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关于含双参数的非线性常微分方程的奇异摄动*

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摘　要

本文应用微分不等式方法研究含双小参数的非线性常微分方程的边值问题，作出渐近解并对余项进行估计.

一、引　　言

微分不等式方法是近年来发展起来的研究非线性奇摄动问题的一个有效方法。前人在这方面已有不少工作[1]~[7]，但却限于含有一个小参数的奇摄动问题。O'Malley[8]~[10]曾应用边界层校正法研究了含两个小参数的线性常微分方程的奇摄动问题。本文采用微分不等式方法研究含两小参数的二阶非线性常微分方程的边值问题，作出渐近解并得出余项估计。

二、Dirichlet问题

我们首先考虑二阶非线性常微分方程的 Dirichlet问题：

$$\varepsilon y''+\mu f(x,\ y)y'+g(x,\ y)=0 \qquad (a<x<b) \tag{2.1}$$

$$y(a)=A, \qquad y(b)=B \tag{2.2}$$

其中ε，μ都是正的小参数。

作如下假设

(H_1)：退化问题$g(x,\ u)=0$存在一个解$u(x)\in C^2[a,\ b]$，

(H_2)：$f(x,\ y)$，$g(x,\ y)$，$g_y(x,\ y)\in C(D_0)$，其中D_0：$a\leqslant x\leqslant b,\ |y-u(x)|\leqslant d(x)$；$d(x)>0$是连续函数，使得

$$d(x)=\begin{cases}|A-u(a)|+\delta & (a\leqslant x\leqslant a+\delta/2)\\ \delta & (a+\delta\leqslant x\leqslant b-\delta)\\ |B-u(b)|+\delta & (b-\delta/2\leqslant x\leqslant b)\end{cases}$$

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$\delta>0$是小常数，

(H_3)：存在常数$m>0$，l_1和l_2使得在D_0中$g_y\leqslant -m$，$l_1\leqslant f(x,\ y)\leqslant l_2$。

下面分别就三种情形(i)$\varepsilon/\mu^2\to 0$，(ii)$\mu^2/\varepsilon\to 0$，(iii)$\varepsilon=\mu^2$进行讨论，我们有如下定理。

定理2.1 若成立$\varepsilon/\mu^2\to 0$当$\mu\to 0$，则边值问题(2.1)～(2.2)在假设$(H_1)\sim(H_3)$下存在一个解$y(x,\ \varepsilon,\ \mu)\in C^2$，且成立

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant W_{1L}(x,\ \varepsilon,\ \mu)+W_{1R}(x,\ \varepsilon,\ \mu)+\gamma_1\mu,$$

这里$W_{1L}=|A-u(a)|\exp[\lambda_1(x-a)]$，$W_{1R}=|B-u(b)|\exp[\lambda_2(x-b)]$，而$\lambda_1$是方程$\varepsilon\lambda^2+\mu l_1\lambda-m=0$的一个负实根，$\lambda_2$是方程$\varepsilon\lambda^2+\mu l_2\lambda-m=0$的一个正实根，且当$l_1>0$时有$\lambda_1=-\frac{\mu}{\varepsilon}l_1+O\left(\frac{1}{\mu}\right)$，当$l_1<0$时$\lambda_1=\frac{m}{\mu l_1}+O\left(\frac{\varepsilon}{\mu^3}\right)$，当$l_2>0$时，$\lambda_2=\frac{m}{\mu l_2}+O\left(\frac{\varepsilon}{\mu^3}\right)$，当$l_2<0$时，$\lambda_2=-\frac{\mu}{\varepsilon}l_2+O\left(\frac{1}{\mu}\right)$，$\gamma_1$是一个充分大的正数。

证明 我们注意到正函数W_{1L}是微分方程$\varepsilon W_{1L}''+\mu l_1W_{1L}'-mW_{1L}=0$满足条件$W_{1L}(a,\ \varepsilon,\ \mu)=|A-u(a)|$，$W_{1L}'(a,\ \varepsilon,\ \mu)=\lambda_1|A-u(a)|$的解，且$W_{1L}'(x,\varepsilon,\mu)<0$，正函数$W_{1R}$是方程$\varepsilon W_{1R}''+\mu l_2W_{1R}'-mW_{1R}=0$满足条件$W_{1R}(b,\ \varepsilon,\ \mu)=|B-u(b)|$，$W_{1R}'(b,\ \varepsilon,\ \mu)=\lambda_2|B-u(b)|$的解，且$W_{1R}'(x,\ \varepsilon,\ \mu)>0$。为简明起见，不妨设$l_1\geqslant 0$，$l_2\geqslant 0$。我们定义两个函数：

