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矩形薄板弹性振动的一般解析解*

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摘　要

本文建立了矩形薄板弹性横向自由振动位型函数微分方程的一般解，可以求解任意边界矩形薄板的振动问题．以四边自由矩形板为例求解了板的频率及其振型．

一、微分方程的解

如图1所示，矩形薄板弹性横向自由振动位型函数的微分方程为[1]

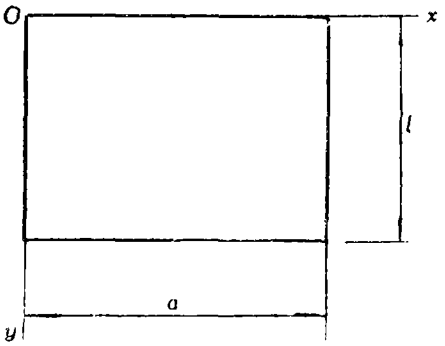

图 1

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}-\gamma^4 W=0 \tag{1.1}$$

式中
$$\gamma^4=\frac{\rho\omega^2}{D} \tag{1.2}$$

ω为板的振动频率、ρ为单位面积质量，D为抗弯刚度．采用分离变量法，设$W=XY$，代入（1.1)式可得

$$Y\frac{\partial^4 X}{\partial x^4}+2\frac{\partial^2 X}{\partial x^2}\frac{\partial^2 Y}{\partial y^2}+X\frac{\partial^4 Y}{\partial y^4}-\gamma^4 XY=0 \tag{1.3}$$

将上式除以XY，然后对y微分一次得

$$\frac{X''}{X}=-\frac{1}{2}\left(\frac{Y^{\text{IV}}}{Y}\right)'\bigg/\left(\frac{Y''}{Y}\right)'$$

上式两边必为一常数，设为$-\alpha^2$，故有

$$X''+\alpha^2 X=0 \tag{1.4}$$

上式的解分为两种：

（1）　当α为零时可得

$$X=A_1+A_2x \tag{1.5}$$

将上式代入(1.3)式可得

* 钱伟长推荐．

$$Y^{IV}-\gamma^4 Y=0$$

上式特征方程的根为$\pm\gamma$和$\pm i\gamma$，即

$$Y=B_1\sinh\gamma y+B_2\cosh\gamma y+B_3\sin\gamma y+B_4\cos\gamma y \tag{1.6}$$

（2） 当α不为零时，由(1.4)式可得

$$X=C_1\sin\alpha x+C_2\cos\alpha x \tag{1.7}$$

将上式代入(1.3)式可得

$$Y^{IV}-2\alpha^2 Y''+(\alpha^4-\gamma^4)Y=0$$

上式特征方程的根又分两种情形：一是当$\gamma>\alpha$时为$\pm\alpha_1$和$\pm i\alpha_2$

$$\alpha_1=\sqrt{\gamma^2+\alpha^2}\ ,\ \alpha_2=\sqrt{\gamma^2-\alpha^2} \tag{1.8}$$

$$Y=D_1\sinh\alpha_1 y+D_2\cosh\alpha_1 y+D_3\sin\alpha_2 y+D_4\cos\alpha_2 y \tag{1.9}$$

另一是当$\gamma<\alpha$时为$\pm\alpha_1$ 和$\pm\alpha_3$

$$\alpha_3=\sqrt{\alpha^2-\gamma^2} \tag{1.10}$$

$$Y=D_1\sinh\alpha_1 y+D_2\cosh\alpha_1 y+D_3\sinh\alpha_3 y+D_4\cosh\alpha_3 y \tag{1.11}$$

