{\partial^2}{\partial x^2}$$

ε^{-1}:
$$\bar{K}_0y_4^{(0)}(x,\tilde{x}_4)=\alpha(x)A_0(x-1)\eta(x-1)\exp\left[-\frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds\right]$$

令
$$\omega'(x)=\eta(x),\quad \omega(x)=\int_1^x\eta(s)ds>0$$

$$\xi(x)=\frac{\int_0^{x-1}\eta(s)ds}{\int_1^x\eta(s)ds}$$

有
$$\eta^2(x)\left(\frac{\partial^2}{\partial\tilde{x}_4^2}-1\right)y_4^{(0)}(x,\tilde{x}_4)=\alpha(x)a_0\sqrt{\eta(0)\eta(x-1)}\exp[-\xi(x)\tilde{x}_4]$$

由假设[H_2]，$\eta(0)\neq\eta(1)$，在 $x=1$ 的某小邻域，可保证$\xi(x)\neq 1$*，这样

$$y_4^{(0)}=C_0(x)\exp[-\tilde{x}_4]+\frac{1}{\eta^2(x)}\alpha(x)a_0\sqrt{\eta(0)\eta(x-1)}\frac{1}{\xi^2(x)-1}$$
$$\cdot\exp[-\xi(x)\tilde{x}_4]=C_0(x)\exp[-\tilde{x}_4]+D_0(x)\exp[-\xi(x)\tilde{x}_4]$$

* 若$\eta(0)=\eta(1)$，特别$\xi(x)=1$有

$$y_4^{(0)}=C_0(x)\exp[-\tilde{x}_4]+\frac{1}{\eta^2(x)}\alpha(x)a_0\sqrt{\eta(0)\eta(x-1)}\left(-\frac{1}{2}\tilde{x}_4\exp[-\tilde{x}_4]\right)$$

以后也可相仿讨论.

ε^0:
$$\eta^2(x)\left(\frac{\partial^2}{\partial\tilde{x}_4^2}-1\right)y_4^{(1)}(x,\tilde{x}_4)=-\bar{K}_1y_4^{(0)}-\alpha(x)A_0'(x-1)\exp[-\xi(x)\tilde{x}_4]$$
$$-\beta(x)A_0(x-1)\exp[-\xi(x)\tilde{x}_4]+\alpha(x)A_1(x-1)\eta(x-1)\exp[-\xi(x)\tilde{x}_4]$$

取$C_0(x)$满足$\bar{K}_1C_0(x)\exp[-\tilde{x}_4]=0$，有
$$C_0(x)=c_0\sqrt{\frac{\eta(1)}{\eta(x)}}$$

这样
$$y_4^{(1)}=C_1(x)\exp[-\tilde{x}_4]+D_1(x,\tilde{x}_4)\exp[-\xi(x)\tilde{x}_4]$$

其中 $D_1(x,\tilde{x}_4)\exp[-\xi(x)\tilde{x}_4]$ 为相应方程特解。

ε^1:
$$\eta^2(x)\left(\frac{\partial^2}{\partial\tilde{x}_4^2}-1\right)y_4^{(2)}(x,\tilde{x}_4)=-\bar{K}_1y_4^{(1)}-\bar{K}_2y_4^{(0)}$$
$$-[\alpha(x)A_1'(x-1)-\alpha(x)A_2(x-1)\eta(x-1)$$
$$+\beta(x)A_1(x-1)]\exp[-\xi(x)\tilde{x}_4]$$

使$C_1(x)$满足 $\bar{K}_1C_1(x)\exp[-\tilde{x}_4]+\bar{K}_2C_0(x)\exp[-\tilde{x}_4]=0$

有
$$C_1(x)=C_1\sqrt{\frac{\eta(1)}{\eta(x)}}+\frac{1}{\sqrt{\eta(x)}}\int_1^x\frac{C_0''(s)}{2\sqrt{\eta(s)}}ds$$