$$\alpha_1(x,\ \varepsilon,\ \mu)=u(x)-W_{1L}-W_{1R}-\gamma_1\mu \tag{2.3}$$

$$\beta_1(x,\ \varepsilon,\ \mu)=u(x)+W_{1L}+W_{1R}+\gamma_1\mu \tag{2.4}$$

不难看出α_1，β_1有如下性质：$\alpha_1\leqslant\beta_1$，$\alpha_1(a,\ \varepsilon,\ \mu)\leqslant A\leqslant\beta_1(a,\ \varepsilon,\ \mu)$，$\alpha_1(b,\ \varepsilon,\ \mu)\leqslant B\leqslant\beta_1(b,\ \varepsilon,\ \mu)$，现在分别在区间$[a,\ a+\delta/2]$，$[a+\delta/2,\ b-\delta/2]$和$[b-\delta/2,\ b]$上，对充分小的$\mu$和$\varepsilon$验证

$$\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\geqslant 0 \tag{2.5}$$

$$\varepsilon\beta_1''+\mu f(x,\ \beta_1)\beta_1'+g(x,\ \beta_1)\leqslant 0 \tag{2.6}$$

为简单起见我们只证明不等式(2.5)[类似地可证明(2.6)]。

当$x\in[a,\ a+\delta/2]$，$W_{1R}=O(\mu^{N+1})=W_{1R}'$，$N$为任意正整数，则

$$\begin{aligned}&\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\\&=\varepsilon u''+\mu fu'-\varepsilon W_{1L}''-\mu fW_{1L}'+g_y(-W_{1L}-\gamma_1\mu)+O(\mu^{N+1})\\&\geqslant\varepsilon u''+\mu fu'-(\varepsilon W_{1L}''+\mu l_1W_{1L}'-mW_{1L})+m\gamma_1\mu-K\mu^{N+1}\\&\geqslant\left[m\gamma_1-\frac{\varepsilon}{\mu}M_1-M-K\mu^N\right]\mu\end{aligned}$$

这里$M_1=\max|u''(x)|$，$M=\max|fu'|$，K为正常数。对充分小的μ，ε/μ^2，取$\gamma_1\geqslant[M_1+M+K]/m$，则

$$\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\geqslant 0$$

当$x\in[a+\delta/2,\ b-\delta/2]$，$W_{1L}=O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)=W_{1L}'$，$W_{1R}=O(\mu^{N+1})=W_{1R}'$，则

$$\begin{aligned}&\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\\&=\varepsilon u''+\mu fu'+g_y(-\gamma_1\mu)+O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)+O(\mu^{N+1})\end{aligned}$$

$$\geqslant \varepsilon u''+\mu f u'+m\gamma_1\mu-K_1\left(\frac{\varepsilon}{\mu}\right)^{N+1}-K\mu^{N+1}$$

$$\geqslant\left[m\gamma_1-\frac{\varepsilon}{\mu}M_1-M-K_1\frac{\varepsilon^{N+1}}{\mu^{N+2}}-K\mu^N\right]\mu$$

取$\gamma_1\geqslant[M_1+M+K_1+K]/m$，则

$$\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\geqslant 0$$

当 $x\in[b-\delta/2,\ b]$，$W_{1L}=O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)=W_{1L}'$，则

$$\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)$$

$$\geqslant\left(m\gamma_1-\frac{\varepsilon}{\mu}M_1-M-K_1\frac{\varepsilon^{N+1}}{\mu^{N+2}}\right)\mu$$

取$\gamma_1\geqslant[M_1+M+K_1]/m$，则

$$\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\geqslant 0$$

故在整个区间$[a,\ b]$上，对于充分小的μ和ε/μ^2，取$\gamma_1\geqslant[M_1+M+K_1+K]/m$，则有

$$\varepsilon\alpha_1''+\mu f(x,\ \alpha_1)\alpha_1'+g(x,\ \alpha_1)\geqslant 0$$

由Nagumo定理[11],[12]，边值问题(2.1)～(2.2)存在一个解$y(x,\ \varepsilon,\ \mu)\in C^2$，且满足

$$\alpha_1(x,\ \varepsilon,\ \mu)\leqslant y(x,\ \varepsilon,\ \mu)\leqslant\beta_1(x,\ \varepsilon,\ \mu)\qquad(a\leqslant x\leqslant b)$$

再由(2.3)～(2.4)可得

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant W_{1L}+W_{1R}+\gamma_1\mu\qquad(a\leqslant x\leqslant b)$$

定理2.2 若$\mu^2/\varepsilon\to 0$当$\varepsilon\to 0$，则在假设(H_1)～(H_3)下，边值问题(2.1)～(2.2)存在一个解$y(x,\ \varepsilon,\ y)\in C^2$，且成立

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant W_{2L}(x,\ \varepsilon,\ \mu)+W_{2R}(x,\ \varepsilon,\ \mu)+c\sqrt{\varepsilon}$$

