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在复合载荷作用下变厚度圆薄板大挠度问题*

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摘 要

本文用变厚度板壳大挠度理论的修正迭代法[1]，对周边固定，在复合载荷下的变厚度圆薄板进行了求解，从而得到了精确度较高的二次近似解析解。将本文的结果退化到特殊情况就可以得到和文[1、2]完全一致的结果。本文还绘出特征曲线进行比较，其结果是理想的。

一、前 言

变厚度圆薄板大挠度问题是一个较复杂的非线性问题，精确解的寻求是困难的。文[3]用动力松驰法研究过锥形圆薄板大挠度问题，文[5]用文[4]提出的方法研究过变厚度扁球壳的非线性稳定问题，文[6]用文[7]采用过的双参数摄动法研究过变厚度圆薄板大挠度问题，文[2]用文[1]提出的变厚度圆薄板大挠度的修正迭代法研究过线性变厚度圆板的非线性问题。

文[1]提出的变厚度板壳大挠度理论的修正迭代法，程序简单易用，方便上机，优越于多参数摄动法和文[4]提出的方法。为了工程需要，本文对线性变厚度圆薄板，周边为固定、在均布载荷和集中载荷共同作用下的大挠度问题，用文[1]的方法得出了二次近似解析解并同均布载荷、集中载荷单独作用下的大挠度解进行了比较，本文结果是理想的，可供工程中设计参考应用。

二、基本方程和边界条件

设变厚度圆薄板半径为a，厚度为h，横向挠度为w，中面薄膜张力为N_r、N_θ，则变厚度圆薄板在复合载荷下大挠度方程为[4]、[8]：

$$Dr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dw}{dr}+r\frac{dD}{dr}\left(\frac{d^2w}{dr^2}+\frac{\mu}{h}\frac{dw}{dr}\right)$$

* 刘人怀推荐。

$$=\frac{1}{2}qr^2+\frac{P}{2\pi r}+rN_r\frac{dw}{dr} \tag{2.1}$$

$$hr\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r^2N_r)=\left[r\frac{d(rN_r)}{dr}-\mu rN_r\right]\frac{dh}{dr}-\frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 \tag{2.2}$$

$$N_\theta=\frac{d}{dr}(rN_r) \tag{2.3}$$

其中　E 为杨氏模量，μ 为波松比，$D=\frac{Eh^3}{12(1-\mu^2)}$.

假设圆板厚度变化规律为：

$$h(r)=h_0f(r)=h_0\left(1+\varepsilon_1\frac{r}{a}+\varepsilon_2\frac{r^2}{a^2}+\varepsilon_3\frac{r^3}{a^3}+\cdots\right) \tag{2.4}$$

其中 h_0 为板中心厚度，ε_1，ε_2，$\varepsilon_3\cdots$ 为变厚度小参数。

本文将完成线性变厚度圆薄板大挠度问题的求解，即取 $\varepsilon=\varepsilon_1\neq0$，$\varepsilon_2=\varepsilon_3=\cdots=0$，那么对不同的 ε 值沿板的直径变化规律如图 1 所示

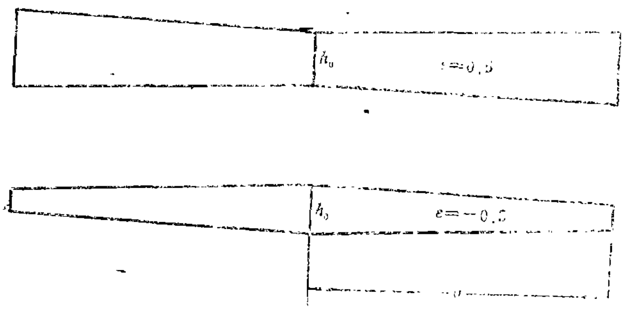

图　1

寻常的边界条件有：

（1）　周边固定夹紧

当 $r=a$；$w=0$，$\frac{dw}{dr}=0$，

$$\frac{d}{dr}(rN_r)-\mu N_r=0 \tag{2.5a,b,c}$$

当 $r=0$；$\frac{dw}{dr}$，N_r 有限　　(2.6a,b)

（2）　周边可移夹紧

当 $r=a$；$w=0$，$\frac{dw}{dr}=0$，$N_r=0$　　(2.7a,b,c)