这样
$$y_4^{(2)}=C_2(x)\exp[-\tilde{x}_4]+D_2(x,\tilde{x}_4)\exp[-\xi(x)\tilde{x}_4]$$

…………

ε^{n-1}:
$$\eta^2(x)\left(\frac{\partial^2}{\partial\tilde{x}_4^2}-1\right)y_4^{(n)}(x,\tilde{x}_4)=-\bar{K}_1y_4^{(n-1)}-\bar{K}_2y_4^{(n-2)}$$
$$-[\alpha(x)A_{n-1}'(x-1)-\alpha(x)A_n(x-1)\eta(x-1)$$
$$+\beta(x)A_{n-1}(x-1)]\exp[-\xi(x)\tilde{x}_4] \tag{2.6}$$
$$y_4^{(n)}=C_n(x)\exp[-\tilde{x}_4]+D_n(x,\tilde{x}_4)\exp[-\xi(x)\tilde{x}_4]$$

其中
$$C_n(x)=C_n\sqrt{\frac{\eta(1)}{\eta(x)}}+\frac{1}{\sqrt{\eta(x)}}\int_1^x\frac{C_{n-1}''(s)}{2\sqrt{\eta(s)}}ds \tag{2.7}$$

满足
$$\bar{K}_1C_n(x)\exp[-\tilde{x}_4]+\bar{K}_2C_{n-1}(x)\exp[-\tilde{x}_4]=0$$

（6） 区域Ⅵ

设
$$\tilde{x}_6=\frac{1}{\varepsilon}\int_x^l\eta(s)ds>0$$

又设解为 $Y_5(x;\ \varepsilon)+y_6(x,\ \tilde{x}_6;\ \varepsilon)$

其中边界层校正函数为
$$y_6(x,\ \tilde{x}_6,\ \varepsilon)\sim\sum_{n=0}^{\infty}y_6^{(n)}(x,\tilde{x}_6)\varepsilon^n$$

当$\tilde{x}_6\to\infty$时，$y_6(x,\ \tilde{x}_6;\ \varepsilon)\to0$

把解代入方程(2.1)，有

$$(K_0^* + \varepsilon K_1^* + \varepsilon^2 K_2^*) y_6(x,\ \tilde{x}_6;\ \varepsilon) = 0$$

其中 $K_0^* = \eta^2(x)\dfrac{\partial^2}{\partial \tilde{x}_6^2} - \eta^2(x)$，$K_1^* = -2\eta(x)\dfrac{\partial^2}{\partial x \partial \tilde{x}_6} - \eta'(x)\dfrac{\partial}{\partial \tilde{x}_6}$，$K_2^* = \dfrac{\partial^2}{\partial x^2}$

相仿区域Ⅲ的运算可得

$$y_6^{(0)} = E_0(x)\exp[-\tilde{x}_6],\ E_0(x) = e_0\sqrt{\frac{\eta(l)}{\eta(x)}}$$

$$y_6^{(1)} = E_1(x)\exp[-\tilde{x}_6],\ E_1(x) = e_1\sqrt{\frac{\eta(l)}{\eta(x)}} - \frac{1}{\sqrt{\eta(x)}}\int_l^x \frac{E_0''(s)}{2\sqrt{\eta(s)}} ds$$

…………

$$y_6^{(n)} = E_n(x)\exp[-\tilde{x}_6],\ E_n(x) = e_n\sqrt{\frac{\eta(l)}{\eta(x)}} - \frac{1}{\sqrt{\eta(x)}}\int_l^x \frac{E_{n-1}''(s)}{2\sqrt{\eta(s)}} ds \qquad (2.8)$$

