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变壁厚柱壳的轴对称问题*

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摘 要

本文给出了壁厚按二次函数变化的轴对称柱壳的一般解。

K. Federhofer[1]曾研究过壁厚按轴向坐标的二次函数变化的轴对称柱壳，并且给出了在某些情况下的解。

本文也研究上述壳体。对于所有可能的情况，给出了控制微分方程的齐次解；对于控制微分方程的非齐次项可表示为自变量的多项式或收敛幂级数的情况，给出了方程的特解。

一、控制微分方程

对于变壁厚轴对称柱壳，令x和θ为其中面上的轴向坐标和环向坐标，z为外法线方向的坐标（图1）；R为中面的半径；u和w为中面上一点的轴向位移和法向位移；Θ为中面转角；N_x和N_θ为轴向力和环向力；Q为横剪力；M_x和M_θ为轴向弯矩和环向弯矩；q_x和q_z为轴向的和法向的外载荷强度；h为壁厚；E为弹性模量；ν为泊桑比。

关于正方向，则按照图1、2、3确定。

变壁厚轴对称柱壳的基本关系式为

$$dN_x/dx+q_x=0 \tag{1.1}$$

$$dQ/dx-N_\theta/R+q_z=0 \tag{1.2}$$

$$dM_x/dx-Q=0 \tag{1.3}$$

$$\Theta=dw/dx \tag{1.4}$$

$$N_x=(Eh/(1-\nu^2))(du/dx+\nu w/R) \tag{1.5}$$

$$N_\theta=(Eh/(1-\nu^2))(\nu du/dx+w/R) \tag{1.6}$$

$$M_x=-(Eh^3/12(1-\nu^2))d\Theta/dx \tag{1.7}$$

由以上诸式可导出如下方程：

$$\frac{d^2\Theta}{dx^2}+\frac{3}{h}\frac{dh}{dx}\frac{d\Theta}{dx}+\frac{12(1-\nu^2)}{Eh^3}Q=0 \tag{1.8}$$

* 钱伟长推荐。

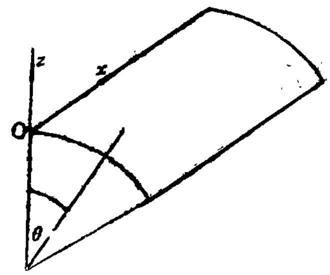

图 1

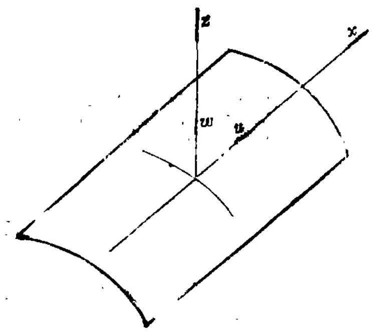

图 2

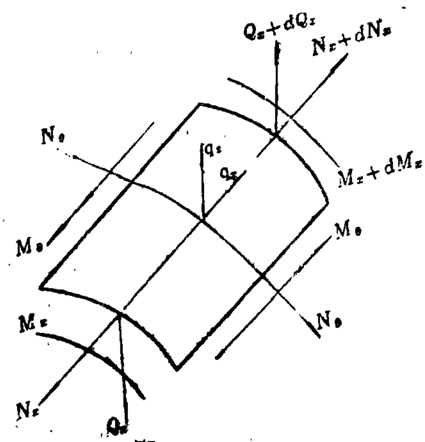

图 3

$$\frac{d^2Q}{dx^2}-\frac{1}{h}\frac{dh}{dx}\frac{dQ}{dx}-\frac{Eh}{R^2}\Theta=\frac{1}{h}\frac{dh}{dx}q_z-\frac{dq_z}{dx}$$

$$-\frac{\nu}{R}q_x+\frac{\nu}{Rh}\frac{dh}{dx}\int_{x_1}^{x}q_x dx-\frac{\nu}{Rh}\frac{dh}{dx}N_{x_1} \tag{1.9}$$

式中，x_1为x的某一参考值，N_{x_1}为$x=x_1$处的轴向力。

令

$$V=Eh^2\Theta \tag{1.10}$$

$$L(\cdots)=\left[h\frac{d^2}{dx^2}-\frac{dh}{dx}\frac{d}{dx}\right](\cdots) \tag{1.11}$$

$$g(x)=h\frac{dq_z}{dx}-\frac{1}{h}\frac{dh}{dx}q_z+\frac{\nu h}{R}q_x-\frac{\nu}{R}\frac{dh}{dx}\int_{x_1}^{x}q_x dx+\frac{\nu}{R}\frac{dh}{dx}N_{x_1} \tag{1.12}$$

则式(1.8)和(1.9)可写为

$$L(V)-2\frac{d^2h}{dx^2}V+12(1-\nu^2)Q=0 \tag{1.13}$$

$$L(Q)-V/R^2=-g(x) \tag{1.14}$$

将式(1.13)代入式(1.14)得

$$LL(V)-2L\left(\frac{d^2h}{dx^2}V\right)+\frac{12(1-\nu^2)}{R^2}V=G(x) \tag{1.15}$$

式中

$$G(x)=12(1-\nu^2)g(x) \tag{1.16}$$

显然，若
$$h=px^2+qx+r \tag{1.17}$$
式中p，q，r为实常数，则
$$d^2h/dx^2=2p=\text{const} \tag{1.18}$$
在式(1.18)的条件下，方程(1.15)可写为
$$[L-2p(1+\sqrt{1-k})][L-2p(1-\sqrt{1-k})]V=G(x) \tag{1.19}$$
式中
$$k=3(1-\nu^2)/(Rp)^2 \tag{1.20}$$
将式(1.11)和(1.17)代入式(1.19)得
$$\left[(px^2+qx+r)\frac{d^2}{dx^2}-(2px+q)\frac{d}{dx}-2p(1+\sqrt{1-k})\right]$$
$$\cdot\left[(px^2+qx+r)\frac{d^2}{dx^2}-(2px+q)\frac{d}{dx}-2p(1-\sqrt{1-k})\right]V=G(x) \tag{1.21}$$
方程(1.21)还可以简化。这分两种情况。

(1) $q^2-4pr\neq 0$

设
$$\xi=1/2+p(x+q/2p)/\sqrt{q^2-4pr} \tag{1.22}$$
则式(1.21)可写为
$$\left[\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2(1+\sqrt{1-k})\right]$$
$$\cdot\left[\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2(1-\sqrt{1-k})\right]V=G_1(\xi) \tag{1.23}$$
式中
$$G_1(\xi)=G(x(\xi)) \tag{1.24}$$

(2) $q^2-4pr=0$

设
$$\tilde{x}=x+q/2p \tag{1.25}$$
则式(1.21)可写为
$$\left[\tilde{x}^2\frac{d^2}{d\tilde{x}^2}-2\tilde{x}\frac{d}{d\tilde{x}}-2(1+\sqrt{1-k})\right]$$
$$\cdot\left[\tilde{x}^2\frac{d^2}{d\tilde{x}^2}-2\tilde{x}\frac{d}{d\tilde{x}}-2(1-\sqrt{1-k})\right]V=G_2(\tilde{x}) \tag{1.26}$$
式中
$$G_2(\tilde{x})=G(x(\tilde{x})) \tag{1.27}$$

二、$q^2-4pr\neq 0$条件下微分方程的解

让我们来求解方程(1.23)。

1. 特解

方程(1.23)可以改写为
$$\xi^2(1-\xi)^2\frac{d^4V}{d\xi^4}+6\xi(1-\xi)\frac{d^2V}{d\xi^2}-6(1-2\xi)\frac{dV}{d\xi}+4kV=G_1(\xi) \tag{2.1}$$
因此