当 $r=0$；$\frac{dw}{dr}$，N_r 有限　　(2.8a,b)

（3）　周边简支

当 $r=a$；$w=0$，$\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}=0$，$N_r=0$　　(2.9a,b,c)

当 $r=0$；$\frac{dw}{dr}$，N_r 有限　　(2.10a,b)

（4）　周边铰支

当 $r=a$；$w=0$，$\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}=0$，$\frac{d(rN_r)}{dr}-\mu N_r=0$　　(2.11a,b,c)

当 $r=0$；$\frac{dw}{dr}$，N_r 有限 (2.12a,b,c)

为了计算方便，引入下列无量纲量

$$x=\frac{r}{a},\quad y=\frac{w}{h_0},\quad \varphi=\frac{dy}{dx},\quad S=\frac{12(1-\mu^2)rN_r a}{Eh_0^3}$$

$$\beta_1 Q=\frac{6a^4(1-\mu^2)q}{Eh_0^4},\qquad \beta_2 Q=\frac{6a^2(1-\mu^2)P}{\pi Eh_0^4}$$

将这些量代入方程(2.1)、(2.2)就得到变厚度圆薄板在复合载荷作用下的无量纲大挠度方程

$$(1+\varepsilon x)^3L(x\varphi)=Q(\beta_1x^2+\beta_2)+S\varphi-3\varepsilon(1+\varepsilon x)^2\left(x\frac{d\varphi}{dx}+\mu\varphi\right) \tag{2.13}$$

$$L(xS)=\frac{\varepsilon\left(x\frac{dS}{dx}-\mu S\right)}{1+\varepsilon x}-\alpha(1+\varepsilon x)\varphi^2 \tag{2.14}$$

其中算子

$$L=x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}$$

而 $\alpha=6(1-\mu^2)$。

由无量纲量表示的边界条件为

（1） 周边固定夹紧

当 $x=1$；$y=0$，$\varphi=0$，$\frac{dS}{dx}-\mu S=0$ (2.15a,b,c)

当 $x=0$；φ，S 有限 (2.16a,b)

（2） 周边可移夹紧

当 $x=1$；$y=0$，$\varphi=0$，$S=0$ (2.17a,b,c)

当 $x=0$；φ，S 有限 (2.18a,b)

（3） 周边简支

当 $x=1$；$y=0$，$\frac{d\varphi}{dx}+\mu\varphi=0$　　$S=0$ (2.19a,b,c)

当 $x=0$；φ，S 有限 (2.20a,b)

（4） 周边铰支

当 $x=1$；$y=0$，$\frac{d\varphi}{dx}+\mu\varphi=0$，$\frac{dS}{dx}-\mu S=0$ (2.21a,b,c)

当 $x=0$；φ，S 有限 (2.22a,b)

三、边值问题的修正迭代解

本文只求在周边固定夹紧条件下的解，即在边界条件(2.15a,b,c)和(2.16a,b)求解方程组(2.13)和(2.14)。

由文[1]，本文迭代程序为

$$L(x\varphi_1)=Q(\beta_1x^2+\beta_2)$$

$$L(xS_1)=-\alpha(1+\varepsilon x)\varphi_1^2$$

$$L(x\varphi_2)=Q(\beta_1x^2+\beta_2)+S_1\varphi_1-3\varepsilon xL(x\varphi_1)-3\varepsilon^2x^2L(x\varphi_1)$$

$$-\varepsilon^3x^3L(x\varphi_1)-3\varepsilon(1+\varepsilon x)^2\left(x\frac{d\varphi_1}{dx}+\mu\varphi_1\right)$$

$$L(xS_2)=\frac{\varepsilon\left(x\frac{dS_1}{dx}-\mu S_1\right)}{1+\varepsilon x}-\alpha(1+\varepsilon x)\varphi_2^2$$

$$\cdots\qquad\cdots\qquad\cdots\qquad\cdots$$

$$L(x\varphi_{n+1})=Q(\beta_1x^2+\beta_2)+S_n\varphi_n-3\varepsilon xL(x\varphi_n)-3\varepsilon^2x^2L(x\varphi_n)$$

$$-\varepsilon^3x^3L(x\varphi_n)-3\varepsilon(1+\varepsilon x)^2\left(x\frac{d\varphi_n}{dx}+\mu\varphi_n\right)$$

$$L(xS_{n+1})=\frac{\varepsilon\left(x\frac{dS_n}{dx}-\mu S_n\right)}{1+\varepsilon x}-\alpha(1+\varepsilon x)\varphi_{n+1}^2$$