（7） **确定常数**

由
$$\begin{cases} Y_2(0;\ \varepsilon) + y_1(0,\ 0;\ \varepsilon) = \varphi(0) \\ Y_5(l;\ \varepsilon) + y_6(l,\ 0;\ \varepsilon) = \gamma \\ Y_2(1;\ \varepsilon) + y_3(1,\ 0;\ \varepsilon) = Y_5(1;\ \varepsilon) + y_4(1,\ 0;\ \varepsilon) \\ Y_2'(1;\ \varepsilon) + \dfrac{d}{dx} y_3(1,\ 0;\ \varepsilon) = Y_5'(1;\ \varepsilon) + \dfrac{d}{dx} y_4(1,\ 0;\ \varepsilon) \end{cases}$$

可以得到一系列线性代数方程组，从而确定常数a_i，b_i，c_i，$e_i(i=0,1,2,\cdots)$，如对a_0, b_0, c_0, e_0就可以由下列方程组确定：

$$\left.\begin{aligned} & -\frac{g(0)}{\eta^2(0)} + a_0 = \varphi(0) \\ & -\frac{1}{\eta^2(l)}\left[\psi(l) + \alpha(l)\left(\frac{g(x)}{\eta^2(x)}\right)'\Big|_{x=l-1} + \beta(l)\frac{g(l-1)}{\eta^2(l-1)}\right] + e_0 = \gamma \\ & b_0 = c_0 + \eta(0)\alpha(1)a_0\frac{1}{\eta^2(0) - \eta^2(1)} \\ & b_0\eta(1) = -c_0\eta(1) - \eta^2(0)\alpha(1)a_0\frac{1}{\eta^2(0) - \eta^2(1)} \end{aligned}\right\} \qquad (2.9)$$

综上所述，我们得到了方程（2.1）满足边值条件（2.2）的形式渐近解：

$$y(x;\ \varepsilon) \sim \begin{cases} y_1(x,\ \tilde{x}_1;\ \varepsilon) + Y_2(x;\ \varepsilon) + y_3(x,\ \tilde{x}_1;\ \varepsilon) & (0<x<1) \\ y_4(x,\ \tilde{x}_4;\ \varepsilon) + Y_5(x;\ \varepsilon) + y_6(x,\ \tilde{x}_6;\ \varepsilon) & (1<x<l) \end{cases}$$

其中
$$y_1(x,\ \tilde{x}_1;\ \varepsilon) \sim \sum_{n=0}^{\infty} A_n(x)\exp[-\tilde{x}_1]\varepsilon^n$$

$$Y_2(x;\ \varepsilon) \sim \sum_{n=0}^{\infty} Y_2^{(n)}(x)\varepsilon^n$$

$$y_3(x,\ \tilde{x}_3;\ \varepsilon)\sim\frac{1}{3}\sum_{n=0}^{\infty}B_n(x)\exp[-\tilde{x}_3]\varepsilon^n$$

$$y_4(x,\ \tilde{x}_4;\ \varepsilon)\sim\frac{1}{\varepsilon}\sum_{h=0}^{\infty}[C_n(x)\exp[-\tilde{x}_4]+D_n(x,\tilde{x}_4)\exp[-\xi(x(\tilde{x}_4]]\varepsilon^n$$

$$Y_6(x;\ \varepsilon)\sim\sum_{h=0}^{\infty}Y_5^{(n)}(x)\varepsilon^n$$

$$y_6(x,\ \tilde{x}_6;\ \varepsilon)\sim\sum_{h=0}^{\infty}E_n(x)\exp[-\tilde{x}_6]$$

$$\tilde{x}_1=\frac{1}{\varepsilon}\int_0^x\eta(x)dx,\ \ \tilde{x}_3=\frac{1}{\varepsilon}\int_x^1\eta(x)dx,\ \tilde{x}_4=\frac{1}{\varepsilon}\int_1^x\eta(x)dx,\ \tilde{x}_6=\frac{1}{\varepsilon}\int_x^l\eta(x)dx$$