(1) 如果

$$G_1(\xi)=\sum_{n=0}^{m}b_n\xi^n \tag{2.2}$$

式中m为正整数或零，则方程(2.1)的特解为

$$V=\sum_{n=0}^{m}a_n\xi^n \tag{2.3}$$

其中a_n由下式确定：

$$(n+2)(n+1)n(n-1)a_{n+2}-2(n+1)^2(n-1)(n-3)a_{n+1}$$
$$+[(n+1)n(n-3)(n-4)+4k]a_n=b_n \tag{2.4}$$

（2） 如果$G(\xi)$为收敛级数

$$G_1(\xi)=\sum_{n=1}^{\infty}b_{-n}\xi^{-n} \tag{2.5}$$

则方程(2.1)的特解为

$$V=\sum_{n=1}^{\infty}a_{-n}\xi^{-n} \tag{2.6}$$

其中a_{-n}由下式确定：

$$(n-2)(n-1)n(n+1)a_{-n+2}-2(n-1)^2(n+1)(n+3)a_{-n+1}$$
$$+[(n-1)n(n+3)(n+4)+4k]a_{-n}=b_{-n} \tag{2.7}$$

可以证明，当$|\xi|\geqslant 1$时，级数(2.6)绝对收敛。

事实上，令

$$\left[\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2(1-\sqrt{1-k})\right]V=V^* \tag{2.8}$$

将式(2.8)代入式(1.23)则得

$$\left[\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2(1+\sqrt{1-k})\right]V^*=G_1(\xi) \tag{2.9}$$

设

$$V^*=\sum_{n=1}^{\infty}c_{-n}\xi^{-n} \tag{2.10}$$

将式(2.5)和(2.10)代入式(2.9)得

$$(n^2-1)c_{-n+1}-[n(n+3)-2(1+\sqrt{1-k})]c_{-n}=b_{-n} \tag{2.11}$$

而将式(2.6)和(2.10)代入式(2.8)则得

$$(n^2-1)a_{-n+1}-[n(n+3)-2(1-\sqrt{1-k})]a_{-n}=c_{-n} \tag{2.12}$$

显然，由式(2.11)和(2.12)可以得到式(2.7)。而且当$n\to\infty$时

$$\lim_{n\to\infty}c_{-n+1}/c_{-n}=1+3/n+O(1/n^2) \tag{2.13}$$

从而

$$\lim_{n\to\infty}a_{-n+1}/a_{-n}=1+3/n+O(1/n^2) \tag{2.14}$$

如此，当$|\xi|\geqslant 1$时，级数(2.6)确为绝对收敛。

2. 齐次解

与方程(1.23)相应的齐次方程为

$$\left[\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2(1+\sqrt{1-k})\right]$$

$$\cdot\left[\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2(1-\sqrt{1-k})\right]V=0 \tag{2.15}$$

根据 k 取值的不同，方程(2.15)的解有三种情况。

(i) $k>1$

在这种情况下，由方程(2.15)可以获得两个超几何方程：

$$\xi(1-\xi)\frac{d^2V}{d\xi^2}-(1-2\xi)\frac{dV}{d\xi}+2(1\pm i\sqrt{k-1})V=0$$

$$(i=\sqrt{-1}) \tag{2.16a,b}$$

方程(2.16a)和(2.16b)为复共轭方程，它们的解是复共轭的。因此，解其中的一个方程，例如(2.16a)，便可得到方程(2.15)的解。

将方程(2.16a)与超几何方程的一般形式

$$\xi(1-\xi)\frac{d^2V}{d\xi^2}+[\gamma-(1+\alpha+\beta)\xi]\frac{dV}{d\xi}-\alpha\beta V=0 \tag{2.17}$$

加以比较，可见在式(2.16a)中

$$\alpha=-3/2+\lambda,\quad \beta=-3/2-\lambda,\quad \gamma=-1 \tag{2.18}$$

其中
$$\lambda=\sqrt{17+8i\sqrt{k-1}}/2 \tag{2.19}$$

因此，方程(2.16a)在奇点 $\xi=0$，1和∞邻域的基本解分别为[2]：

$$\left.\begin{aligned}
V_{1(0)}&=\xi^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1;\ 2-\gamma;\ \xi)\\
&=\xi^2F(1/2+\lambda,\ 1/2-\lambda;\ 3;\ \xi)\\
V_{2(0)}&=\xi^{1-\gamma}f(\alpha-\gamma+1,\ \beta-\gamma+1;\ 2-\gamma;\ \xi)\\
&=\xi^2f(1/2+\lambda,\ 1/2-\lambda;\ 3;\ \xi)\\
V_{1(1)}&=(1-\xi)^{\gamma-\alpha-\beta}F(\gamma-\beta,\ \gamma-\alpha;\ \gamma+1-\alpha-\beta;\ 1-\xi)\\
&=(1-\xi)^2F(1/2+\lambda,\ 1/2-\lambda;\ 3;\ 1-\xi)\\
V_{2(1)}&=(1-\xi)^{\gamma-\alpha-\beta}f(\gamma-\beta,\ \gamma-\alpha;\ \gamma+1-\alpha-\beta;\ 1-\xi)\\
&=(1-\xi)^2f(1/2+\lambda,\ 1/2-\lambda;\ 3;\ 1-\xi)\\
V_{1(\infty)}&=\xi^{-\alpha}F(\alpha,\ \alpha-\gamma+1;\ \alpha-\beta+1;\ 1/\xi)\\
&=\xi^{3/2-\lambda}F(-3/2+\lambda,\ 1/2+\lambda;\ 1+2\lambda;\ 1/\xi)\\
V_{2(\infty)}&=\xi^{-\beta}F(\beta,\ \beta-\gamma+1;\ \beta-\alpha+1;\ 1/\xi)\\
&=\xi^{3/2+\lambda}F(-3/2-\lambda,\ 1/2-\lambda;\ 1-2\lambda;\ 1/\xi)
\end{aligned}\right\} \tag{2.20}$$

式(2.20)中的两个特殊函数为

(a) 超几何函数

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n\qquad(|z|<1) \tag{2.21}$$

其中

$$\left.\begin{aligned}&(\theta)_0=1\\&(\theta)_n=\theta(\theta+1)(\theta+2)\cdots(\theta+n-1)\qquad(n\geqslant1)\end{aligned}\right\}\tag{2.22}$$

因此，在式(2.21)中 $c\neq0,-1,-2,\cdots$，除非 a（或 b）也等零或负整数，并且 $a\geqslant c$（或 $b\geqslant c$）；

$$\text{(b)}\quad f(a,b;c;z)=F(a,b;c;z)\ln z+\sum_{n=1}^{\infty}e_nz^n-\sum_{n=1}^{c-1}g_nz^{-n}\qquad(|z|<1,\ c=1,2,3\cdots)\tag{2.23}$$

其中

$$\begin{aligned}e_n=&\frac{(a)_n(b)_n}{n!(c)_n}[\psi(a+n)-\psi(a)+\psi(b+n)-\psi(b)-\psi(c+n)\\&+\psi(c)-\psi(1+n)+\psi(1)]\\=&\frac{(a)_n(b)_n}{n!(c)_n}\sum_{m=1}^{n}\left(\frac{1}{a+m-1}+\frac{1}{b+m-1}-\frac{1}{c+m-1}-\frac{1}{m}\right)\end{aligned}$$

$$(a\neq0,\ b\neq0)\tag{2.24}$$

$\psi(x)$ 是伽玛函数 $\Gamma(x)$ 的对数导数

$$\psi(x)=\frac{d\ln\Gamma(x)}{dx}=\frac{\Gamma'(x)}{\Gamma(x)}\tag{2.25}$$

$$g_n=\begin{cases}0 & (c=1)\\ \dfrac{(n-1)!(1-c)_n}{(1-a)_n(1-b)_n} & \left(\begin{aligned}&c=2,3,4,\cdots\\&a,b\neq1,2,\cdots,c-1\end{aligned}\right)\end{cases}\tag{2.26}$$