如将(1.3)式除以XY，然后对x微分一次，则同样可得相似的另一类解。适当选取各种解的组合可以求解各种不同边界矩形板的自由振动问题。

二、一般解的建立

满足四边以及四角为任意条件的矩形板的振动问题，本文取为：

$$\begin{aligned}
W=&\sum_m\left[A_m\frac{\sinh\alpha_1(b-y)}{\sinh\alpha_1 b}+B_m\frac{\sinh\alpha_1 y}{\sinh\alpha_1 b}\right]\sin\alpha x+\sum_{m<M}\left[C_m\frac{\sin\alpha_2(b-y)}{\sin\alpha_2 b}\right.\\
&\left.+D_m\frac{\sin\alpha_2 y}{\sin\alpha_2 b}\right]\sin\alpha x+\sum_{m>M}\left[C_m\frac{\sinh\alpha_3(b-y)}{\sinh\alpha_3 b}+D_m\frac{\sinh\alpha_3 y}{\sinh\alpha_3 b}\right]\sin\alpha x\\
&+\sum_n\left[E_n\frac{\sinh\beta_1(a-x)}{\sinh\beta_1 a}+F_n\frac{\sinh\beta_1 x}{\sinh\beta_1 a}\right]\sin\beta y+\sum_{n<N}\left[G_n\frac{\sin\beta_2(a-x)}{\sin\beta_2 a}\right.\\
&\left.+H_n\frac{\sin\beta_2 x}{\sin\beta_2 a}\right]\sin\beta y+\sum_{n>N}\left[G_n\frac{\sinh\beta_3(a-x)}{\sinh\beta_3 a}+H_n\frac{\sinh\beta_3 x}{\sinh\beta_3 a}\right]\sin\beta y\\
&+\left[A\frac{\sinh\gamma(b-y)}{\sinh\gamma b}+B\frac{\sinh\gamma y}{\sinh\gamma b}\right]\frac{a-x}{a}+\left[C\frac{\sinh\gamma(b-y)}{\sinh\gamma b}\right.\\
&\left.+D\frac{\sinh\gamma y}{\sinh\gamma b}\right]\frac{x}{a}+\left[E\frac{\sinh\gamma(a-x)}{\sinh\gamma a}+F\frac{\sinh\gamma x}{\sinh\gamma a}\right]\frac{b-y}{b}+\left[G\frac{\sinh\gamma(a-x)}{\sinh\gamma a}\right.\\
&\left.+H\frac{\sinh\gamma x}{\sinh\gamma a}\right]\frac{y}{b}+I\left[\frac{\sin\gamma(b-y)}{\sin\gamma b}\frac{a-x}{a}+\frac{\sin\gamma(a-x)}{\sin\gamma a}\frac{b-y}{b}\right]\\
&+J\left[\frac{\sin\gamma(b-y)}{\sin\gamma b}\frac{x}{a}+\frac{\sin\gamma x}{\sin\gamma a}\frac{b-y}{b}\right]+K\left[\frac{\sin\gamma y}{\sin\gamma b}\frac{a-x}{a}+\frac{\sin\gamma(a-x)}{\sin\gamma a}\frac{y}{b}\right]\\
&+L\left[\frac{\sin\gamma y}{\sin\gamma b}\frac{x}{a}+\frac{\sin\gamma x}{\sin\gamma a}\frac{y}{b}\right]
\end{aligned} \tag{2.1}$$

式中 $$\alpha=\frac{m\pi}{a}\qquad(m=1,2,3,\cdots) \tag{2.2}$$

$$\beta=\frac{n\pi}{b}\qquad(n=1,2,3,\cdots)\tag{2.3}$$

$$\beta_1=\sqrt{\gamma^2+\beta^2},\quad \beta_2=\sqrt{\gamma^2-\beta^2},\quad \beta_3=\sqrt{\beta^2-\gamma^2}\tag{2.4}$$

$$M=\frac{\gamma a}{\pi},\quad N=\frac{\gamma b}{\pi}\tag{2.5}$$

(2.1)式含有$4m+4n+12$个积分常数。(2.1)式的第一部分能满足$y=0$和$y=b$两个边为任意的问题；第二部分能满足$x=0$和$x=a$两个边为任意的问题；第三部分能满足四个角为任意的问题。由于每个边有挠度或等效剪力，斜度或弯矩二个边界条件，将每个边界条件所建立的方程式中非正弦级数均展成正弦级数，则利用正弦级数的正交性可得$4m$和$4n$个方程式，加上每个角有挠度或反力，角的两边的斜度或弯矩三个条件，总共可以求解$4m+4n+12$个未知数。(2.1)式需展成的正弦级数如下

$$\frac{\sinh\alpha'(b-y)}{\sinh\alpha' b}=\sum_n\frac{2n\pi}{(n\pi)^2+(\alpha' b)^2}\sin\beta y\tag{2.6}$$