这里 $W_{2L}=|A-u(a)|\exp[\lambda_3(x-a)]$，$W_{2R}=|B-u(b)|\exp[\lambda_4(x-b)]$，而$\lambda_3$是$\varepsilon\lambda^2+\mu l_1\lambda-m=0$的负实根，$\lambda_4$是$\varepsilon\lambda^2+\mu l_2\lambda-m=0$的正实根，且有$\lambda_3=-\sqrt{\frac{m}{\varepsilon}}+O\left(\frac{\mu}{\varepsilon}\right)$，$\lambda_4=\sqrt{\frac{m}{\varepsilon}}+O\left(\frac{\mu}{\varepsilon}\right)$，$c$为正常数。

证明 定理2.2的证明与定理2.1的证明方法相同。我们定义

$$\alpha_2(x,\ \varepsilon,\ \mu)=u(x)-W_{2L}-W_{2R}-c\sqrt{\varepsilon}\tag{2.7}$$

$$\beta_2(x,\ \varepsilon,\ \mu)=u(x)+W_{2L}+W_{2R}+c\sqrt{\varepsilon}\tag{2.8}$$

显然$\alpha_2\leqslant\beta_2$，$\alpha_2(a,\ \varepsilon,\ \mu)\leqslant A\leqslant\beta_2(a,\ \varepsilon,\ \mu)$，$\alpha_2(b,\ \varepsilon,\ \mu)\leqslant B\leqslant\beta_2(b,\ \varepsilon,\ \mu)$。

当$x\in[a,\ a+\delta/2]$，$W_{2R}=O(\sqrt{\varepsilon}^{N+1})=W_{2R}'$，则

$$\varepsilon\alpha_2''+\mu f(x,\ \alpha_2)\alpha_2'+g(x,\ \alpha_2)$$

$$=\varepsilon u''+\mu f u'-\varepsilon W_{2L}''-\mu f W_{2L}'+g_y(-W_{2L}-c\sqrt{\varepsilon})+O(\sqrt{\varepsilon}^{N+1})$$

$$\geqslant\varepsilon u''+\mu f u'-(\varepsilon W_{2L}''+\mu l_1 W_{2L}'-mW_{2L})+mc\sqrt{\varepsilon}-K\sqrt{\varepsilon}^{N+1}$$

$$\geqslant\left[cm-\sqrt{\varepsilon}M_1-\frac{\mu}{\sqrt{\varepsilon}}M-\sqrt{\varepsilon}^N K\right]\sqrt{\varepsilon}$$

因ε，$\frac{\mu^2}{\varepsilon}$充分小，不妨设$\sqrt{\varepsilon}M_1\leqslant\frac{1}{3cm}$，$\frac{\mu}{\sqrt{\varepsilon}}M\leqslant\frac{1}{3cm}$，$\sqrt{\varepsilon}^N K\leqslant\frac{1}{3cm}$，则

$$\varepsilon\alpha_2''+\mu f(x,\ \alpha_2)\alpha_2'+g(x,\ \alpha_2)\geqslant 0$$

当$x\in[a+\delta/2,\ b-\delta/2]$，$W_{2L}$，$W_{2L}'=O(\sqrt{\varepsilon}^{N+1})=W_{2R}$，$W_{2R}'$，则

$$\begin{aligned}&\varepsilon\alpha_2''+\mu f(x,\ \alpha_2)\alpha_2'+g(x,\ \alpha_2)\\&=\varepsilon u''+\mu fu'-g_y c\sqrt{\varepsilon}+O(\sqrt{\varepsilon}^{N+1})\\&\geqslant\left(cm-M_1\sqrt{\varepsilon}-\frac{\mu}{\sqrt{\varepsilon}}M-K\sqrt{\varepsilon}^N\right)\sqrt{\varepsilon}\geqslant 0\end{aligned}$$

当$x\in[b-\delta/2,\ b]$，$W_{2L}=O(\sqrt{\varepsilon}^{N+1})=W_{2L}'$，则

$$\begin{aligned}&\varepsilon\alpha_2''+\mu f(x,\ \alpha_2)\alpha_2'+g(x,\ \alpha_2)\\&=\varepsilon u''+\mu fu'-\varepsilon W_{2R}''-\mu fW_{2R}'+g_y(-W_{2R}-c\sqrt{\varepsilon})+O(\sqrt{\varepsilon}^{N+1})\\&\geqslant\varepsilon u''+\mu fu'-(\varepsilon W_{2R}''+\mu l_2W_{2R}'-mW_{2R})+cm\sqrt{\varepsilon}-K\sqrt{\varepsilon}^{N+1}\\&\geqslant\left[cm-M_1\sqrt{\varepsilon}-\frac{\mu}{\sqrt{\varepsilon}}M-K\sqrt{\varepsilon}^N\right]\sqrt{\varepsilon}\geqslant 0\end{aligned}$$