式(2.20)中的 V 是复函数。分离其实部和虚部，便可得到方程(2.15)在每个奇点邻域的四个线性独立的实函数解。

这里要研究两种情况。

(1) $q^2-4pr>0$

在这种条件下，ξ 是实量。

让我们首先研究在奇点 $\xi=0$ 和 $\xi=1$ 邻域的解。此时

$$a=1/2+\lambda,\ b=1/2-\lambda,\ c=3\tag{2.27}$$

设

$$d_n^{(1)}+id_n^{(2)}=(a)_n(b)_n=\left(\frac{1}{2}+\lambda\right)_n\left(\frac{1}{2}-\lambda\right)_n\tag{2.28}$$

则

$$d_n^{(1)}+id_n^{(2)}=\left(n-\frac{1}{2}+\lambda\right)\left(n-\frac{1}{2}-\lambda\right)(d_{n-1}^{(1)}+id_{n-1}^{(2)})\qquad(n\geqslant1)\tag{2.29}$$

注意式(2.19)得

$$\left.\begin{aligned}d_n^{(1)}&=(n^2-n-4)d_{n-1}^{(1)}+2\sqrt{k-1}\,d_{n-1}^{(2)}\\d_n^{(2)}&=(n^2-n-4)d_{n-1}^{(2)}-2\sqrt{k-1}\,d_{n-1}^{(1)}\end{aligned}\right\}\qquad(n\geqslant1)\tag{2.30}$$

因为 $d_0^{(1)}=1$，$d_0^{(2)}=0$，所以由式(2.30)可以确定所有的系数 $d_n^{(1)}$ 和 $d_n^{(2)}$。设

$$e_n=(e_n^{(1)}+ie_n^{(2)})/n!(3)_n\tag{2.31}$$

则

$$e_n^{(1)}+ie_n^{(2)}=(d_n^{(1)}+id_n^{(2)})\sum_{m=1}^{n}\left[\frac{1}{m-1/2+\lambda}+\frac{1}{m-1/2-\lambda}-\frac{1}{m+2}-\frac{1}{m}\right]$$

$$=(d_n^{(1)}+id_n^{(2)})\sum_{m=1}^{n}\Big[\frac{(2m-1)(m^2-m-4)}{(m^2-m-4)^2+4(k-1)}-\frac{2(m+1)}{m(m+2)}+\frac{2i\sqrt{k-1}}{(m^2-m-4)^2+4(k-1)}\Big] \tag{2.32}$$

因而

$$\left.\begin{aligned}e_n^{(1)}&=d_n^{(1)}\sum_{m=1}^{n}\Big[\frac{(2m-1)(m^2-m-4)}{(m^2-m-4)^2+4(k-1)}-\frac{2(m+1)}{m(m+2)}\Big]\\&\quad-d_n^{(2)}\sum_{m=1}^{n}\frac{2\sqrt{k-1}}{(m^2-m-4)^2+4(k-1)}\\e_n^{(2)}&=d_n^{(2)}\sum_{m=1}^{n}\Big[\frac{(2m-1)(m^2-m-4)}{(m^2-m-4)^2+4(k-1)}-\frac{2(m+1)}{m(m+2)}\Big]\\&\quad+d_n^{(1)}\sum_{m=1}^{n}\frac{2\sqrt{k-1}}{(m^2-m-4)^2+4(k-1)}\end{aligned}\right\} \tag{2.33}$$

设 $$g_n=g_n^{(1)}+ig_n^{(2)} \tag{2.34}$$

则

$$\left.\begin{aligned}&g_1^{(1)}=\frac{2}{3+k},\quad g_1^{(2)}=-\frac{\sqrt{k-1}}{3+k}\\&g_2^{(1)}=\frac{3-k}{2k(3+k)},\quad g_2^{(2)}=-\frac{3\sqrt{k-1}}{2k(3+k)}\end{aligned}\right\} \tag{2.35}$$

如此，方程(2.15)在奇点$\xi=0,1$邻域的四个线性独立的实函数解可表达为

$$\left.\begin{aligned}&V_{1(l)}^{(r)}=z^2\sum_{n=0}^{\infty}\frac{d_n^{(1)}}{n!(3)_n}z^n,\quad V_{2(l)}^{(r)}=z^2\sum_{n=0}^{\infty}\frac{d_n^{(2)}}{n!(3)_n}z^n\\&V_{3(l)}^{(r)}=z^2\sum_{n=0}^{\infty}\frac{d_n^{(1)}}{n!(3)_n}z^n\ln z+\sum_{n=1}^{\infty}\frac{e_n^{(1)}}{n!(3)_n}z^n-\sum_{n=1}^{2}g_n^{(1)}z^{-n}\\&V_{4(l)}^{(r)}=z^2\sum_{n=0}^{\infty}\frac{d_n^{(2)}}{n!(3)_n}z^n\ln z+\sum_{n=1}^{\infty}\frac{e_n^{(2)}}{n!(3)_n}z^n-\sum_{n=1}^{2}g_n^{(2)}z^{-n}\end{aligned}\right\} \tag{2.36}$$

$(l=0,1;\ z=\xi,1-\xi)$

式中$(\cdots)^{(r)}$表示$(\cdots)$为实函数。

对于奇点$\xi=\infty$，设

$$t_n^{(1)}+it_n^{(2)}=\frac{(a)_n(b)_n}{n!(c)_n}=\frac{(-3/2+\lambda)_n(1/2+\lambda)_n}{n!(1+2\lambda)_n} \tag{2.37}$$

则 $$t_n^{(1)}+it_n^{(2)}=\frac{(n-5/2+\lambda)(n-1/2+\lambda)}{n(n+2\lambda)}(t_{n-1}^{(1)}+i\,t_{n-1}^{(2)})\qquad(n\geqslant1) \tag{2.38}$$

现在将 λ 写为另一种形式

$$\lambda=\eta_1+i\eta_2 \tag{2.39}$$

其中

$$\eta_1,\eta_2=\sqrt{(\sqrt{225+64k}\pm17)/2}/2 \tag{2.40}$$

则 $t_n^{(1)}$，$t_n^{(2)}$ 可写为

$$\left.\begin{aligned}
t_n^{(1)}=&\frac{1}{n[(n+2\eta_1)^2+4\eta_2^2]}\Big\langle\Big\{\Big[\Big(n-\frac{5}{2}+\eta_1\Big)\Big(n-\frac{1}{2}+\eta_1\Big)\\
&-\eta_2^2\Big](n+2\eta_1)+2\eta_2^2(2n-3+2\eta_1)\Big\}t_{n-1}^{(1)}\\
&-\eta_2\{(n+2\eta_1)(2n-3+2\eta_1)-2[(n-5/2+\eta_1)\\
&\cdot(n-1/2+\eta_1)-\eta_2^2]\}t_{n-1}^{(2)}\Big\rangle\\
t_n^{(2)}=&\frac{1}{n[(n+2\eta_1)^2+4\eta_2^2]}\Big\langle\Big\{\Big[\Big(n-\frac{5}{2}+\eta_1\Big)\Big(n-\frac{1}{2}+\eta_1\Big)\\
&-\eta_2^2\Big](n+2\eta_1)+2\eta_2^2(2n-3+2\eta_1)\Big\}t_{n-1}^{(2)}\\
&+\eta_2\{(n+2\eta_1)(2n-3+2\eta_1)-2[(n-5/2+\eta_1)\\
&\cdot(n-1/2+\eta_1)-\eta_2^2]\}t_{n-1}^{(1)}\Big\rangle\\
&\qquad\qquad\qquad\qquad(n\geqslant1)
\end{aligned}\right\} \tag{2.41}$$