对于 φ_1 有下面边值问题

$$L(x\varphi_1)=Q(\beta_1x^2+\beta_2) \tag{3.1}$$

当 $x=1$；$y_1=0$，$\varphi_1=0$ (3.2a,b)

当 $x=0$；$y_1=y_0$，φ_1 有限 (3.3a,b)

其中 y_0 为圆薄板中心无量纲挠度

解得：

$$\varphi_1=\frac{\beta_1Q(x^3-x)}{8}+\frac{1}{2}\beta_2Qx\ln x \tag{3.4}$$

$$y_1=\frac{1}{32}\beta_1Q(x^4-2x^2+1)+\frac{1}{8}\beta_2Q(2x^2\ln x-x^2+1) \tag{3.5}$$

当 $x=0$ 时，由(3.3a)可得

$$Q=\frac{32}{\beta_1+4\beta_2}y_0 \tag{3.6}$$

将(3.6)代入(3.4)可得

$$\varphi_1=[\alpha_1(x^3-x)+\alpha_2x\ln x]y_0 \tag{3.7}$$

对于 S_1 有下面的边值问题

$$L(xS_1)=-\alpha(1+\varepsilon x)\varphi_1^2 \tag{3.8}$$

当 $x=1$，$\frac{dS_1}{dx}-\mu S_1=0$ (3.9)

当 $x=0$；S_1 有限 (3.10)

解此边值问题得：

$$S_1=-\frac{1}{3}\alpha y_0^2\left\{\frac{1}{16}\alpha_1^2x^7-\frac{1}{4}\alpha_1\left(\alpha_1+\frac{5}{12}\alpha_2\right)x^5+\frac{3}{8}\left(\alpha_1^2+\frac{3}{2}\alpha_1\alpha_2+\frac{7}{8}\alpha_2^2\right)x^3\right.$$

$$\left.+\frac{1}{4}\alpha_1\alpha_2x^5\ln x-\frac{3}{8}\alpha_2\left(2\alpha_1+\frac{3}{2}\alpha_2\right)x^3\ln x+\frac{3}{8}\alpha_2^2x^3\ln^2x+f_1x\right\}$$

$$-\frac{16}{315}\alpha\varepsilon y_0^2\left\{\frac{5}{16}\alpha_1^2x^8-\frac{9}{8}\alpha_1\left(\alpha_1+\frac{12}{35}\alpha_2\right)x^6+\frac{21}{16}\left(\alpha_1^2+\frac{16}{15}\alpha_1\alpha_2+\frac{98}{225}\alpha_2^2\right)x^4\right.$$

$$\left.+\frac{9}{8}\alpha_1\alpha_2x^6\ln x-\frac{21}{16}\alpha_2\left(2\alpha_1+\frac{16}{15}\alpha_2\right)x^4\ln x+\frac{21}{16}\alpha_2^2x^4\ln^2x+f_2x\right\} \tag{3.11}$$

其中：

$$f_1=\frac{(\mu-7)\alpha_1^2}{16(1-\mu)}-\frac{(\mu-5)\alpha_1}{4(1-\mu)}\left(\alpha_1+\frac{5}{12}\alpha_2\right)+\frac{3(\mu-3)}{8(1-\mu)}\left(\alpha_1^2+\frac{3}{2}\alpha_1\alpha_2+\frac{7}{8}\alpha_2^2\right)$$

$$-\frac{\alpha_1\alpha_2}{4(1-\mu)}+\frac{3\alpha_2}{8(1-\mu)}\left(2\alpha_1+\frac{3}{2}\alpha_2\right)$$

$$f_2=\frac{5(\mu-8)\alpha_1^2}{16(1-\mu)}-\frac{9(\mu-6)\alpha_1}{8(1-\mu)}\left(\alpha_1+\frac{12}{35}\alpha_2\right)+\frac{21(\mu-4)}{16(1-\mu)}\left(\alpha_1^2+\frac{16}{15}\alpha_1\alpha_2\right.$$

$$\left.+\frac{98}{225}\alpha_2^2\right)-\frac{9\alpha_1\alpha_2}{8(1-\mu)}+\frac{21\alpha_2}{16(1-\mu)}\left(2\alpha_1+\frac{16}{15}\alpha_2\right)$$