$$\frac{\sinh\alpha' y}{\sinh\alpha' b}=-\sum_n\frac{2n\pi\cos n\pi}{(n\pi)^2+(\alpha' b)^2}\sin\beta y\tag{2.7}$$

$$\frac{\sin\alpha'(b-y)}{\sin\alpha' b}=\sum_n\frac{2n\pi}{(n\pi)^2-(\alpha' b)^2}\sin\beta y\tag{2.8}$$

$$\frac{\sin\alpha' y}{\sin\alpha' b}=-\sum_n\frac{2n\pi\cos n\pi}{(n\pi)^2-(\alpha' b)^2}\sin\beta y\tag{2.9}$$

$$\frac{b-y}{b}=\sum_n\frac{2}{n\pi}\sin\beta y\tag{2.10}$$

$$\frac{y}{b}=-\sum_n\frac{2\cos n\pi}{n\pi}\sin\beta y\tag{2.11}$$

三、计 算 例

以四边自由的矩形板为例，边界条件是：

$$(M_x)_{x=0}=0,\ (M_x)_{x=a}=0\tag{3.1}$$

$$(M_y)_{y=0}=0,\ (M_y)_{y=b}=0\tag{3.2}$$

$$(V_x)_{x=0}=0,\ (V_x)_{x=a}=0\tag{3.3}$$

$$(V_y)_{y=0}=0,\ (V_y)_{y=b}=0\tag{3.4}$$

式中

$$M_x=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)$$

$$M_y=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)$$

$$V_x=-D\left[\frac{\partial^3 W}{\partial x^3}+(2-\nu)\frac{\partial^3 W}{\partial x\partial y^2}\right]$$

$$V_y=-D\left[\frac{\partial^3 W}{\partial y^3}+(2-\nu)\frac{\partial^3 W}{\partial y\partial x^2}\right]$$

角点条件是

$$R_{(0,0)}=0,\ R_{(a,0)}=0,\ R_{(0,b)}=0,\ R_{(a,b)}=0\tag{3.5}$$

$$M_{x(0,0)}=0, M_{x(a,0)}=0, M_{x(0,b)}=0, M_{x(a,b)}=0 \tag{3.6}$$

$$M_{y(0,0)}=0, M_{y(a,0)}=0, M_{y(0,b)}=0, M_{y(a,b)}=0 \tag{3.7}$$

式中 $R=2D(1-\nu)\dfrac{\partial^2 W}{\partial x\partial y}$

利用对称和反对称条件可使求解基频的问题大大简化。对于$x=a/2$和$y=b/2$为对称或反对称振型应有

$$W_{x=0}=\pm W_{x=a};\quad W_{y=0}=\pm W_{y=b} \tag{3.8}$$

$$W_{(0,0)}=\pm W_{(a,0)};\quad W_{(0,0)}=\pm W_{(0,b)};\quad W_{(0,b)}=\pm W_{(a,b)} \tag{3.9}$$

对于正方形板的情形，由直线$x=y$的对称或反对称条件又有

$$W_{x=0}=\pm W_{y=0} \tag{3.10}$$

正负号同时书写时，上号为对称情形，下号为反对称情形。将(2.1)式代入以上各式，首先由(3.6)和(3.7)式可得

$$A=E=I,\ C=F=J,\ B=G=K,\ D=H=L \tag{3.11}$$

由(3.9)式可得

$$I=\pm J=\pm K=L \tag{3.12}$$

应用以上各式，则由(3.1)和(3.2)式并应用到(1.8)，(1.10)，(2.4)，(2.6)至(2.9)式以及利用正弦级数的正交性可得

$$G_n=E_n\frac{\gamma^2+(1-\nu)\beta^2}{\gamma^2-(1-\nu)\beta^2}+I\frac{1\mp\cos n\pi}{\gamma^2-(1-\nu)\beta^2}\frac{4\nu n\pi\gamma^4 b^2}{(\gamma b)^4-(n\pi)^4} \tag{3.13}$$

$$H_n=F_n\frac{\gamma^2+(1-\nu)\beta^2}{\gamma^2-(1-\nu)\beta^2}\pm I\frac{1\mp\cos n\pi}{\gamma^2-(1-\nu)\beta^2}\frac{4\nu n\pi\gamma^4 b^2}{(\gamma b)^4-(n\pi)^4} \tag{3.14}$$