所以对于充分小的ε和μ^2/ε，在整个区间$[a,\ b]$上有

$$\varepsilon\alpha_2''+\mu f(x,\ \alpha_2)\alpha_2'+g(x,\ \alpha_2)\geqslant 0$$

类似地可得

$$\varepsilon\beta_2''+\mu f(x,\ \beta_2)\beta_2'+g(x,\ \beta_2)\leqslant 0$$

故由Nagumo定理[11,12]知边值问题(2.1)～(2.2)存在一个解$y(x,\ \varepsilon,\ \mu)\in C^2$，且成立

$$\alpha_2(x,\ \varepsilon,\ \mu)\leqslant y(x,\ \varepsilon,\ \mu)\leqslant\beta_2(x,\ \varepsilon,\ \mu)\qquad(a\leqslant x\leqslant b)$$

因而有

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant W_{2L}+W_{2R}+c\sqrt{\varepsilon}\qquad(a\leqslant x\leqslant b)$$

定理2.3 若$\varepsilon=\mu^2$，则在假设(H_1)～(H_3)下，边值问题(2.1)～(2.2)存在一个解$y(x,\ \mu)\in C^2$，且成立

$$|y(x,\ \mu)-u(x)|\leqslant W_L(x,\ \mu)+W_R(x,\ \mu)+\gamma\mu$$

这里 $W_L=|A-u(a)|\exp[\lambda_5(x-a)]$，$W_R=|B-u(b)|\exp[\lambda_6(x-b)]$，而$\lambda_5$是$\mu^2\lambda^2+\mu l_1\lambda-m=0$的负实根，$\lambda_6$是$\mu^2\lambda^2+\mu l_2\lambda-m=0$的正实根，$\gamma$为充分大的正数。

证明 作函数

$$\alpha(x,\ \mu)=u(x)-W_L-W_R-\gamma\mu$$
$$\beta(x,\ \mu)=u(x)+W_L+W_R+\gamma\mu$$

不难验证，$\alpha\leqslant\beta$，$\alpha(a,\ \mu)\leqslant A\leqslant\beta(a,\ \mu)$，$\alpha(b,\ \mu)\leqslant B\leqslant\beta(b,\ \mu)$，且对于充分小的$\mu$分别在区间$[a,\ a+\delta/2]$，$[a+\delta/2,\ b-\delta/2]$和$[b-\delta/2,\ b]$上成立

$$\mu^2\alpha''+\mu f(x,\ \alpha)\alpha'+g(x,\ \alpha)\geqslant 0,$$
$$\mu^2\beta''+\mu f(x,\ \beta)\beta'+g(x,\ \beta)\leqslant 0.$$

故边值问题存在一个解$y(x,\ \mu)\in C^2$，且成立

$$\alpha(x,\ \mu)\leqslant y(x,\ \mu)\leqslant\beta(x,\ \mu)$$

因而就有

$$|y(x,\ \mu)-u(x)|\leqslant W_L+W_R+\gamma\mu$$

定理2.4 假设

$(\mathrm{H}_1)'$：退化问题 $g(x, u)=0$ 存在一个解 $u(x)\in C^2$，且使得 $u''(x)\geqslant 0$，$u'f\geqslant 0$，$u(a)\leqslant A$，$u(b)\leqslant B$，

$(\mathrm{H}_2)'$：$f(x, y)$，$g(x, y)$，$g_y(x, y)\in C(D_1)$，$D_1: a\leqslant x\leqslant b$，$0\leqslant y-u(x)\leqslant d(x)$，

$(\mathrm{H}_3)'$：在D_1中$g_y(x, y)\leqslant -m$，$l_1\leqslant f(x, y)\leqslant l_2$

则在$\varepsilon/\mu^2\to 0$，$\mu^2/\varepsilon\to 0$以及$\varepsilon=\mu^2$三种情形下，边值问题(2.1)～(2.2)存在解$y(x,\varepsilon,\mu)\in C^2$，且分别有如下关系式成立

$$0\leqslant y(x, \varepsilon, \mu)-u(x)\leqslant W_{1L}+W_{1R}+\gamma_1\mu \quad (\text{若}\varepsilon/\mu^2\to 0\text{当}\mu\to 0) \tag{2.9}$$

$$0\leqslant y(x, \varepsilon, \mu)-u(x)\leqslant W_{2L}+W_{2R}+c\sqrt{\varepsilon} \quad \left(\text{若}\frac{\mu^2}{\varepsilon}\to 0\text{当}\varepsilon\to 0\right) \tag{2.10}$$

$$0\leqslant y(x, \mu)-u(x)\leqslant W_L+W_R+\gamma\mu \quad (\text{若当}\varepsilon=\mu^2) \tag{2.11}$$