$A_n(x)$，$B_n(x)$，$C_n(x)$，$E_n(x)$ 分别由 (2.4)，(2.5)，(2.7)，(2.8) 给定，$D_n(x,\tilde{x}_4)\cdot\exp[-\xi(x)\tilde{x}_4]$ 是满足方程 (2.6) 的某特解；常数a_i，b_i，c_i，e_i 由类似于 (2.9) 的线性代数方程组确定。

三、余 项 估 计

方程 (2.1) 在$0<x<1$上为

$$\varepsilon^2y''(x;\ \varepsilon)-\eta^2(x)y(x;\ \varepsilon)=g(x) \tag{3.1}$$

有形式渐近解

$$y(x;\ \varepsilon)\sim\sum_{n=0}^{\infty}\left[Y_2^{(n)}(x)+A_n(x)\exp\left[-\frac{1}{\varepsilon}\int_0^x\eta(s)ds\right]\right]\varepsilon^n$$

$$+\frac{1}{\varepsilon}\sum_{n=0}^{\infty}B_n(x)\exp\left[-\frac{1}{\varepsilon}\int_x^1\eta(s)ds\right]\varepsilon^n$$

由Borel-Ritt定理[6]，存在ε的解析函数$A(\varepsilon)$，$B(\varepsilon)$使

$$A(\varepsilon)\sim\sum_{n=0}^{\infty}[Y_2^{(n)}(0)+A_n(0)]\varepsilon^n$$

$$B(\varepsilon)\sim\sum_{n=0}^{\infty}Y_2^{(n)}(1)\varepsilon^n+\frac{1}{\varepsilon}\sum_{n=1}^{\infty}B_n(1)\varepsilon^n$$

这样我们有边值问题

$$\left.\begin{aligned}&\varepsilon^2\bar{y}''(x;\ \varepsilon)-\eta^2(x)\bar{y}(x;\ \varepsilon)=g(x)\qquad(0<x<1)\\&\bar{y}(0;\ \varepsilon)=A(\varepsilon),\ \ \bar{y}(1;\ \varepsilon)=\frac{1}{\varepsilon}B_0(1)+B(\varepsilon)\end{aligned}\right\} \tag{3.2}$$

它的解可记为

$$\bar{y}(x;\varepsilon)=\sum_{n=0}^{N}\left[Y_2^{(n)}+A_n(x)\exp\left[-\frac{1}{\varepsilon}\int_0^x\eta(s)ds\right]\right]\varepsilon^n$$

$$+\frac{1}{\varepsilon}\sum_{n=0}^{N+1}B_n(x)\exp\left[-\frac{1}{\varepsilon}\int_x^1\eta(s)ds\right]\varepsilon^n+R_N \tag{3.3}$$

代入问题（3.2）得到

$$\left.\begin{array}{ll}\varepsilon^2R_N''-\eta^2(x)R_N=\varepsilon^{N+1}F_N(x;\varepsilon) & (0<x<1)\\ R_N(0;\varepsilon)=0,\ R_N(1;\varepsilon)=\varepsilon^{N+1}G_N(1;\varepsilon) & \end{array}\right\} \tag{3.4}$$

其中 $$F_N(x;\varepsilon)=-\left[Y_2^{(N-1)''}+\varepsilon Y_2^{(N)}(x)+\varepsilon A_N''(x)\exp\left[-\frac{1}{\varepsilon}\int_0^x\eta(s)ds\right]\right.$$

$$\left.+\varepsilon B_{N+1}''(x)\exp\left[-\frac{1}{\varepsilon}\int_x^1\eta(s)ds\right]\right]=O(1)$$

$$\varepsilon^{N+1}G_N(1;\varepsilon)=B(\varepsilon)+\frac{1}{\varepsilon}B_0(1)-\sum_{n=0}^{N}Y_2^{(n)}\varepsilon^n-\frac{1}{\varepsilon}\sum_{n=0}^{N+1}B_n(1)\varepsilon^n$$

$$\sim\sum_{n=N+1}^{\infty}Y_2^{(n)}(1)\varepsilon^n+\frac{1}{\varepsilon}\sum_{n=N+2}^{\infty}B_n(1)\varepsilon^n=\varepsilon^{N+1}O(1)$$