$$C_m=A_m\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}+I\frac{1\mp\cos m\pi}{\gamma^2-(1-\nu)\alpha^2}\frac{4\nu m\pi\gamma^4 a^2}{(\gamma a)^4-(m\pi)^4} \tag{3.15}$$

$$D_m=B_m\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}\pm I\frac{1\mp\cos m\pi}{\gamma^2-(1-\nu)\alpha^2}\frac{4\nu m\pi\gamma^4 a^2}{(\gamma a)^4-(m\pi)^4} \tag{3.16}$$

再由(3.8)式并应用到(2.10)和(2.11)式可得

$$B_m=\pm A_m,\ F_n=\pm E_n \tag{3.17}$$

最后由(3.3)的第一式和(3.5)的第一式可得

$$\begin{aligned}
&\sum_m A_m\Big\{\alpha[(1-\nu)\alpha^2+(2-\nu)\gamma^2]\frac{2n\pi(1\mp\cos n\pi)}{(n\pi)^2+(\alpha_1 b)^2}\\
&\quad+\alpha[(1-\nu)\alpha^2-(2-\nu)\gamma^2]\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}\frac{2n\pi(1\mp\cos n\pi)}{(n\pi)^2+(\alpha_3 b)^2}\Big\}\\
&\quad+E_n\Bigg\{[(1-\nu)\beta^2-\gamma^2]\beta_1\left(\coth\beta_1 a\mp\frac{1}{\sinh\beta_1 a}\right)\\
&\quad+\frac{[\gamma^2+(1-\nu)\beta^2]^2}{\gamma^2-(1-\nu)\beta^2}\begin{bmatrix}\beta_2\left(\cot\beta_2 a\mp\dfrac{1}{\sin\beta_2 a}\right),\ 当n<N\\ \beta_3\left(\cot\beta_3 a\mp\dfrac{1}{\sinh\beta_3 a}\right),\ 当n>N\end{bmatrix}\Bigg\}\\
&\quad+I\Big\{\sum_m\alpha[(1-\nu)\alpha^2-(2-\nu)\gamma^2]\frac{1\mp\cos m\pi}{\gamma^2-(1-\nu)\alpha^2}\frac{4\nu m\pi\gamma^4 a^2}{(\gamma a)^4-(m\pi)^2}\frac{2n\pi(1\mp\cos n\pi)}{(n\pi)^2+(\alpha_3 b)^2}
\end{aligned}$$

$$+4\nu n\pi\gamma^4b^2\frac{1\mp\cos n\pi}{(\gamma b)^4-(n\pi)^4}\frac{\gamma^2+(1-\nu)\beta^2}{\gamma^2-(1-\nu)\beta^2}\begin{bmatrix}\beta_2\left(\cot\beta_2a\mp\dfrac{1}{\sin\beta_2a}\right),\text{ 当}n<N\\ \beta_3\left(\coth\beta_3a\mp\dfrac{1}{\sinh\beta_3a}\right),\text{ 当}n>N\end{bmatrix}$$

$$-\left(\coth\gamma a\mp\frac{1}{\sinh\gamma a}-\cot\gamma a\pm\frac{1}{\sin\gamma a}\right)(1\mp\cos n\pi)\frac{2\gamma^3}{n\pi}$$

$$-(1\mp1)(2-\nu)(1\mp\cos n\pi)\frac{4n\pi}{a}\frac{\gamma^4b^2}{(\gamma b)^4-(n\pi)^4}\Bigg\}=0 \tag{3.18}$$

$$\sum_m A_m\alpha\left\{\alpha_1\left(\coth\alpha_1b\mp\frac{1}{\sinh\alpha_1b}\right)+\frac{\gamma^2+(1-\nu)\alpha^2}{\gamma^2-(1-\nu)\alpha^2}\begin{bmatrix}\alpha_2\left(\cot\alpha_2b\mp\dfrac{1}{\sin\alpha_2b}\right),\text{ 当}m<M\\ \alpha_3\left(\coth\alpha_3b\mp\dfrac{1}{\sinh\alpha_3b}\right),\text{当}m>M\end{bmatrix}\right\}$$