对 φ_2 有下面边值问题

$$L(x\varphi_2)=Q(\beta_1x^2+\beta_2)+S_1\varphi_1-(3\varepsilon x+3\varepsilon^2x^2+\varepsilon^3x^3)L(x\varphi_1)$$
$$-3\varepsilon(1+\varepsilon x)^2\left(x\frac{d\varphi_1}{dx}+\mu\varphi_1\right) \tag{3.12}$$

当 $x=1$; $y_2=0$, $\varphi_2=0$ (3.13a,b)

当 $x=0$; $y_2=y_0$, φ_2 有限 (3.14a,b)

其中：

$$S_1\varphi_1=-\frac{1}{3}\alpha y_0^3[\alpha_1l_{17}x^{10}-\alpha_1(l_{17}+l_{16})x^8+\alpha_1(l_{15}+l_{13})x^6-\alpha_1(l_{13}-f_1)x^4$$
$$-\alpha_1f_1x^2+(\alpha_1l_{26}+\alpha_2l_{17})x^8\ln x+(\alpha_1l_{33}+\alpha_2l_{26})x^6\ln^2x$$
$$-(\alpha_1l_{23}+\alpha_1l_{26}+\alpha_2l_{15})x^6\ln x-(\alpha_1l_{33}+l_{23}\alpha_2)x^4\ln^2x$$
$$+(\alpha_1l_{23}+\alpha_2l_{13})x^4\ln x+\alpha_2l_{33}x^4\ln^3x+\alpha_2f_1x^2\ln x]-\frac{16}{315}\alpha\varepsilon y_0^3[\alpha_1m_{18}x^{11}$$
$$-\alpha_1(m_{18}+m_{16})x^9+\alpha_1(m_{16}+m_{14})x^7-\alpha m_{14}x^5+\alpha_1f_2x^4-\alpha_1f_2x^2$$
$$+(\alpha_1m_{26}+\alpha_2m_{18})x^9\ln x-(\alpha_1m_{24}+\alpha_1m_{26}+\alpha_2m_{16})x^7\ln x+(\alpha_1m_{24}$$
$$+\alpha_2m_{14})x^5\ln x+\alpha_2f_2x^2\ln x+(\alpha_1m_{34}+\alpha_2m_{26})x^7\ln^2x-(\alpha_1m_{34}$$
$$+\alpha_2m_{24})x^5\ln^2x+\alpha_2m_{34}x^5\ln^3x]$$

解此边值问题可得

$$\varphi_2=\frac{1}{8}Q\beta_1(x^3-x)+\frac{1}{2}Q\beta_2x\ln x-\eta\varepsilon y_0[\beta_2(x^2-x)+\frac{3}{8}\varepsilon\beta_2(x^3-x)$$

$$+\frac{1}{15}(3\beta_1+\varepsilon^2\beta_2)(x^4-x)+\frac{1}{8}\varepsilon\beta_1(x^5-x)+\frac{1}{35}\varepsilon^2\beta_1(x^6-x)]$$

$$-3\varepsilon y_0\Big\{(3+\mu)\Big[\frac{1}{15}\alpha_1(x^4-x)+\frac{1}{12}\varepsilon\alpha_1(x^5-x)+\frac{1}{35}\varepsilon^2\alpha_1(x^6-x)\Big]$$

$$-(1+\mu)\Big[\frac{1}{3}\Big(\alpha_1+\frac{4}{3}\alpha_2\Big)(x^2-x)+\frac{1}{4}\varepsilon\Big(\alpha_1+\frac{3}{4}\alpha_2\Big)(x^3-x)$$