证明　我们先考虑(2.9)，作函数

$$\tilde{\alpha}_1=u(x),$$
$$\beta_1=u(x)+W_{1L}+W_{1R}+\gamma_1\mu,$$

由假设的条件以及定理2.1的证明不难看出只须证明

$$\varepsilon\tilde{\alpha}_1''+\mu f(x, \tilde{\alpha}_1)\tilde{\alpha}_1'+g(x, \tilde{\alpha}_1)\geqslant 0$$

而

$$\varepsilon\tilde{\alpha}_1''+\mu f(x, \tilde{\alpha}_1)\tilde{\alpha}_1'+g(x, \tilde{\alpha}_1)$$
$$=\varepsilon u''+\mu f u'+g(x, u)\geqslant 0$$

故知边值问题(2.1)～(2.2)存在一个解$y(x, \varepsilon, \mu)\in C^2$，且满足

$$u(x)\leqslant y(x, \varepsilon, \mu)\leqslant u(x)+W_{1L}+W_{1R}+\gamma_1\mu$$

即

$$0\leqslant y(x, \varepsilon, \mu)-u(x)\leqslant W_{1L}+W_{1R}+\gamma_1\mu$$

若$\mu^2/\varepsilon\to 0$当$\varepsilon\to 0$，作函数

$$\tilde{\alpha}_2=u(x)$$
$$\beta_2=u(x)+W_{2L}+W_{2R}+c\sqrt{\varepsilon}$$

若$\varepsilon=\mu^2$，作函数

$$\tilde{\alpha}=u(x)$$
$$\beta=u(x)+W_L+W_R+\gamma\mu$$

我们也不难得到定理的结论。

定理2.5　假设

$(\mathrm{H}_1)''$：退化问题 $g(x, u)=0$ 存在一个解 $u(x)\in C^2$，且使得 $u''(x)\leqslant 0$，$u'f\leqslant 0$，$u(a)\geqslant A$，$u(b)\geqslant B$，

$(\mathrm{H}_2)''$：$f(x, y)$，$g(x, y)$，$g_y(x, y)\in C(D_2)$，其中 D_2：$a\leqslant x\leqslant b$，$-d(x)\leqslant y-u(x)\leqslant 0$，

$(\mathrm{H}_3)''$：在D_2中$g_y(x, y)\leqslant -m$，$l_1\leqslant f(x, y)\leqslant l_2$，

则边值问题(2.1)～(2.2)存在解$y(x, \varepsilon, \mu)\in C^2$，且成立

$$-W_{1L}-W_{1R}-\gamma_1\mu\leqslant y-u(x)\leqslant 0 \quad \left(\text{若}\frac{\varepsilon}{\mu^2}\to 0\text{当}\mu\to 0\right) \tag{2.12}$$

$$-W_{2L}-W_{2R}-c\sqrt{\varepsilon}\leqslant y-u(x)\leqslant 0 \quad \left(\text{若}\frac{\mu^2}{\varepsilon}\to 0\text{当}\varepsilon\to 0\right) \tag{2.13}$$

$$-W_L-W_R-\gamma\mu\leqslant y-u(x)\leqslant 0 \qquad (\text{若}\varepsilon=\mu^2) \tag{2.14}$$

证明 若$\varepsilon/\mu^2\to 0$当$\mu\to 0$，作函数

$$\alpha_1=u(x)-W_{1L}-W_{1R}-\gamma_1\mu,$$
$$\tilde{\beta}_1=u(x),$$

则$\varepsilon\tilde{\beta}''_1+\mu f(x,\ \tilde{\beta}_1)\tilde{\beta}'_1+g(x,\ \tilde{\beta}_1)=\varepsilon u''+\mu f u'\leqslant 0$，由假设的条件及定理2.2的证明不难看出Nagumo定理的条件均满足，故可知边值问题(2.1)～(2.2)存在一个解$y(x,\ \varepsilon,\ \mu)\in C^2$，且成立

$$-W_{1L}-W_{1R}-\gamma_1\mu\leqslant y-u(x)\leqslant 0$$

若$\mu^2/\varepsilon\to 0$当$\varepsilon\to 0$，$\varepsilon=\mu^2$分别作函数

$$\alpha_2=u(x)-W_{2L}-W_{2R}-c\sqrt{\varepsilon}\,,\quad \tilde{\beta}_2=u(x)$$
$$\alpha=u(x)-W_L-W_R-\gamma\mu,\quad \tilde{\beta}=u(x)$$

类似地可证明(2.13)，(2.14)式。

三、Robin 问题

现在，我们转到考虑 Robin 问题上来，考虑如下问题：

$$\varepsilon y''+\mu f(x,\ y)y'+g(x,\ y)=0 \tag{3.1}$$
$$y(a)-p_1y'(a)=A,\quad y(b)=B \tag{3.2}$$