$$+\sum_n E_n\beta\left\{\beta_1\left(\coth\beta_1a\mp\frac{1}{\sinh\beta_1a}\right)+\frac{\gamma^2+(1-\nu)\beta^2}{\gamma^2-(1-\nu)\beta^2}\begin{bmatrix}\beta_2\left(\cot\beta_2a\mp\dfrac{1}{\sin\beta_2a}\right),\text{ 当}n<N\\ \beta_3\left(\coth\beta_3a\mp\dfrac{1}{\sinh\beta_3a}\right),\text{ 当}n>N\end{bmatrix}\right\}$$

$$+I\Bigg\{\sum_m\frac{1\mp\cos m\pi}{\gamma^2-(1-\nu)\alpha^2}\frac{4\nu m\pi\alpha\gamma^4a^2}{(\gamma a)^4-(m\pi)^4}\begin{bmatrix}\alpha_2\left(\cot\alpha_2b\mp\dfrac{1}{\sin\alpha_2b}\right),\text{当}m<M\\ \alpha_3\left(\coth\alpha_3b\mp\dfrac{1}{\sinh\alpha_3b}\right),\text{ 当}m>M\end{bmatrix}$$

$$+\sum_n\frac{1\mp\cos n\pi}{\gamma^2-(1-\nu)\beta^2}\frac{4\nu n\pi\beta\gamma^4b^2}{(\gamma b)^4-(n\pi)^4}\begin{bmatrix}\beta_2\left(\cot\beta_2a\mp\dfrac{1}{\sin\beta_2a}\right),\text{ 当}n<N\\ \beta_3\left(\coth\beta_3a\mp\dfrac{1}{\sinh\beta_2a}\right),\text{ 当}n>N\end{bmatrix}$$

$$-(1\mp1)\frac{\gamma}{a}\left(\coth\gamma b\mp\frac{1}{\sinh\gamma b}+\cot\gamma b\mp\frac{1}{\sin\gamma b}\right)$$

$$-(1\mp1)\frac{\gamma}{b}\left(\coth\gamma a\mp\frac{1}{\sinh\gamma a}+\cot\gamma a\mp\frac{1}{\sin\gamma a}\right)\Bigg\}=0 \tag{3.19}$$

如果再应用(3.4)的第一式，还可以求得和(3.18)式相类似的等式，则连同(3.19)式可以求解A_m，E_n和I的问题，相应地可以求得对$x=a/2$和$y=b/2$分别为对称或反对称共四种情形。现在我们仅求解对$x=a/2$和$y=b/2$均为对称或均为反对称的情形，同时令$a=b$，即正方情形，再则由(3.10)式，并取m和n的总项数相等可得

$$E_n=(\pm)A_n \tag{3.20}$$

上式中正负号加一括号是为了区别另外两种情形，故总共有四种情形。由于利用了这些对称和反对称条件，故(3.3)，(3.4)和(3.5)的其他各式均自动满足。将上式代入(3.18)和(3.19)式，令$a=b$并应用到(2.5)式可得

$$\sum_m A_m4mn\left[\frac{(1-\nu)m^2+(2-\nu)M^2}{m^2+n^2+M^2}+\frac{M^2+(1-\nu)m^2}{M^2-(1-\nu)m^2}\frac{(1-\nu)m^2-(2-\nu)M^2}{m^2+n^2-M^2}\right]$$

$$(\pm)A_n\left\{\left[(1-\nu)n^2-M^2\right]n_1\pi\left(\coth n_1\pi\mp\frac{1}{\sinh n_1\pi}\right)\right.$$

$$+\frac{[M^2+(1-\nu)n^2]^2}{M^2-(1-\nu)n^2}\begin{bmatrix} n_2\pi\left(\cot n_2\pi\mp\frac{1}{\sin n_2\pi}\right), & 当n<M \\ n_3\pi\left(\coth n_3\pi\mp\frac{1}{\sinh n_3\pi}\right), & 当n>M \end{bmatrix}\Bigg\}$$

$$+I\Bigg\{\sum_m\frac{32\nu}{\pi}\frac{m^2n}{m^2+n^2-M^2}\frac{M^4}{M^4-m^4}\frac{(1-\nu)m^2-(2-\nu)M^2}{M^2-(1-\nu)m^2}$$