$$+\frac{1}{15}\varepsilon^2\Big(\alpha_1+\frac{8}{15}\alpha_2\Big)(x^4-x)-\frac{1}{15}\varepsilon^2\alpha_2x^4\ln x-\frac{1}{4}\varepsilon\alpha_2x^3\ln x-\frac{1}{3}\alpha_2x^2\ln x\Big]$$

$$+\frac{1}{15}\varepsilon^2\alpha_2(x^4-x)+\frac{1}{4}\varepsilon\alpha_2(x^3-x)+\frac{1}{3}\alpha_2(x^2-x)\Big\}-\frac{1}{3}\alpha y_0^3\ \{g_{110}(x^{11}-x)$$

$$-g_{90}(x^9-x)+g_{70}(x^7-x)-g_{50}(x^5-x)-g_{30}(x^3-x)+g_{91}x^9\ln x$$
$$-g_{71}x^7\ln x+g_{51}x^5\ln x+g_{31}x^3\ln x+g_{72}x^7\ln^2x-g_{52}x^5\ln^2x+g_{53}x^5\ln^3x\}$$

$$-\frac{16}{315}\varepsilon\alpha y_0^3\{h_{120}(x^{12}-x)-h_{100}(x^{10}-x)+h_{80}(x^8-x)-h_{60}(x^6-x)$$

$$+h_{50}(x^5-x)-h_{30}(x^3-x)+h_{101}x^{10}\ln x-h_{81}x^8\ln x+h_{61}x^6\ln x$$
$$+h_{31}x^3\ln x+h_{82}x^8\ln^2x-h_{62}x^6\ln^2x+h_{63}x^6\ln^3x\} \qquad (3.15)$$

其中

$$\eta=\frac{32}{\beta_1+4\beta_2},\ l_{17}=\frac{1}{16}\alpha_1,\ l_{15}=\frac{1}{4}\alpha_1\Big(\alpha_1+\frac{5}{12}\alpha_2\Big)$$

$$l_{13}=\frac{3}{8}\Big(\alpha_1^2+\frac{3}{2}\alpha_1\alpha_2+\frac{7}{8}\alpha_2^2\Big),\ l_{25}=\frac{1}{4}\alpha_1\alpha_2,\ l_{23}=\frac{3}{8}\alpha_2\Big(2\alpha_1+\frac{3}{2}\alpha_2\Big)$$

$$l_{33}=\frac{3}{8}\alpha_2^2,\ m_{18}=\frac{5}{16}\alpha_1^2,\ m_{16}=\frac{9}{8}\alpha_1\Big(\alpha_1+\frac{12}{35}\alpha_2\Big)$$

$$m_{14}=\frac{21}{16}\Big(\alpha_1^2+\frac{16}{15}\alpha_1\alpha_2+\frac{98}{225}\alpha_2^2\Big),\ m_{26}=\frac{9}{8}\alpha_1\alpha_2$$

$$m_{24}=\frac{21}{16}\alpha_2\Big(2\alpha_1+\frac{16}{15}\alpha_2\Big),\ m_{34}=\frac{21}{16}\alpha_2^2$$

$$g_{110}=\frac{1}{120}\alpha_1l_{17},\quad g_{90}=\frac{1}{80}\alpha_1(l_{17}+l_{15})+\frac{9}{3200}(\alpha_1l_{25}+\alpha_2l_{17})$$

$$g_{70}=\frac{1}{48}\alpha_1(l_{15}+l_{13})+\frac{7}{1152}(\alpha_1l_{25}+\alpha_2l_{15})+\frac{37}{13824}(\alpha_1l_{33}+\alpha_2l_{25})$$

$$g_{50}=\frac{1}{24}\alpha_1(l_{13}-f_1)+\frac{5}{288}(\alpha_1l_{23}+\alpha_2l_{13})+\frac{19}{1728}(\alpha_1l_{33}+\alpha_2l_{23})+\frac{65}{6912}\alpha_2l_{33}$$

$$g_{30}=\frac{1}{8}\Big(\alpha_1+\frac{3}{4}\alpha_2\Big)f_1,\quad g_{91}=\frac{1}{80}(\alpha_1l_{25}+\alpha_2l_{17})$$

$$g_{71}=\frac{1}{48}(\alpha_1 l_{23}+\alpha_1 l_{25}+\alpha_2 l_{15})+\frac{7}{576}(\alpha_1 l_{33}+\alpha_2 l_{25})$$