因为 $t_0^{(1)}=1$，$t_0^{(2)}=0$，所以由式(2.41)可确定所有的系数 $t_n^{(1)}$ 和 $t_n^{(2)}$。类似地，设

$$\bar{t}_n^{(1)}+i\bar{t}_n^{(2)}=\frac{(-3/2-\lambda)_n(1/2-\lambda)_n}{n!(1-2\lambda)_n} \tag{2.42}$$

则

$$\left.\begin{aligned}
\bar{t}_n^{(1)}=&\frac{1}{n[(n-2\eta_1)^2+4\eta_2^2]}\Big\langle\Big\{\Big[\Big(n-\frac{5}{2}-\eta_1\Big)\Big(n-\frac{1}{2}-\eta_1\Big)\\
&-\eta_2^2\Big](n-2\eta_1)+2\eta_2^2(2n-3-2\eta_1)\Big\}\bar{t}_{n-1}^{(1)}\\
&+\eta_2\{(n-2\eta_1)(2n-3-2\eta_1)-2[(n-5/2-\eta_1)\\
&\cdot(n-1/2-\eta_1)-\eta_2^2]\}\bar{t}_{n-1}^{(2)}\Big\rangle\\
\bar{t}_n^{(2)}=&\frac{1}{n[(n-2\eta_1)^2+4\eta_2^2]}\Big\langle\Big\{\Big[\Big(n-\frac{5}{2}-\eta_1\Big)\Big(n-\frac{1}{2}-\eta_1\Big)\\
&-\eta_2^2\Big](n-2\eta_1)+2\eta_2^2(2n-3-2\eta_1)\Big\}\bar{t}_{n-1}^{(2)}\\
&-\eta_2\{(n-2\eta_1)(2n-3-2\eta_1)-2[(n-5/2-\eta_1)\\
&\cdot(n-1/2-\eta_1)-\eta_2^2]\}\bar{t}_{n-1}^{(1)}\Big\rangle\\
&\qquad\qquad\qquad\qquad(n\geqslant1)
\end{aligned}\right\} \tag{2.43}$$

因为 $\bar{t}_0^{(1)}=1$，$\bar{t}_0^{(2)}=0$，所以由式(2.43)可确定所有的系数 $\bar{t}_n^{(1)}$ 和 $\bar{t}_n^{(2)}$。

注意

$$y^{\pm i\theta}=\cos(\theta\ln y)\pm i\sin(\theta\ln y) \tag{2.44}$$

则得

$$\left.\begin{aligned}
V_{1(\infty)}^{(r)} &= \xi^{3/2-\eta_1}\sum_{n=0}^{\infty}[t_n^{(1)}\cos(\eta_2\ln\xi)+t_n^{(2)}\sin(\eta_2\ln\xi)]\xi^{-n}\\
V_{2(\infty)}^{(r)} &= \xi^{3/2-\eta_1}\sum_{n=0}^{\infty}[t_n^{(2)}\cos(\eta_2\ln\xi)-t_n^{(1)}\sin(\eta_2\ln\xi)]\xi^{-n}\\
V_{3(\infty)}^{(r)} &= \xi^{3/2-\eta_1}\sum_{n=0}^{\infty}[\bar{t}_n^{(1)}\cos(\eta_2\ln\xi)-\bar{t}_n^{(2)}\sin(\eta_2\ln\xi)]\xi^{-n}\\
V_{4(\infty)}^{(r)} &= \xi^{3/2-\eta_1}\sum_{n=0}^{\infty}[\bar{t}_n^{(2)}\cos(\eta_2\ln\xi)+\bar{t}_n^{(1)}\sin(\eta_2\ln\xi)]\xi^{-n}
\end{aligned}\right\} \tag{2.45}$$

（2） $q^2-4pr<0$

在这种条件下 ξ 是复量，将式(2.20)中的函数V 分离为实部和虚部的手续较为复杂。较好的办法是将方程(2.16a)加以变换，而求具有实宗量的解。

由式(1.23)

$$\xi=1/2-iy \tag{2.46}$$

式中 $$y=p(x+q/2p)/\sqrt{4pr-q^2} \tag{2.47}$$

设 $$\zeta=-4y^2 \tag{2.48}$$

则方程(2.16a)可变换为

$$\zeta(1-\zeta)\frac{d^2V}{d\zeta^2}+\frac{1}{2}(1+\zeta)\frac{dV}{d\zeta}+\frac{1}{2}(1+i\sqrt{k-1})V=0 \tag{2.49}$$

比较式(2.49)与(2.17)，并且注意式(2.19)，可见在式(2.49)中

$$\alpha=-3/4+\lambda/2,\ \beta=-3/4-\lambda/2,\ \gamma=1/2 \tag{2.50}$$

因为，方程(2.49)的基本解为

$$\left.\begin{aligned}
V_{1(0)}&=F(\alpha,\ \beta;\ \gamma;\ \zeta)=F\left(-\frac{3}{4}+\frac{\lambda}{2},\ -\frac{3}{4}-\frac{\lambda}{2};\ \frac{1}{2};\ \zeta\right)\\
V_{2(0)}&=\zeta^{1-\gamma}F(1+\alpha-\gamma,\ 1+\beta-\gamma;\ 2-\gamma;\ \zeta)\\
&=\zeta^{1/2}F\left(-\frac{1}{4}+\frac{\lambda}{2},\ -\frac{1}{4}-\frac{\lambda}{2};\ \frac{3}{2};\ \zeta\right)\\
V_{1(1)}&=(1-\zeta)^{\gamma-\alpha-\beta}F(\gamma-\beta,\ \gamma-\alpha;\ 1+\gamma-\alpha-\beta;\ 1-\zeta)\\
&=(1-\zeta)^2F\left(\frac{5}{4}+\frac{\lambda}{2},\ \frac{5}{4}-\frac{\lambda}{2};\ 3;\ 1-\zeta\right)\\
V_{2(1)}&=(1-\zeta)^{\gamma-\alpha-\beta}f(\gamma-\beta,\ \gamma-\alpha;\ 1+\gamma-\alpha-\beta;\ 1-\zeta)\\
&=(1-\zeta)^2f\left(\frac{5}{4}+\frac{\lambda}{2},\ \frac{5}{4}-\frac{\lambda}{2};\ 3;\ 1-\zeta\right)\\
V_{1(\infty)}&=\zeta^{-\alpha}F(\alpha,\ 1-\gamma+\alpha;\ 1-\beta+\alpha;\ 1/\zeta)\\
&=\zeta^{3/4-\lambda/2}F\left(-\frac{3}{4}+\frac{\lambda}{2},\ -\frac{1}{4}+\frac{\lambda}{2};\ 1+\lambda;\ \frac{1}{\zeta}\right)\\
V_{2(\infty)}&=\zeta^{-\beta}F\left(\beta,\ 1-\gamma+\beta;\ 1-\alpha+\beta;\ \frac{1}{\zeta}\right)\\
&=\zeta^{3/4+\lambda/2}F\left(-\frac{3}{4}-\frac{\lambda}{2},\ -\frac{1}{4}-\frac{\lambda}{2};\ 1-\lambda;\ \frac{1}{\zeta}\right)
\end{aligned}\right\} \tag{2.51}$$