所以 $$G_N(1;\varepsilon)=O(1)$$

利用常微分方程的极值原理[5]或 Nagumo的微分不等式的有关结论[6]，我们对边值问题（3.4）的解估计，把（3.4）写成

$$\left.\begin{array}{ll}\varepsilon^2R_N''=\eta^2(x)R_N+\varepsilon^{N+1}F_N(x;\varepsilon)=F(x,R_N;\varepsilon) & (0<x<1)\\ R_N(0;\varepsilon)=0,\ R_N(1;\varepsilon)=\varepsilon^{N+1}G_N(1;\varepsilon) & \end{array}\right\} \tag{3.5}$$

由于 $$\partial F/\partial R_N=\eta^2(x)\geqslant\delta^2$$

从[6]定理 2.3 的推论2.1知

$$|R_N|\leqslant M/\delta^2$$

其中 $$M=\max\{\max_{[0,1]}|F(x,0;\varepsilon)|,\ \delta^2\varepsilon^{N+1}|G_N(1;\varepsilon)|\}$$

$$=O(\varepsilon^{N+1})$$

所以 $$R_N=O(\varepsilon^{N+1})$$

这样方程（2.1）在$0<x<1$上存在唯一的一致有效渐近解(3.3)。

方程（2.1）在$1<x<l$上为

$$\varepsilon^2y''(x;\varepsilon)-\eta^2(x)y(x;\varepsilon)=f(x;\varepsilon)$$

其中 $$f(x;\varepsilon)=\psi(x)-\alpha(x)\left[Y_2'(x-1;\varepsilon)+\frac{d}{dx}y_1\left(x-1,\ \frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds;\varepsilon\right)\right.$$

$$+\frac{d}{dx}y_3\left(x-1,\ \frac{1}{\varepsilon}\int_{x-1}^{1}\eta(s)ds;\ \varepsilon\right)\Big]-\beta(x)\Big[Y_2(x-1;\varepsilon)$$

$$+y_1\left(x-1,\ \frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds;\ \varepsilon\right)+y_3\left(x-1,\ \frac{1}{\varepsilon}\int_{x-1}^{1}\eta(s)ds;\ \varepsilon\right)\Big]$$

$$=\psi(x)-\alpha(x)\ \frac{d}{dx}\Big[\sum_{n=0}^{N}Y_2^{(n)}(x-1)\varepsilon^n$$

$$+\sum_{n=0}^{N}A_n(x-1)\exp\left[-\frac{1}{\varepsilon}\int_0^{x-1}\eta(s)\,ds\right]\varepsilon^n$$

$$+\frac{1}{\varepsilon}\sum_{n=0}^{N+1}B_n(x-1)\exp\left[-\frac{1}{\varepsilon}\int_{x-1}^{1}\eta(s)\,ds\right]\varepsilon^n+R_N(x-1;\varepsilon)\Big]$$

$$-\beta(x)\Big[\sum_{n=0}^{N}Y_2^{(n)}(x-1)\varepsilon^n+\sum_{n=0}^{N}A_n(x-1)\exp\left[-\frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds\right]\varepsilon^n$$

$$+\frac{1}{\varepsilon}\sum_{n=0}^{N+1}B_n(x-1)\exp\left[-\frac{1}{\varepsilon}\int_{x-1}^{1}\eta(s)\,ds\right]\varepsilon^n+R_N(x-1;\varepsilon)\Big]$$