$$g_{51}=\frac{1}{24}(\alpha_1 l_{23}+\alpha_2 l_{13})+\frac{5}{144}(\alpha_1 l_{23}+\alpha_2 l_{33})+\frac{19}{576}\alpha_1 l_{33}$$

$$g_{31}=\frac{1}{8}\alpha_2 f_1,\quad g_{72}=\frac{1}{48}(\alpha_1 l_{33}+\alpha_2 l_{25})$$

$$g_{52}=\frac{1}{24}(\alpha_1 l_{33}+\alpha_2 l_{23})+\frac{5}{96}\alpha_2 l_{33}$$

$$g_{53}=\frac{1}{24}\alpha_2 l_{33},\quad h_{120}=\frac{1}{143}\alpha_1 m_{18}$$

$$h_{100}=\frac{1}{99}\alpha_1(m_{18}+m_{16})+\frac{20}{9801}(\alpha_1 m_{26}+\alpha_2 m_{18})$$

$$h_{80}=\frac{1}{63}\alpha_1(m_{16}+m_{14})+\frac{16}{3969}(\alpha_1 m_{24}+\alpha_1 m_{26}+\alpha_2 m_{16})+\frac{386}{250047}(\alpha_1 m_{34}+\alpha_2 m_{26})$$

$$h_{60}=\frac{1}{35}\alpha_1 m_{14}+\frac{12}{1225}(\alpha_1 m_{24}+\alpha_2 m_{14})+\frac{218}{42875}(\alpha_1 m_{34}+\alpha_2 m_{24})+\frac{5328}{1500625}\alpha_2 m_{34}$$

$$h_{50}=\frac{1}{24}\alpha_1 f_2,\quad h_{30}=\frac{1}{8}\left(\alpha_1+\frac{3}{4}\alpha_2\right)f_2$$

$$h_{101}=\frac{1}{99}(\alpha_1 m_{26}+\alpha_2 m_{18})$$

$$h_{81}=\frac{1}{63}(\alpha_1 m_{24}+\alpha_1 m_{26}+\alpha_2 m_{16})+\frac{32}{3969}(\alpha_1 m_{34}+\alpha_2 m_{26})$$

$$h_{61}=\frac{1}{35}(\alpha_1 m_{24}+\alpha_2 m_{14})+\frac{24}{1225}(\alpha_1 m_{34}+\alpha_2 m_{24})+\frac{654}{42875}\alpha_2 m_{34}$$

$$h_{31}=\frac{1}{8}\alpha_2 f_2,\quad h_{82}=\frac{1}{63}(\alpha_1 m_{34}+\alpha_2 m_{26})$$

$$h_{62}=\frac{1}{35}(\alpha_1 m_{34}+\alpha_2 m_{24})+\frac{36}{1225}\alpha_2 m_{34},\quad h_{63}=\frac{1}{35}\alpha_2 m_{34}$$

于是

$$y=\frac{1}{32}\beta_1 Q(x^4-2x^2+1)+\frac{1}{8}\beta_2 Q(2x^2\ln x-x^2+1)-\eta\varepsilon y_0\Big[\frac{1}{6}\beta_2(2x^3-3x^2+1)$$

$$+\frac{3}{32}\varepsilon\beta_2(x^4-2x^2+1)+\frac{1}{150}(3\beta_1+\varepsilon^2\beta_2)(2x^5-5x^2+3)+\frac{1}{48}\varepsilon\beta_1(x^6-3x^2+2)$$

$$+\frac{1}{490}\varepsilon^2\beta_1(2x^7-7x^2+5)\Big]-3\varepsilon y_0\Big\{(3+\mu)\Big[\frac{1}{150}\alpha_1(2x^5-5x^2+3)$$

$$+\frac{1}{72}\varepsilon\alpha_1(x^6-3x^2+2)+\frac{1}{490}\varepsilon^2\alpha_1(2x^7-7x^2+5)\Big]$$

$$-(1+\mu)\Big[\frac{1}{54}(3\alpha_1+4\alpha_2)(2x^3-3x^2+1)$$

$$+\frac{1}{64}\varepsilon(4\alpha_1+3\alpha_2)(x^4-2x^2+1)+\frac{1}{2250}\varepsilon^2(15\alpha_1+8\alpha_2)(2x^5-5x^2+3)$$