$$+\frac{8\nu nM^4}{M^4-n^4}\frac{M^2+(1-\nu)n^2}{M^2-(1-\nu)n^2}\begin{bmatrix} n^2\left(\cot n_2\pi\mp\frac{1}{\sin n_2\pi}\right), & 当n<M \\ n_3\left(\coth n_3\pi\mp\frac{1}{\sinh n_3\pi}\right), & 当n>M \end{bmatrix}$$

$$-\frac{4M^3}{n}\left(\coth M\pi\mp\frac{1}{\sinh M\pi}-\cot M\pi\pm\frac{1}{\sin M\pi}\right)-(1\mp1)(2-\nu)\frac{8nM^4}{M^4-n^4}\Bigg\}=0 \tag{3.21}$$

对$x=y$为反对称时，$I=0$，对称时为：

$$\sum_m A_m m\pi\Bigg\{m_1\left(\coth m_1\pi\mp\frac{1}{\sinh m_1\pi}\right)+\frac{M^2+(1-\nu)m^2}{M^2-(1-\nu)m^2}\begin{bmatrix} m_2\left(\cot m_2\pi\mp\frac{1}{\sin m_2\pi}\right), & 当m<M \\ m_3\left(\coth m_3\pi\mp\frac{1}{\sinh m_3\pi}\right), & 当m>M \end{bmatrix}\Bigg\}$$

$$+I\Bigg\{\sum_m\frac{8\nu m^2}{M^2-(1-\nu)m^2}\frac{M^4}{M^4-m^4}\begin{bmatrix} m_2\left(\cot m_2\pi\mp\frac{1}{\sin m_2\pi}\right), & 当m<M \\ m_3\left(\coth m_3\pi\mp\frac{1}{\sinh m_2\pi}\right), & 当m>M \end{bmatrix}$$

$$-(1\mp1)M\left(\coth M\pi\mp\frac{1}{\sinh M\pi}+\cot M\pi\mp\frac{1}{\sin M\pi}\right)\Bigg\}=0 \tag{3.22}$$

式中 $m_1=\sqrt{M^2+m^2}$，$m_2=\sqrt{M^2-m^2}$，$m_3=\sqrt{m^2-M^2}$

$n_1=\sqrt{M^2+n^2}$，$n_2=\sqrt{M^2-n^2}$，$n_3=\sqrt{n^2-M^2}$

以上二式当$x=a/2$和$y=a/2$为对称时m和n仅取奇数值，反对称时仅取偶数值。

令(3.21)和(3.22)式中未知数A_m和I的系数矩阵行列式等于零，可以求得确定各种类型振动频率的M值。此外由上式可以解出A_m/A_1，I/A_1，将此比值以及(3.11)式至(3.17)式和(3.20)式代入(2.1)式可求得沿板边$y=0$和对角线$x=y$振幅曲线为

$$W_{y=0}=\sum_m A_m\frac{2M^2}{M^2-(1-\nu)m^2}\sin\frac{m\pi x}{a}+I\Bigg\{\sum_m\frac{8\nu}{\pi}\frac{m}{M^2-(1-\nu)m^2}$$

$$\cdot\frac{M^4}{M^4-m^4}\sin\frac{m\pi x}{a}+2\left(\frac{a-x}{a}\pm\frac{x}{a}\right)+\frac{\sinh M\pi(1-x/a)}{\sinh M\pi}$$

$$\pm\frac{\sinh M\pi x/a}{\sinh M\pi}+\frac{\sin M\pi(1-x/a)}{\sin M\pi}\pm\frac{\sin M\pi x/a}{\sin M\pi}\Bigg\} \tag{3.23}$$

$$W_{x=y}=2\Bigg\{\sum_m A_m\left[\frac{\sinh m_1\pi(1-x/a)}{\sinh m_1\pi}\pm\frac{\sinh m_1\pi x/a}{\sinh m_1\pi}\right]\sin\frac{m\pi x}{a}$$