有形式渐近解

$$y(x;\varepsilon)\sim\sum_{n=0}^{\infty}\Big[Y_5^{(n)}(x)+E_n(x)\exp\left[-\frac{1}{\varepsilon}\int_x^l\eta(s)\,ds\right]\Big]\varepsilon^n$$

$$+\frac{1}{\varepsilon}\sum_{n=0}^{\infty}\Big[C_n(x)\exp\left[-\frac{1}{\varepsilon}\int_1^x\eta(s)\,ds\right]$$

$$+D_n\left(x,\ \frac{1}{\varepsilon}\int_1^x\eta(s)\,ds\right)\exp\left[-\frac{1}{\varepsilon}\int_0^{x-1}\eta(s)\,ds\right]\Big]\varepsilon^n \tag{3.6}$$

由 Borel-Ritt 定理，存在ε的解析函数$C(\varepsilon)$，$D(\varepsilon)$使

$$C(\varepsilon)\sim\sum_{n=0}^{\infty}Y_5^{(n)}\varepsilon^n+\frac{1}{\varepsilon}\sum_{n=1}^{\infty}[C_n(1)+D_n(1,0)]\varepsilon^n$$

$$D(\varepsilon)\sim\sum_{n=0}^{\infty}\Big[Y_5^{(n)}(l)+E_n(l)\Big]\varepsilon^n$$

这样我们有边值问题

$$\left.\begin{aligned}&\varepsilon^2\tilde{y}''(x;\varepsilon)-\eta^2(x)\tilde{y}(x;\varepsilon)=f(x;\varepsilon)\qquad(1<x<l)\\&\tilde{y}(1;\varepsilon)=C(\varepsilon)+\frac{1}{\varepsilon}[C_0(1)+D_0(1,0)]\\&\tilde{y}(l;\varepsilon)=D(\varepsilon)\end{aligned}\right\} \tag{3.7}$$

它的解可记为

$$\tilde{y}(x;\varepsilon)=\sum_{n=0}^{N} Y_5^{(n)}(x)\varepsilon^n+\frac{1}{\varepsilon}\sum_{n=0}^{N+1}\Big[C_n(x)\exp\Big[-\frac{1}{\varepsilon}\int_1^x\eta(s)ds\Big]$$

$$+D_n\Big(x,\ \frac{1}{\varepsilon}\int_1^x\eta(s)ds\Big)\exp\Big[-\frac{1}{\varepsilon}\int_0^{x-1}\eta(s)ds\Big]\Big]\varepsilon^n$$

$$+\sum_{n=0}^{N}E_n(x)\exp\Big[-\frac{1}{\varepsilon}\int_x^l\eta(s)ds\Big]+\tilde{R}_N \tag{3.8}$$

代入问题 (3.7) 得到

$$\left.\begin{aligned}&\varepsilon^2\tilde{R}_N''-\eta^2(x)\tilde{R}_N=\varepsilon^{N+1}\tilde{F}_N(x;\varepsilon)\qquad(1<x<l)\\&\tilde{R}_N(1;\varepsilon)=\varepsilon^{N+1}\tilde{E}_N(1;\varepsilon)\\&\tilde{R}_N(l;\varepsilon)=0\end{aligned}\right\} \tag{3.9}$$

其中 $$\varepsilon^{N+1}\tilde{F}_N(x;\varepsilon)=-\varepsilon^{N+2}Y_6^{(N)''}(x)-\varepsilon^{N+1}Y_6^{(N-1)''}(x)-\beta(x)R_N(x-1;\varepsilon)$$

$$-\alpha(x)[Y_2^{(N+1)}(x-1)\varepsilon^{N+1}+Y_2^{(N+2)}(x-1)\varepsilon^{N+2}]$$

$$-\alpha(x)R'_{N+2}(x-1;\varepsilon)-\alpha(x)[A'_{N+1}(x-1)\exp[-\xi(x)\tilde{x}_4]\varepsilon^{N+}$$

$$-A'_{N+1}(x-1)\exp[-\xi(x)\tilde{x}_4]\eta(x-1)\varepsilon^N$$

$$+A'_{N+2}(x-1)\exp[-\xi(x)\tilde{x}_4]\varepsilon^{N+2}$$

$$-A_{N+2}(x-1)\exp[-\xi(x)\tilde{x}_4]\eta(x-1)\varepsilon^{N+1}]$$

$$-\varepsilon^{N+1}[(K_1^*+\varepsilon K_2^*)y_6^{(N)}(x,\tilde{x}_6)+K_2^*y_6^{(N-1)}(x,\tilde{x}_6)]$$