$$-\frac{1}{375}\varepsilon^2\alpha_2(5x^5\ln x-x^5+1)-\frac{1}{64}\varepsilon\alpha_2(4x^4\ln x-x^4+1)$$

$$-\frac{1}{27}\alpha_2(3x^3\ln x-x^3+1)\Big]+\frac{1}{150}\alpha_2\varepsilon^2(2x^5-5x^2+3)+\frac{1}{16}\varepsilon\alpha_2(x^4-2x^2+1)$$

$$+\frac{1}{18}\alpha_2(2x^3-3x^2+1)\Big\}-\frac{1}{3}\alpha y_0^3\Big\{\frac{1}{12}g_{110}(x^{12}-6x^2+5)$$

$$-\frac{1}{10}g_{90}(x^{10}-5x^2+4)+\frac{1}{8}g_{10}(x^8-4x^2+3)-\frac{1}{6}g_{50}(x^6-3x^2+2)$$

$$-\frac{1}{4}g_{30}(x^4-2x^2+1)+\frac{1}{100}g_{91}(10x^{10}\ln x-x^{10}+1)-\frac{1}{64}g_{71}(8x^8\ln x-x^8+1)$$

$$+\frac{1}{36}g_{51}(6x^6\ln x-x^6+1)+\frac{1}{16}g_{31}(4x^4\ln x-x^4+1)$$

$$+\frac{1}{512}g_{72}(64x^8\ln^2x-16x^8\ln x+2x^8-2)-\frac{1}{216}g_{62}(36x^6\ln^2x-12x^6\ln x$$

$$+2x^6-2)+\frac{1}{216}g_{63}(36x^6\ln^3x-18x^6\ln^2x+6x^6\ln x-x^6+1)\Big\}$$

$$-\frac{16}{315}\alpha\varepsilon y_0^3\Big\{\frac{1}{26}h_{120}(2x^{13}-13x^2+11)-\frac{1}{22}h_{100}(2x^{11}-11x^2+9)$$

$$+\frac{1}{18}h_{80}(2x^9-9x^2+7)-\frac{1}{14}h_{60}(2x^7-7x^2+5)+\frac{1}{6}h_{50}(x^6-3x^2+2)$$

$$-\frac{1}{4}h_{30}(x^4-2x^2+1)+\frac{1}{121}h_{101}(11x^{11}\ln x-x^{11}+1)-\frac{1}{81}h_{81}(9x^9\ln x-x^9+1)$$

$$+\frac{1}{49}h_{61}(7x^7\ln x-x^7+1)+\frac{1}{16}h_{31}(4x^4\ln x-x^4+1)+\frac{1}{729}h_{82}(81x^9\ln^2x-18x^9\ln x$$

$$+2x^9-2)-\frac{1}{343}h_{62}(49x^7\ln^2x-14x^7\ln x+2x^7-2)$$

$$+\frac{1}{2401}h_{63}(343x^7\ln^3x-147x^7\ln^2x+42x^7\ln x-6x^7+6)\Big\} \qquad (3.16)$$

令 $x=0$ 可得

$$Q=\frac{32}{\beta_1+4\beta_2}\Big\{\Big\{y_0+\eta\varepsilon y_0\Big[\frac{1}{6}\beta_2+\frac{3}{32}\varepsilon\beta_2+\frac{1}{50}(3\beta_1+\varepsilon^2\beta_2)+\frac{1}{24}\varepsilon\beta_1+\frac{1}{98}\varepsilon^2\beta_1\Big]$$

$$+3\varepsilon y_0\Big\{(3+\mu)\Big(\frac{1}{50}\alpha_1+\frac{1}{36}\varepsilon\alpha_1+\frac{1}{98}\varepsilon^2\alpha_1\Big)-(1+\mu)\Big[\frac{1}{54}(3\alpha_1+4\alpha_2)$$

$$+\frac{1}{64}\varepsilon(4\alpha_1+3\alpha_2)+\frac{1}{750}\varepsilon^2(15\alpha_1+8\alpha_2)-\frac{1}{375}\varepsilon^2\alpha_2-\frac{1}{64}\varepsilon\alpha_2-\frac{1}{27}\alpha_2\Big]$$