$$
\begin{aligned}
&+\sum_m\left[A_m\frac{M^2+(1-\nu)m^2}{M^2-(1-\nu)m^2}+I\frac{8\nu}{\pi}\frac{m}{M^2-(1-\nu)m^2}\frac{M^4}{M^4-m^4}\right]\left\{\begin{array}{l}\dfrac{\sin m_2\pi(1-x/a)}{\sin m_2\pi}\pm\dfrac{\sin m_2\pi x/a}{\sin m_2\pi},\text{当}m<M\\[2ex]\dfrac{\sinh m_3\pi(1-x/a)}{\sinh m_3\pi}\pm\dfrac{\sinh m_3\pi x/a}{\sinh m_3\pi},\text{当}m>M\end{array}\right\}\sin\frac{m\pi x}{a}+I\left[\frac{\sinh M\pi(1-x/a)}{\sinh M\pi}\pm\frac{\sinh M\pi x/a}{\sinh M\pi}\right.\\
&\left.+\frac{\sin M\pi(1-x/a)}{\sin M\pi}\pm\frac{\sin M\pi x/a}{\sin M\pi}\right]\left(\frac{a-x}{a}\pm\frac{x}{a}\right)\Bigg\}
\end{aligned}
\tag{3.24}
$$

取 $\nu=0.333$，m 和 n 各取八项，采用反逆解法，由(3.21)和(3.22)式求得四种类型确定基频的 M 值见表1和表2。由(3.23)式和（3.24）式求得沿板边和对角线的振幅曲线见表3和表4。

表 1　对 $x=a/2$ 和 $y=a/2$ 均为对称的 M 值

	对 $x=y$ 为对称			对 $x=y$ 为反对称		
本　文	1.5846	5.5580	3.5362	1.3956	3.4345	4.0249
文献[1]	1.5731	2.5217	3.5227	1.3956	3.4348	4.0248
文献[2]	1.5565			1.4433		

表 2　对 $x=a/2$ 和 $y=a/2$ 均为反对称的 M 值

	对 $x=y$ 为对称		对 $x=y$ 为反对称	
本　文	1.1756	3.9337	2.6277	4.5295
文献[1]	1.1551	3.9145	2.6271	4.5303
文献[2]	1.1925			

表中文献[1]是用解析解的迭加法，取 $\nu=0.333$ 求得的，演算过程比较复杂。文献[2]是用能量法，取 $\nu=0.225$ 计算的结果。本文的理论分析简单，计算方法容易掌握，便于工程实际应用。

表 3　对 $x=a/2$，$y=a/2$ 和 $x=y$ 均为对称的振幅曲线

x/a	$M=1.3956$		$M=3.5362$	
	$W_{y=0}$	$W_{x=y}$	$W_{y=0}$	$W_{x=y}$
0	1	1	1	1
1/16	0.8380	0.6980	0.8691	0.6321
1/8	0.6711	0.4411	0.6426	0.0847
3/16	0.5102	0.2321	0.4156	−0.3966
1/4	0.3634	0.0697	0.2425	−0.5697
5/16	0.2277	0.0491	0.2406	−0.3207
3/8	0.1132	−0.1291	0.3795	0.2363
7/16	0.0513	−0.1748	0.5185	0.7939
1/2	0.0229	−0.1896	0.6164	1.0260

表 4 对$x=a/2$和$y=a/2$为反对称，$x=y$为对称的振幅曲线

x/a	$M=1.1756$		$M=3.9337$	
	$W_{y=0}$	$W_{x=y}$	$W_{y=0}$	$W_{x=y}$
0	1	1	1	1
1/16	0.9130	0.8254	0.7127	0.6362
1/8	0.8184	0.6487	0.4889	0.4513
3/16	0.7129	0.4768	0.4028	0.1188
1/4	0.5934	0.3198	0.4401	−0.3238
5/16	0.4602	0.1866	0.5179	−0.5808
3/8	0.3144	0.0851	0.4823	−0.4730
7/16	0.1596	0.0216	0.3009	−0.1627
1/2	0	0	0	0

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A General Analytical Solution for Elastic Vibration of Rectangular Thin Plates

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Abstract

A general solution of differential equation for lateral displacement function of rectangular elastic thin plates in free vibration is established in this paper. It can be used to solve the vibration problem of rectangular plate with arbitrary boundaries. As an example, the frequency and its vibration mode of a rectangular plate with four edges free have been solved.