分离式(2.51)中复函数V的实部和虚部，则可将方程(2.49)在奇点$\zeta=0$，1和∞邻域的四个线性独立的实函数解写为

$$
\left.
\begin{aligned}
&V_{1(0)}=\sum_{n=0}^{\infty}\frac{D_n^{(1)}}{n!(1/2)_n}\zeta^n,\qquad V_{2(0)}=\sum_{n=0}^{\infty}\frac{D_n^{(2)}}{n!(1/2)_n}\zeta^n\\
&V_{3(0)}=\zeta^{1/2}\sum_{n=0}^{\infty}\frac{\bar{D}_n^{(1)}}{n!(3/2)_n}\zeta^n,\quad V_{4(0)}=\zeta^{1/2}\sum_{n=0}^{\infty}\frac{\bar{D}_n^{(2)}}{n!(3/2)_n}\zeta^n\\
&V_{1(1)}=(1-\zeta)^2\sum_{n=0}^{\infty}\frac{\tilde{D}_n^{(1)}}{n!(3)_n}(1-\zeta)^n\\
&V_{2(1)}=(1-\zeta)^2\sum_{n=0}^{\infty}\frac{\tilde{D}_n^{(2)}}{n!(3)_n}(1-\zeta)^n\\
&V_{3(1)}=(1-\zeta)^2\sum_{n=0}^{\infty}\frac{\tilde{D}_n^{(1)}}{n!(3)_n}(1-\zeta)^n\ln(1-\zeta)\\
&\qquad+\sum_{n=1}^{\infty}\frac{E_n^{(1)}}{n!(3)_n}(1-\zeta)^n-\sum_{n=1}^{2}G_n^{(1)}(1-\zeta)^{-n}\\
&V_{4(1)}=(1-\zeta)^2\sum_{n=0}^{\infty}\frac{\tilde{D}_n^{(2)}}{n!(3)_n}(1-\zeta)^n\ln(1-\zeta)\\
&\qquad+\sum_{n=1}^{\infty}\frac{E_n^{(2)}}{n!(3)_n}(1-\zeta)^n-\sum_{n=1}^{2}G_n^{(2)}(1-\zeta)^{-n}\\
&V_{1(\infty)}=\zeta^{3/4+\eta_1/2}\sum_{n=0}^{\infty}\left[T_n^{(1)}\cos\left(\frac{\eta_2}{2}\ln\zeta\right)+T_n^{(2)}\sin\left(\frac{\eta_2}{2}\ln\zeta\right)\right]\zeta^{-n}\\
&V_{2(\infty)}=\zeta^{3/4-\eta_1/2}\sum_{n=0}^{\infty}\left[T_n^{(2)}\cos\left(\frac{\eta_2}{2}\ln\zeta\right)-T_n^{(1)}\sin\left(\frac{\eta_2}{2}\ln\zeta\right)\right]\zeta^{-n}\\
&V_{3(\infty)}=\zeta^{3/4+\eta_1/2}\sum_{n=0}^{\infty}\left[\boldsymbol{T}_n^{(1)}\cos\left(\frac{\eta_2}{2}\ln\zeta\right)-\boldsymbol{T}_n^{(2)}\sin\left(\frac{\eta_2}{2}\ln\zeta\right)\right]\zeta^{-n}\\
&V_{4(\infty)}=\zeta^{3/4+\eta_1/2}\sum_{n=0}^{\infty}\left[\boldsymbol{T}_n^{(2)}\cos\left(\frac{\eta_2}{2}\ln\zeta\right)+\boldsymbol{T}_n^{(1)}\sin\left(\frac{\eta_2}{2}\ln\zeta\right)\right]\zeta^{-n}
\end{aligned}
\right\}\tag{2.52}
$$

式中

$$
\left.
\begin{aligned}
&D_0^{(1)}=1,\ D_0^{(2)}=0\\
&D_n^{(1)}=(n^2-7n/2+2)D_{n-1}^{(1)}+\sqrt{k-1}\,D_{n-1}^{(2)}/2\qquad(n\geqslant1)\\
&D_n^{(2)}=(n^2-7n/2+2)D_{n-1}^{(2)}-\sqrt{k-1}\,D_{n-1}^{(1)}/2\qquad(n\geqslant1)
\end{aligned}
\right\}\tag{2.53}
$$

$$\left.\begin{aligned}&\bar{D}_0^{(1)}=1,\quad \bar{D}_0^{(2)}=0\\&\bar{D}_n^{(1)}=(n^2-5n/2+1/2)\bar{D}_{n-1}^{(1)}+\sqrt{k-1}\,\bar{D}_{n-1}^{(2)}/2\qquad(n\geqslant1)\\&\bar{D}_n^{(2)}=(n^2-5n/2+1/2)\bar{D}_{n-1}^{(2)}-\sqrt{k-1}\,\bar{D}_{n-1}^{(1)}/2\qquad(n\geqslant1)\end{aligned}\right\}\tag{2.54}$$

$$\left.\begin{aligned}&\tilde{D}_0^{(1)}=1,\ \tilde{D}_0^{(2)}=0\\&\tilde{D}_n^{(1)}=(n^2+n/2-1)\tilde{D}_{n-1}^{(1)}+\sqrt{k-1}\,\tilde{D}_{n-1}^{(2)}/2\qquad(n\geqslant1)\\&\tilde{D}_n^{(2)}=(n^2+n/2-1)\tilde{D}_{n-1}^{(2)}-\sqrt{k-1}\,\tilde{D}_{n-1}^{(1)}/2\qquad(n\geqslant1)\end{aligned}\right\}\tag{2.55}$$

$$\left.\begin{aligned}E_n^{(1)}=&\ \tilde{D}_n^{(1)}\sum_{m=1}^{n}\left[\frac{(4m+1)(2m^2+m-2)}{(2m^2+m-2)^2+k-1}-\frac{2(m+1)}{m(m+2)}\right]\\&-\tilde{D}_n^{(2)}\sum_{m=1}^{n}\frac{2\sqrt{k-1}}{(2m^2+m-2)^2+k-1}\\E_n^{(2)}=&\ \tilde{D}_n^{(2)}\sum_{m=1}^{n}\left[\frac{(4m+1)(2m^2+m-2)}{(2m^2+m-2)^2+k-1}-\frac{2(m+1)}{m(m+2)}\right]\\&+\tilde{D}_n^{(1)}\sum_{m=1}^{n}\frac{2\sqrt{k-1}}{(2m^2+m-2)^2+k-1}\end{aligned}\right\}\tag{2.56}$$

$$G_1^{(1)}=\frac{8}{k+3},\quad G_1^{(2)}=\frac{4\sqrt{k-1}}{k+3},\quad G_2^{(1)}=\frac{8(3-k)}{k(k+3)},\quad G_2^{(2)}=\frac{24\sqrt{k-1}}{k(k+3)}\tag{2.57}$$

$$\left.\begin{aligned}&T_0^{(1)}=1,\quad T_0^{(2)}=0\\&T_n^{(1)}=\frac{1}{n[(n+\eta_1)^2+\eta_2^2]}\left\langle\left\{\left[\left(n-\frac{7}{4}+\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}+\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right.\right.\\&\qquad\left.\cdot(n+\eta_1)+\frac{\eta_2^2}{2}(2n-3+\eta_1)\right\}T_{n-1}^{(1)}-\eta_2\left\{\frac{1}{2}(n+\eta_1)(2n-3+\eta_1)\right.\\&\qquad\left.\left.-\left[\left(n-\frac{7}{4}+\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}+\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right\}T_{n-1}^{(2)}\right\rangle\qquad(n\geqslant1)\\&T_n^{(2)}=\frac{1}{n[(n+\eta_1)^2+\eta_2^2]}\left\langle\left\{\left[\left(n-\frac{7}{4}+\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}+\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right.\right.\\&\qquad\left.\cdot(n+\eta_1)+\frac{\eta_2^2}{2}(2n-3+\eta_1)\right\}T_{n-1}^{(2)}+\eta_2\left\{\frac{1}{2}(n+\eta_1)(2n-3+\eta_1)\right.\\&\qquad\left.\left.-\left[\left(n-\frac{7}{4}+\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}+\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right\}T_{n-1}^{(1)}\right\rangle\qquad(n\geqslant1)\end{aligned}\right\}\tag{2.58}$$