$$-\varepsilon^{N+1}\overline{K}_2[C_N(x)\exp[-\tilde{x}_4]+D_N(x,\tilde{x}_4)\exp[-\xi(x)\tilde{x}_4]]$$

$$-\varepsilon^{N+1}(\overline{K}_1+\varepsilon\overline{K}_2)[C_{N+1}(x)\exp[-\tilde{x}_4]$$

$$+D_{N+1}(x,\tilde{x}_4)\exp[-\xi(x)\tilde{x}_4]]$$

$$-\alpha(x)A_{N+1}(x-1)\exp[-\xi(x)\tilde{x}_4]\eta(x-1)\varepsilon^N$$

$$-\alpha(x)\ \frac{d}{dx}\Big(\frac{1}{\varepsilon}\sum_{n=0}^{N+3}B_n(x-1)\exp\Big[-\frac{1}{\varepsilon}\int_{x-1}^1\eta(s)ds\Big]\varepsilon^n\Big)$$

$$-\beta(x)\Big(\frac{1}{\varepsilon}\sum_{n=0}^{N+1}B_n(x-1)\exp\Big[-\frac{1}{\varepsilon}\int_{x-1}^1\eta(s)ds\Big]\varepsilon^n\Big) \tag{3.10}$$

由于 $R_N=O(\varepsilon^{N+1})$，从(3.5)经积分知$R'_N=O(\varepsilon^{N-1})$，所以 (3.10) 右端中 $R'_{N+2}(x-1;\ \varepsilon)=O(\varepsilon^{N+1})$，所以

$$\tilde{F}_N(x;\varepsilon)=O(1)$$

另 (3.9) 中

$$\varepsilon^{N+1}\tilde{E}_N(1;\varepsilon)\sim\sum_{n=N+1}^{\infty}Y_5^{(n)}(1)\varepsilon^n+\frac{1}{\varepsilon}\sum_{n=N+2}^{\infty}[C_n(1)+D_n(1,0)]\varepsilon^n=\varepsilon^{N+1}O(1)$$

所以 $$\widetilde{E}_N(1;\varepsilon)=O(1)$$

相仿(3.4)，利用极值原理估计 (3.9) 的解$\tilde{R}_N$，由于$-\eta^2(x)/\varepsilon^2<0$，(3.9)的解存在唯一，我们有

$$\widetilde{M}=\max\{\max\varepsilon^{N+1}|\widetilde{F}_N(x;\varepsilon)|,\ \delta^2\varepsilon^{N+1}|\widetilde{E}_N(1;\varepsilon)|\}=O(\varepsilon^{N+1})$$

$$|\tilde{R}_N|\leqslant\widetilde{M}/\delta^2$$

所以 $$\tilde{R}_N=O(\varepsilon^{N+1})$$

这样方程(2.1)在$1<x<l$存在唯一的一致有效渐近解(3.8)。

综上所述我们有如下定理：

定理 对于方程(2.1)，边值条件(2.2)，若假设[H_1]，[H_2]成立，则在二阶导数存在处，有一致有效渐近解

$$y(x;\varepsilon)=\begin{cases}\bar{y}(x;\varepsilon) & (0<x<1)\\ \tilde{y}(x;\varepsilon) & (1<x<l).\end{cases}$$

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Singular Perturbation Solution of Boundary-Value Problem for a Second-Order Differential-Difference Equation

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Abstract

In this paper, the method of two-variables expansion is used to construct boundary layer terms of asymptotic solution of the boundary-value problem for a second-order DDE. The n-order formal asymptotic solution is obtained and the error is estimated. Thus the existence of uniformly valid asymptotic solution is proved.