以及

$$\varepsilon y''+\mu f(x,\ y)y'+g(x,\ y)=0 \tag{3.1$'$}$$
$$y(a)-p_1y'(a)=A,\quad y(b)+p_2y'(b)=B \tag{3.2$'$}$$

其中ε，μ是正的小参数，p_1，p_2为正常数。

首先我们考虑边值问题(3.1)～(3.2)，我们有如下定理：

定理3.1 假设

(A_1) 退化问题$g(x,\ u)=0$存在一个解$u(x)\in C^2$，

(A_2) $f(x,\ y)$，$g(x,\ y)$，$g_y(x,\ y)\in C(\bar{D})$，$\bar{D}:a\leqslant x\leqslant b$，$|y-u(x)|\leqslant \bar{d}(x)$，

$$\bar{d}(x)=\begin{cases}\delta & [a,\ b-\delta]\\ |B-u(b)|+\delta & \left[b-\dfrac{\delta}{2},\ b\right]\end{cases}$$

(A_3) 在$\bar{D}$中$g_y(x,\ y)\leqslant -m$，$l_1\leqslant f(x,\ y)\leqslant l_2$，

则在$\mu\to 0$时$\varepsilon/\mu^2\to 0$的情形下，边值问题(3.1)～(3.2)存在一个解$y(x,\ \varepsilon,\ \mu)\in C^2$，且成立

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant V_{1L}(x,\ \varepsilon,\ \mu)+W_{1R}(x,\ \varepsilon,\ \mu)+\gamma_1\mu \tag{3.3}$$

这里$V_{1L}=\dfrac{1}{-\lambda_1 p_1}|A-u(a)+p_1u'(a)|\exp[\lambda_1(x-a)]$。

证明 注意到正函数V_{1L}是微分方程$\varepsilon V''_{1L}+\mu l_1V'_{1L}-mV_{1L}=0$满足条件$V_{1L}(a,\ \varepsilon,\ \mu)=\dfrac{1}{-\lambda_1 p_1}|A-u(a)-p_1u'(a)|$，$V'_{1L}(a,\ \varepsilon,\ \mu)=-\dfrac{1}{p_1}|A-u(a)+p_1u'(a)|$的解，且$V'_{1L}(x,\ \varepsilon,\ \mu)<0$。作函数

$$\alpha(x,\ \varepsilon,\ \mu)=u(x)-V_{1L}-W_{1R}-\gamma_1\mu \tag{3.4}$$
$$\beta(x,\ \varepsilon,\ \mu)=u(x)+V_{1L}+W_{1R}+\gamma_1\mu \tag{3.5}$$

不难验证：$\alpha\leqslant\beta$，$\alpha(a,\ \varepsilon,\ \mu)-p_1\alpha'(a,\ \varepsilon,\ \mu)\leqslant A\leqslant\beta(a,\ \varepsilon,\ \mu)-p_1\beta'(a,\ \varepsilon,\ \mu)$，$\alpha(b,\ \varepsilon,\ \mu)\leqslant B\leqslant\beta(b,\ \varepsilon,\ \mu)$，与前述定理相同，可分别在区间 $[a,\ a+\delta/2]$，$[a+\delta/2,\ b-\delta/2]$ 和 $[b-\delta/2,\ b]$ 上证明

$$\varepsilon\alpha''+\mu f(x,\ \alpha)\alpha'+g(x,\ \alpha)\geqslant 0 \tag{3.6}$$

$$\varepsilon\beta''+\mu f(x,\ \beta)\beta'+g(x,\ \beta)\leqslant 0 \tag{3.7}$$

事实上，当 $x\in[a,\ a+\delta/2]$，$W_{1R}=O(\mu^{N+1})=W'_{1R}$，则

$$\begin{aligned}&\varepsilon\alpha''+\mu f(x,\ \alpha)\alpha'+g(x,\ \alpha)\\&=\varepsilon u''+\mu fu'-\varepsilon V''_{1L}-\mu fV'_{1L}+g_y(-V_{1L}-\gamma_1\mu)+O(\mu^{N+1})\\&\geqslant\varepsilon u''+\mu fu'-(\varepsilon V''_{1L}+\mu l_1V'_{1L}-mV_{1L})+m\gamma_1\mu-K\mu^{N+1}\geqslant 0,\end{aligned}$$