$$\left.\begin{aligned}T_n^{(1)}=&\frac{1}{n[(n-\eta_1)^2+\eta_2^2]}\left\langle\left\{\left[\left(n-\frac{7}{4}-\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}-\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right.\right.\\&\left.\cdot(n-\eta_1)+\frac{\eta_2^2}{2}(2n-3-\eta_1)\right\}T_{n-1}^{(1)}+\eta_2\left\{\frac{1}{2}(n-\eta_1)(2n-3-\eta_1)\right.\\&\left.\left.-\left[\left(n-\frac{7}{4}-\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}-\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right\}T_{n-1}^{(2)}\right\rangle\qquad(n\geqslant1)\\T_n^{(2)}=&\frac{1}{n[(n-\eta_1)^2+\eta_2^2]}\left\langle\left\{\left[\left(n-\frac{7}{4}-\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}-\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right.\right.\\&\left.\cdot(n-\eta_1)+\frac{\eta_2^2}{2}(2n-3-\eta_1)\right\}T_{n-1}^{(2)}-\eta_2\left\{\frac{1}{2}(n-\eta_1)(2n-3-\eta_1)\right.\\&\left.\left.-\left[\left(n-\frac{7}{4}-\frac{\eta_1}{2}\right)\left(n-\frac{5}{4}-\frac{\eta_1}{2}\right)-\frac{\eta_2^2}{4}\right]\right\}T_{n-1}^{(1)}\right\rangle\qquad(n\geqslant1)\end{aligned}\right\}\tag{2.59}$$

(ii)　$k<1$

由式(2.15)可得如下两个超几何方程：

$$\xi(1-\xi)\frac{d^2V}{d\xi^2}-(1-2\xi)\frac{dV}{d\xi}+2(1\pm\sqrt{1-k})V=0\tag{2.60a,b}$$

下面分两种情况来研究方程(2.60)的解。

(1)　$q^2-4pr>0$

比较式(2.60)与式(2.17)可见，在式(2.60a)中

$$\alpha_1=-3/2+\lambda_1,\ \beta_1=-3/2-\lambda_1,\ \gamma_1=-1\tag{2.61}$$

其中　$\lambda_1=\sqrt{17+8\sqrt{1-k}}/2$　(2.62)

而在式(2.60b)中

$$\alpha_2=-3/2+\lambda_2,\ \beta_2=-3/2-\lambda_2,\ \gamma_2=-1\tag{2.63}$$

其中　$\lambda_2=\sqrt{17-8\sqrt{1-k}}/2$　(2.64)

因为ξ为实量，所以方程(2.15)在奇点$\xi=0$，1和∞邻域的四个线性独立的实函数解为

$$\left.\begin{aligned}V_{1(0)}^{*}&=\xi^{1-\gamma_1}F(\alpha_1-\gamma_1+1,\ \beta_1-\gamma_1+1;\ 2-\gamma_1;\ \xi)\\&=\xi^2F(1/2+\lambda_1,\ 1/2-\lambda_1;\ 3;\ \xi)\\V_{2(0)}&=\xi^{1-\gamma_1}f(\alpha_1-\gamma_1+1,\ \beta_1-\gamma_1+1;\ 2-\gamma_1;\ \xi)\\&=\xi^2f(1/2+\lambda_1,\ 1/2-\lambda_1;\ 3;\ \xi)\\V_{3(0)}&=\xi^{1-\gamma_2}F(\alpha_2-\gamma_2+1,\ \beta_2-\gamma_2+1;\ 2-\gamma_2;\ \xi)\\&=\xi^2F(1/2+\lambda_2,\ 1/2-\lambda_2;\ 3;\ \xi)\\V_{4(0)}&=\xi^{1-\gamma_2}f(\alpha_2-\gamma_2+1,\ \beta_2-\gamma_2+1;\ 2-\gamma_2;\ \xi)\\&=\xi^2f(1/2+\lambda_2,\ 1/2-\lambda_2;\ 3;\ \xi)\\V_{1(1)}&=(1-\xi)^{\gamma_1-\alpha_1-\beta_1}F(\gamma_1-\beta_1,\ \gamma_1-\alpha_1;\ \gamma_1+1-\alpha_1-\beta_1;\ 1-\xi)\\&=(1-\xi)^2F(1/2+\lambda_1,\ 1/2-\lambda_1;\ 3;\ 1-\xi)\\V_{2(1)}&=(1-\xi)^{\gamma_1-\alpha_1-\beta_1}f(\gamma_1-\beta_1,\ \gamma_1-\alpha_1;\ \gamma_1+1-\alpha_1-\beta_1;\ 1-\xi)\\&=(1-\xi)^2f(1/2+\lambda_1,\ 1/2-\lambda_1;\ 3;\ 1-\xi)\\V_{3(1)}&=(1-\xi)^{\gamma_2-\alpha_2-\beta_2}F(\gamma_2-\beta_2,\ \gamma_2-\alpha_2;\ \gamma_2+1-\alpha_2-\beta_2;\ 1-\xi)\\&=(1-\xi)^2F(1/2+\lambda_2,\ 1/2-\lambda_2;\ 3;\ 1-\xi)\end{aligned}\right\}\tag{2.65}$$

$$\left.\begin{aligned}
V_{4(1)}&=(1-\xi)^{\gamma_2-\alpha_2-\beta_2}f(\gamma_2-\beta_2,\ \gamma_2-\alpha_2;\ \gamma_2+1-\alpha_2-\beta_2;\ 1-\xi)\\
&=(1-\xi)^2 f(1/2+\lambda_2,\ 1/2-\lambda_2;\ 3;\ 1-\xi)\\
V_{1(\infty)}&=\xi^{-\alpha_1}F(\alpha_1,\ \alpha_1-\gamma_1+1;\ \alpha_1-\beta_1+1;\ 1/\xi)\\
&=\xi^{3/2-\lambda_1}F(-3/2+\lambda_1,\ 1/2+\lambda_1;\ 1+2\lambda_1;\ 1/\xi)\\
V_{2(\infty)}&=\xi^{-\beta_1}F(\beta_1,\ \beta_1-\gamma_1+1;\ \beta_1-\alpha_1+1;\ 1/\xi)\\
&=\xi^{3/2+\lambda_1}F(-3/2-\lambda_1,\ 1/2-\lambda_1;\ 1-2\lambda_1;\ 1/\xi)\\
V_{3(\infty)}&=\xi^{-\alpha_2}F(\alpha_2,\ \alpha_2-\gamma_2+1;\ \alpha_2-\beta_2+1;\ 1/\xi)\\
&=\xi^{3/2-\lambda_2}F(-3/2+\lambda_2,\ 1/2+\lambda_2;\ 1+2\lambda_2;\ 1/\xi)\\
V_{4(\infty)}&=\xi^{-\beta_2}F(\beta_2,\ \beta_2-\gamma_2+1;\ \beta_2-\alpha_2+1;\ 1/\xi)\\
&=\xi^{3/2+\lambda_2}F(-3/2-\lambda_2,\ 1/2-\lambda_2;\ 1-2\lambda_2;\ 1/\xi)
\end{aligned}\right\}$$