$$+\frac{1}{50}\varepsilon^2\alpha_2+\frac{1}{16}\varepsilon\alpha_2+\frac{1}{18}\alpha_2\Big\}+\frac{1}{3}\alpha y_0^3\Big(\frac{5}{12}g_{110}-\frac{2}{5}g_{90}+\frac{3}{8}g_{70}-\frac{1}{3}g_{50}$$

$$-\frac{1}{4}g_{30}+\frac{1}{100}g_{91}-\frac{1}{64}g_{71}+\frac{1}{36}g_{51}+\frac{1}{16}g_{31}-\frac{1}{256}g_{72}+\frac{1}{108}g_{62}+\frac{1}{216}g_{63}\Big)$$

$$+\frac{16}{315}\varepsilon\alpha y_0^3\Big[\frac{11}{26}h_{120}-\frac{9}{22}h_{100}+\frac{7}{18}h_{80}-\frac{5}{14}h_{60}+\frac{1}{3}h_{50}-\frac{1}{4}h_{30}$$

$$+\frac{1}{121}h_{101}-\frac{1}{81}h_{81}+\frac{1}{49}h_{61}+\frac{1}{16}h_{31}-\frac{2}{729}h_{82}+\frac{2}{343}h_{62}+\frac{6}{2401}h_{63}\Big]\Big\}\Big\} \qquad (3.17)$$

取 $\varepsilon=\frac{1}{3}$，$\mu=0.25$

则当 $\beta_1=\beta_2=1$ 时（复合载荷情况）

$$Q=10.53189y_0+3.31982y_0^3 \qquad (3.18)$$

当 $\beta_1=1$，$\beta_2=0$ 时（均布载荷情况）

$$Q=57.67346y_0+20.48929y_0^3 \qquad (3.19)$$

当 $\beta_1=0$，$\beta_2=1$ 时（中心集中载荷情况）

$$Q=12.85149y_0+4.07347y_0^3 \qquad (3.20)$$

式(3.19)、(3.20)分别与文[1]，文[2]结果相符。

图 2为式(3.18)、(3.19)、(3.20)的特征曲线。由图 2 可看出，载荷按同样的比例增加时，在均布载荷作用下中心挠度增大的慢，而在集中载荷作用下增加快，在复合载荷作用下其中心挠度不等于两个上述中心挠度的叠加，它的增加比上述两种载荷分别作用下的中心挠度要快。这些现象与物理意义完全吻合。

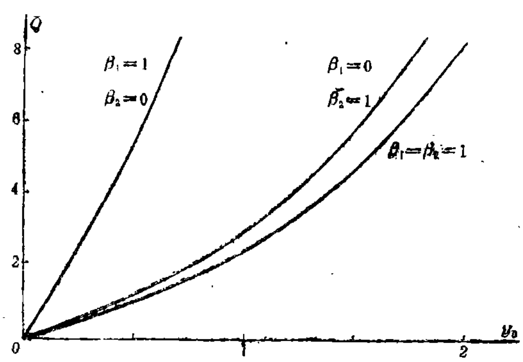

图 2

四、结　语

本文用变厚度板壳修正迭代法，对挠度求出了二次近似解析解，由本文(3.17)式不但可以得到变厚度圆薄板在均布载荷和集中载荷分别作用下的大挠度特征关系式，而且当取$\varepsilon=0$时还可以分别得到等厚度圆板在复合载荷，均布载荷和集中载荷作用下的大挠度特征关系式，其结果与用摄动法得到的等厚度圆板大挠度的特征关系式是完全一致的，为此本文结果更具有普遍性。

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Large Deflection Problem of Circular Plates with Variable Thickness under the Action of Combined Loads

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Abstract

By using the modified iteration method of large deflection theory of plates with variable thickness[1], we solve the problem of circular plates with variable thickness subjected to combined loads under the boundary conditions of the clamped edges and get comparatively more accurate second-order approximate analytical solution. If the results of this paper are degraded into the special cases, the results coinciding with those of papers [1,2] can be obtained. In this paper, the characteristic curves are plotted and some comparisons are made. The results of this paper are satisfactory.