当 $x\in[a+\frac{\delta}{2},\ b-\frac{\delta}{2}]$，$V_{1L}=O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)=V'_{1L}$，$W_{1R}=O(\mu^{N+1})=W'_{1R}$，则

$$\begin{aligned}&\varepsilon\alpha''+\mu f(x,\ \alpha)\alpha'+g(x,\ \alpha)\\&=\varepsilon u''+\mu fu'-g_y\gamma_1\mu+O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)+O(\mu^{N+1})\\&\geqslant\varepsilon u''+\mu fu'+m\gamma_1\mu-K\left(\frac{\varepsilon}{\mu}\right)^{N+1}-K\mu^{N+1}\geqslant 0\end{aligned}$$

当 $x\in[b-\frac{\delta}{2},\ b]$，$V_{1L}=O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)=V'_{1L}$，则

$$\begin{aligned}&\varepsilon\alpha''+\mu f(x,\ \alpha)\alpha'+g(x,\ \alpha)\\&=\varepsilon u''+\mu fu'-\varepsilon W''_{1R}-\mu fW'_{1R}+g_y(-W_{1R}-\gamma_1\mu)+O\left(\left(\frac{\varepsilon}{\mu}\right)^{N+1}\right)\\&\geqslant\varepsilon u''+\mu fu'-(\varepsilon W''_{1R}+\mu l_2W'_{1R}-mW_{1R})+m\gamma_1\mu-K_1\left(\frac{\varepsilon}{\mu}\right)^{N+1}\\&\geqslant\left[m\gamma_1-\frac{\varepsilon}{\mu}M_1-M-K_1\frac{\varepsilon^{N+1}}{\mu^{N+2}}\right]\mu\geqslant 0.\end{aligned}$$

故在整个区间 $[a,\ b]$ 上(3.6)式成立，类似地可证(3.7)式，所以由微分不等式定理[13]知边值问题(3.1)～(3.2)存在一个解 $y(x,\ \varepsilon,\ \mu)\in C^2$，且满足：

$$\alpha(x,\ \varepsilon,\ \mu)\leqslant y(x,\ \varepsilon,\ \mu)\leqslant\beta(x,\ \varepsilon,\ \mu)$$

因而有

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant V_{1L}+W_{1R}+\gamma_1\mu$$

注1 若 $\frac{\mu^2}{\varepsilon}\to 0$ 当 $\varepsilon\to 0$ 时，则在假设 (A_1)～(A_3) 下，可以证明边值问题 (3.1)～(3.2) 存在一个解 $y(x,\ \varepsilon,\ \mu)\in C^2$，且成立

$$|y(x,\ \varepsilon,\ \mu)-u(x)|\leqslant V_{2L}+W_{2R}+c\sqrt{\varepsilon}$$

这里 $V_{2L}=-|A-u(a)+p_1u'(a)|\exp[\lambda_3(x-a)]/\lambda_3p_1$。若 $\varepsilon=\mu^2$，也可以证明边值问题 (3.1)～(3.2)存在一个解 $y(x,\ \mu)\in C^2$，且成立

$$|y(x,\ \mu)-u(x)|\leqslant V_L+W_R+\gamma\mu,$$

其中 $V_L=-\frac{1}{\lambda_5p_1}|A-u(a)+p_1u'(a)|\exp[\lambda_5(x-a)]$.

注2 对于边值问题(3.1)～(3.2)，也有类似于定理2.4和定理2.5的结论。

注3 对于如下 Robin 问题

$$\varepsilon y''+\mu f(x,y)y'+g(x,y)=0$$
$$y(a)=A,\quad y(b)+p_2y'(b)=B$$

($p_2>0$为常数)，可以通过变量转换$x\to a+b-x$来处理。

下面我们来考虑边值问题(3.1)′～(3.2)′。有下面定理

定理3.2 假设

(A1)′：退化问题$g(x,u)=0$存在一个解$u(x)\in C^2$，

(A2)′：$f(x,y)$，$g(x,y)$，$g_y(x,y)\in C(\tilde{D})$，$\tilde{D}:a\leqslant x\leqslant b$，$|y-u(x)|\leqslant\tilde{d}(x)$，而 $\tilde{d}(x)\equiv\delta$，

(A3)′：在$\tilde{D}$中$g_y(x,y)\leqslant -m$，$l_1\leqslant f(x,y)\leqslant l_2$，

则在$\mu\to 0$时$\varepsilon/\mu^2\to 0$的情形，边值问题(3.1)′～(3.2)′存在一个解$y(x,\varepsilon,\mu)\in C^2$，且成立

$$|y(x,\varepsilon,\mu)-u(x)|\leqslant V_{1L}+V_{1R}+\gamma_1\mu$$

这里$V_{1R}=|B-u(b)-p_2u'(b)|\exp[\lambda_2(x-b)]/\lambda_2p_2$。

证明 注意到正函数V_{1R}是方程$\varepsilon V''_{1R}+\mu l_2V'_{1R}-mV_{1R}=0$满足条件$V_{1R}(b,\varepsilon,\mu)=|B-u(b)-p_2u'(b)|/\lambda_2p_2$，$V'_{1R}(b,\varepsilon,\mu)=|B-u(b)-p_2u'(b)|/p_2$的解。作函数

$$\alpha(x,\varepsilon,\mu)=u(x)-V_{1L}-V_{1R}-\gamma_1\mu \tag{3.8}$$
$$\beta(x,\varepsilon,\mu)=u(x)+V_{1L}+V_{1R}+\gamma_1\mu \tag{3.9}$$