（2） $q^2-4pr<0$

利用式(2.46)和(2.48)，可将方程(2.60)变换为

$$\zeta(1-\zeta)\frac{d^2V}{d\zeta^2}+\frac{1}{2}(1+\zeta)\frac{dV}{d\zeta}+\frac{1}{2}(1\pm\sqrt{1-k})V=0 \tag{2.66}$$

因此

$$\left.\begin{aligned}
V_{1(0)}&=F\left(-\frac{3}{4}+\frac{\lambda_1}{2},\ -\frac{3}{4}-\frac{\lambda_1}{2};\ \frac{1}{2};\ \zeta\right)\\
V_{2(0)}&=\zeta^{1/2}F\left(-\frac{1}{4}+\frac{\lambda_1}{2},\ -\frac{1}{4}-\frac{\lambda_1}{2};\ \frac{3}{2};\ \zeta\right)\\
V_{3(0)}&=F\left(-\frac{3}{4}+\frac{\lambda_2}{2},\ -\frac{3}{4}-\frac{\lambda_2}{2};\ \frac{1}{2};\ \zeta\right)\\
V_{4(0)}&=\zeta^{1/2}F\left(-\frac{1}{4}+\frac{\lambda_2}{2},\ -\frac{1}{4}-\frac{\lambda_2}{2};\ \frac{3}{2};\ \zeta\right)\\
V_{1(1)}&=(1-\zeta)^2F\left(\frac{5}{4}+\frac{\lambda_1}{2},\ \frac{5}{4}-\frac{\lambda_1}{2};\ 3;\ 1-\zeta\right)\\
V_{2(1)}&=(1-\zeta)^2f\left(\frac{5}{4}+\frac{\lambda_1}{2},\ \frac{5}{4}-\frac{\lambda_1}{2};\ 3;\ 1-\zeta\right)\\
V_{3(1)}&=(1-\zeta)^2F\left(\frac{5}{4}+\frac{\lambda_2}{2},\ \frac{5}{4}-\frac{\lambda_2}{2};\ 3;\ 1-\zeta\right)\\
V_{4(1)}&=(1-\zeta)^2f\left(\frac{5}{4}+\frac{\lambda_2}{2},\ \frac{5}{4}-\frac{\lambda_2}{2};\ 3;\ 1-\zeta\right)\\
V_{1(\infty)}&=\zeta^{3/4-\lambda_1/2}F\left(-\frac{3}{4}+\frac{\lambda_1}{2},\ -\frac{1}{4}+\frac{\lambda_1}{2};\ 1+\lambda_1;\ \frac{1}{\zeta}\right)\\
V_{2(\infty)}&=\zeta^{3/4+\lambda_1/2}F\left(-\frac{3}{4}-\frac{\lambda_1}{2},\ -\frac{1}{4}-\frac{\lambda_1}{2};\ 1-\lambda_1;\ \frac{1}{\zeta}\right)\\
V_{3(\infty)}&=\zeta^{3/4-\lambda_2/2}F\left(-\frac{3}{4}+\frac{\lambda_2}{2},\ -\frac{1}{4}+\frac{\lambda_2}{2};\ 1+\lambda_2;\ \frac{1}{\zeta}\right)\\
V_{4(\infty)}&=\zeta^{3/4-\lambda_2/2}F\left(-\frac{3}{4}-\frac{\lambda_2}{2},\ -\frac{1}{4}-\frac{\lambda_2}{2};\ 1-\lambda_2;\ \frac{1}{\zeta}\right)
\end{aligned}\right\} \tag{2.67}$$

(iii) $k=1$

当$k=1$时，方程(2.15)变为

$$L_1L_1(V)=0 \tag{2.68}$$

式中

$$L_1=\xi(1-\xi)\frac{d^2}{d\xi^2}-(1-2\xi)\frac{d}{d\xi}+2 \tag{2.69}$$

方程(2.68)为重超几何方程。文[3]给出了这种方程的解法。

这里也分两种情况来加以研究。

(1) $q^2-4pr>0$

令

$$L_1(V)=V^* \tag{2.70}$$

将式(2.70)代入式(2.68)得

$$L_1(V^*)=0 \tag{2.71}$$

显然，方程(2.68)的解由方程(2.71)的两个齐次解和方程(2.70)的两个特解所组成。因此，方程(2.68)在每个奇点邻域的四个线性独立的实函数解分别为[3]

$$\left.\begin{aligned}
V^*_{1(0)}&=\xi^2F\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ \xi\right)\\
V^*_{2(0)}&=\xi^2f\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ \xi\right)\\
V_{1(0)}&=\xi^2\phi\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ \xi\right)\\
V_{2(0)}&=\xi^2\varphi\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ \xi\right)\\
V^*_{1(1)}&=(1-\xi)^2F\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ 1-\xi\right)\\
V^*_{2(1)}&=(1-\xi)^2f\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ 1-\xi\right)\\
V_{1(1)}&=(1-\xi)^2\phi\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ 1-\xi\right)\\
V_{2(1)}&=(1-\xi)^2\varphi\left(\frac{1}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 3;\ 1-\xi\right)\\
V^*_{1(\infty)}&=\xi^{(3-\sqrt{17})/2}F\left(-\frac{3}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}+\frac{\sqrt{17}}{2};\ 1+\sqrt{17};\ \frac{1}{\xi}\right)\\
V^*_{2(\infty)}&=\xi^{(3+\sqrt{17})/2}F\left(-\frac{3}{2}-\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 1-\sqrt{17};\ \frac{1}{\xi}\right)\\
V_{1(\infty)}&=\xi^{(3-\sqrt{17})/2}\phi_1\left(-\frac{3}{2}+\frac{\sqrt{17}}{2},\ \frac{1}{2}+\frac{\sqrt{17}}{2};\ 1+\sqrt{17};\ \frac{1}{\xi}\right)\\
V_{2(\infty)}&=\xi^{(3+\sqrt{17})/2}\phi_1\left(-\frac{3}{2}-\frac{\sqrt{17}}{2},\ \frac{1}{2}-\frac{\sqrt{17}}{2};\ 1-\sqrt{17};\ \frac{1}{\xi}\right)
\end{aligned}\right\} \tag{2.72}$$

式中

$$\phi(a,\ b;\ c;\ z)=\sum_{n=1}^{\infty}D^*_nz^n \tag{2.73}$$

$$\varphi(a,\ b;\ c;\ z)=\phi(a,\ b;\ c;\ z)\ln z+\sum_{n=1}^{\infty}E_n^*z^n-\sum_{n=1}^{c-1}G_n^*z^{-n} \tag{2.74}$$

$$\phi_1(a,\ b;\ c;\ z)=\sum_{n=1}^{\infty}D_n^{**}z^n \tag{2.75}$$

而

$$D_n^*=\left(\sum_{j=1}^{n}(a)_{n,j}(b)_{n,j}\right)\Big/n!(c)_n \tag{2.76}$$

$$(\theta)_{n,j}=(\theta)_n/(\theta+j-1) \tag{2.77}$$

$$E_n^*=\sum_{j=1}^{n}\left[\frac{(a+1+j)_{n-j-1}(b+1+j)_{n-j-1}}{(c+j)_{n-j}(1+j)_{n-j}}(a+b+2j)-\frac{(a+j)_{n-j}(b+j)_{n-j}}{(c-1+j)_{n-j+1}(j)_{n-j+1}}(c-1+2j)\right]D_j^*+\sum_{j=1}^{n-1}\frac{(a+1+j)_{n-j-1}(b+1+j)_{n-j-1}}{(c+j)_{n-j}(1+j)_{n-j}}e_j \tag{2.78}$$

$$G_n^*=-\sum_{j=1}^{n}\frac{(c-n)_{n-j}(1-n)_{n-j}}{(a-n)_{n-j+1}(b-n)_{n-j+1}}g_j \tag{2.79}$$

$$D_n^{**}=-\frac{(a)_n(b)_n}{n!(c)_n}\sum_{j=1}^{n}\frac{1}{j(c-1+j)} \tag{2.80}$$

（2） $q^2-4pr<0$

利用式(2.46)和(2.48)可将方程(2.60)变换为

$$L_2L_2(V)=0 \tag{2.81}$$

式中 $$L_2=\zeta(1-\zeta)\frac{d^2}{d\zeta^2}+\frac{1}{2}(1+\zeta)\frac{d}{d\zeta}+\frac{1}{2} \tag{2.82}$$

设 $$L_2(V)=V^* \tag{2.83}$$

则 $$L_2(V^*)=0 \tag{2.84}$$

因此

$$\left.\begin{aligned}V_{1(0)}^*&=F\left(-\frac{3}{4}+\frac{\sqrt{17}}{4},\ -\frac{3}{4}-\frac{\sqrt{17}}{4};\ \frac{1}{2};\ \zeta\right)\\V_{2(0)}^*&=\zeta^{1/2}F\left(-\frac{1}{4}+\frac{\sqrt{17}}{4},\ -\frac{1}{4}-\frac{\sqrt{17}}{4};\ \frac{3}{2};\ \zeta\right)\\V_{1(0)}&=\phi\left(-\frac{3}{4}+\frac{\sqrt{17}}{4},\ -\frac{3}{4}-\frac{\sqrt{17}}{4};\ \frac{1}{2};\ \zeta\right)\\V_{2(0)}&=\zeta^{1/2}\phi\left(-\frac{1}{4}+\frac{\sqrt{17}}{4},\ -\frac{1}{4}-\frac{\sqrt{17}}{4};\ \frac{3}{2};\ \zeta\right)\end{aligned}\right\}$$

$$\left.\begin{aligned}
V^*_{1(1)}&=(1-\zeta)^2F\left(\frac{5}{4}+\frac{\sqrt{17}}{4},\ \frac{5}{4}-\frac{\sqrt{17}}{4};\ 3;\ 1-\zeta\right)\\
V^*_{2(1)}&=(1-\zeta)^2f\left(\frac{5}{4}+\frac{\sqrt{17}}{4},\ \frac{5}{4}-\frac{\sqrt{17}}{4};\ 3;\ 1-\zeta\right)\\
V_{1(1)}&=(1-\zeta)^2\phi\left(\frac{5}{4}+\frac{\sqrt{17}}{4},\ \frac{5}{4}-\frac{\sqrt{17}}{4};\ 3;\ 1-\zeta\right)\\
V_{2(1)}&=(1-\zeta)^2\varphi\left(\frac{5}{4}+\frac{\sqrt{17}}{4},\ \frac{5}{4}-\frac{\sqrt{17}}{4};\ 3;\ 1-\zeta\right)\\
V^*_{1(\infty)}&=\zeta^{(3-\sqrt{17})/4}\,F\left(-\frac{3}{4}+\frac{\sqrt{17}}{4},\ -\frac{1}{4}+\frac{\sqrt{17}}{4};\ 1+\frac{\sqrt{17}}{2};\ \frac{1}{\zeta}\right)\\
V^*_{2(\infty)}&=\zeta^{(3+\sqrt{17})/4}\,F\left(-\frac{3}{4}-\frac{\sqrt{17}}{4},\ -\frac{1}{4}-\frac{\sqrt{17}}{4};\ 1-\frac{\sqrt{17}}{2};\ \frac{1}{\zeta}\right)\\
V_{1(\infty)}&=\zeta^{(3-\sqrt{17})/4}\,\phi_1\left(-\frac{3}{4}+\frac{\sqrt{17}}{4},\ -\frac{1}{4}+\frac{\sqrt{17}}{4};\ 1+\frac{\sqrt{17}}{2};\ \frac{1}{\zeta}\right)\\
V_{2(\infty)}&=\zeta^{(3+\sqrt{17})/4}\,\phi_1\left(-\frac{3}{4}-\frac{\sqrt{17}}{4},\ -\frac{1}{4}-\frac{\sqrt{17}}{4};\ 1-\frac{\sqrt{17}}{2};\ \frac{1}{\zeta}\right)
\end{aligned}\right\}\tag{2.85}$$

三、$q^2-4pr=0$条件下微分方程的解

让我们来求解方程(1.26)。

1. 特解

方程(1.26)可以改写为

$$\bar{x}^4\frac{d^4V}{d\bar{x}^4}-6\bar{x}^2\frac{d^2V}{d\bar{x}^2}+12\bar{x}\frac{dV}{d\bar{x}}+4kV=G_2(\bar{x})\tag{3.1}$$

如果

$$G_2(\bar{x})=\sum_{n=m_1}^{m_2}b^*_n\bar{x}^n\tag{3.2}$$

其中m_1和m_2为整数或零，则方程(3.1)的特解为

$$V=\sum_{n=m_1}^{m_2}a^*_n\bar{x}^n\tag{3.3}$$

式中

$$a^*_n=\frac{b^*_n}{(n+1)n(n-3)(n-4)+4k}\tag{3.4}$$

2. 齐次解

与方程(1.26)相应的齐次方程为

$$\left[\bar{x}^2\frac{d^2}{d\bar{x}^2}-2\bar{x}\frac{d}{d\bar{x}}-2(1+\sqrt{1-k})\right]\left[\bar{x}^2\frac{d^2}{d\bar{x}^2}-2\bar{x}\frac{d}{d\bar{x}}-2(1-\sqrt{1-k})\right]V=0\tag{3.5}$$

根据k所取数值的不同，方程(3.5)的解有三种情况

(i) $k>1$

$$\left.\begin{array}{l} V_1=\tilde{x}^{3/2+\eta_1}\cos(\eta_2\ln\tilde{x}),\ V_2=\tilde{x}^{3/2+\eta_1}\sin(\eta_2\ln\tilde{x}) \\ V_3=\tilde{x}^{3/2-\eta_1}\cos(\eta_2\ln\tilde{x}),\ V_4=\tilde{x}^{3/2-\eta_1}\sin(\eta_2\ln\tilde{x}) \end{array}\right\} \tag{3.6}$$

式中，η_1和η_2见式(2.40)。

(ii) $k<1$

$$\left.\begin{array}{l} V_1=\tilde{x}^{3/2+\lambda_1},\ V_2=\tilde{x}^{3/2-\lambda_1} \\ V_3=\tilde{x}^{3/2+\lambda_2},\ V_4=\tilde{x}^{3/2-\lambda_2} \end{array}\right\} \tag{3.7}$$

式中λ_1和λ_2请见式(2.62)和(2.64)

(iii) $k=1$

$$\left.\begin{array}{l} V_1=\tilde{x}^{(3+\sqrt{17})/2},\quad V_2=\tilde{x}^{(3-\sqrt{17})/2} \\ V_3=\tilde{x}^{(3+\sqrt{17})/2}\ln\tilde{x},\ V_4=\tilde{x}^{(3-\sqrt{17})/2}\ln\tilde{x} \end{array}\right\} \tag{3.8}$$

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Axisymmetric Problems of Cylindrical Shells with Variable Wall Thickness

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Abstract

The purpose of this paper is to give the general solutions for axisymmetric cylindrical shells with parabolically varying wall thickness.