易知$\alpha\leqslant\beta$，$\alpha(a,\varepsilon,\mu)-p_1\alpha'(a,\varepsilon,\mu)\leqslant A\leqslant\beta(a,\varepsilon,\mu)-p_1\beta'(a,\varepsilon,\mu)$，和$\alpha(b,\varepsilon,\mu)+p_2\alpha'(b,\varepsilon,\mu)\leqslant B\leqslant\beta(b,\varepsilon,\mu)+p_2\beta'(b,\varepsilon,\mu)$，且在区间$[a,a+\delta/2]$，$[a+\delta/2,b-\delta/2]$以及$[b-\delta/2,b]$上不难证明

$$\varepsilon\alpha''+\mu f(x,\alpha)\alpha'+g(x,\alpha)\geqslant 0$$
$$\varepsilon\beta''+\mu f(x,\beta)\beta'+g(x,\beta)\leqslant 0$$

故由微分不等式定理[13]知边值问题(3.1)′～(3.2)′存在一个解$y(x,\varepsilon,\mu)\in C^2$，且成立

$$\alpha(x,\varepsilon,\mu)\leqslant y(x,\varepsilon,\mu)\leqslant\beta(x,\varepsilon,\mu)$$

故再由(3.8)～(3.9)知

$$|y(x,\varepsilon,\mu)-u(x)|\leqslant V_{1L}+V_{1R}+\gamma_1\mu$$

注4 对于$\frac{\mu^2}{\varepsilon}\to 0$当$\varepsilon\to 0$时以及$\varepsilon=\mu^2$的情形，有类似于定理2.2和定理2.3的结论。

注5 边值问题(3.1)′～(3.2)′也有类似定理2.4和定理2.5的结论。

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参 考 文 献

[1] Chang, K. W. and F. A. Howes, Nonlinear perturbation Phenomena: Theory and application, *Appl. Math. Sci.*, Springer-Verlag, New York Inc., **56** (1984).

[2] O'Donnell, Mark A., Boundary and Corner layer behavior in singularly perturbed semilinear systems of boundary value problems, *SIAM. J. Math. Anal.*, **15** (1984), 317—332.

[3] Howes, F. A., Differential inequalities and applications to nonlinear singular perturbation problems, *J. Diff. Equ.*, **20** (1976), 133—149.

[4] Chang, K. W., Diagonalization method for a vector boundary problems of singular perturbation type, *J. Math. Anal. Appl.*, **48** (1974), 652—665.

[5] Howes, F. A., Effective characterization of the asymptotic behavior of solution of singularly perturbed boundary value problem, *SIAM J. Appl. Math.*, **30**, 2 (1976).

[6] Howes, F. A., Singularly perturbed semilinear Robin problem, *Studies in Appl. Math.*, **67** (1982), 125—139.

[7] Mo Jia-qi, The estimation of singularly perturbed solution for a second order quasilinear equation via differential inequalities, *J. Math. Res. and Exposition*, **1** (1986), 58—64.

[8] O'Malley, R. E. Jr., Singular perturbation of boundary value problem for linear ordinary differential equations involving two parameters, *J. Math. Anal. Appl.*, **19** (1967), 291—308.

[9] O'Malley, R. E. Jr., Boundary value problems for linear systems of ordinary differential equations involving many small parameters, *J. Math. Mech.*, **18** (1969), 835—856.

[10] O'Malley, R. E. Jr., On initial value problems for nonlinear systems of differential equations with two small parameters, *Arch. Ration Mech. Anal.*, **40** (1971), 209—222.

[11] Nagumo, M., Über die Differentialgleichung $y''=f(x, y, y)$, *Proc. Phys. Soc., Japan*, **19** (1937), 861—866.

[12] Jackson, L. K., Subfunctions and second-order ordinary differential inequalities, *Advances in Math.*, **2** (1968), 307—363.

[13] Heidel, J. W., A Second-order nonlinear boundary value problem, *J. Math. Anal. and Appl.*, **48** (1974), 493—503.

On the Singular Perturbation of a Nonlinear Ordinary Differential Equation with Two Parameters

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Abstract

In this paper, the method of differential inequalities has been applied to study the boundary value problems of nonlinear ordinary differential equation with two parameters. The asymptotic solutions have been found and the remainders have been estimated.