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边缘有弹性点支矩形板的横向振动*

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摘　要

本文研究了两对边简支、另两对边任意支承的自由边有弹性点支的矩形板的横向振动问题，提供了一种求其固有频率和振型的高精度解法，自由边上弹性点支的个数及位置均可任意．本文用脉冲函数表示弹性点支的反力和力矩，利用 Fourier 级数将脉冲函数沿边缘展开，从而得到了满足全部边界条件的特征方程，可求得任意精度的任意阶固有频率及振型．

一、引　　言

在建筑、桥梁等工程中，一些结构往往可简化为边缘有弹性点支的矩形板，因而研究其横向振动很有实际意义．对于有点支的矩形板，由于点支处剪力或弯矩的不连续性，因而难以求得解析解，只能近似解之．早期对边缘有点支矩形板的研究主要集中于角点有刚性点支的情况[1]~[3]，多用差分法求解．直到近十年来，人们开始注重研究点支位于边缘任意位置的情况．Kerstens[4] 曾引进拉格朗日乘子来表示未知的点支反力，用修正的瑞雷-李兹法研究了边缘有任意多个刚性点支的矩形板，选择的基函数（梁函数）并不能满足板的振动微分方程和全部边界条件．Gorman[5],[6]用划分成子板的迭加法[7]研究了边缘有刚性对称点支矩形板的振动，对称性的要求限制了其应用范围．最近，Bapat[8]应用柔度函数法[9]研究了两对边简支、另两对边自由中点各有一刚性点支的矩形板，当点支位置任意或点支个数多于1个时，要选择合适的柔度函数是困难的．到目前为止，所有的这些研究还都限于刚性点支的情况，本文考虑的点支是弹性的，刚性点支是其特殊情况，因而具有一般性也更易于符合工程的实际情况．

二、微 分 方 程 及 其 解

考虑如图1所示的两对简支、另两对边任意支承的自由边有弹性点支的矩形板，为便于表述，称简支边为横边，称另两对边为纵边，图中a，b分别表示矩形板横边和纵边的长度．

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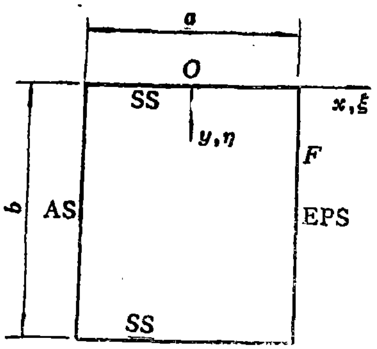

图1 两对边简支、自由边有弹性点支的矩形板

直角坐标系中，板横向振动的微分方程为

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=0 \tag{2.1}$$

式中，$w=w(x, y, t)$是板的横向位移，x，y为直角坐标，$-a/2\leqslant x\leqslant a/2$，$0\leqslant y\leqslant b$，$t$是时间，$h$是板厚度，$\rho$是材料密度，$D$是弯曲刚度，$D=Eh^3/12(1-\mu^2)$，$E$为杨氏弹性模量，$\mu$为泊松比。

如令

$$\xi=x/a,\quad \eta=y/b \tag{2.2}$$

$$w(x, y, t)=W(\xi, \eta)\sin\omega t \tag{2.3}$$

式中，$W(\xi, \eta)$为振型函数，ω为固有圆频率。将（2.2）、（2.3）代入（2.1）得到

$$\frac{\partial^4 W}{\partial \xi^4}+\frac{2}{\beta^2}\frac{\partial^4 W}{\partial \xi^2\partial \eta^2}+\frac{1}{\beta^4}\frac{\partial^4 W}{\partial \eta^4}-\frac{\lambda^4}{\beta^4}W=0 \tag{2.4}$$

式中，$\beta=a/b$是矩形板横纵边长的比率，$\lambda^2=\omega b^2\sqrt{\rho h/D}$是无量纲频率参数。

对于两对边简支矩形板，W有Levy 形式的解

$$W(\xi, \eta)=\sum_{m=1}^{\infty}X_m(\xi)\sin m\pi\eta \tag{2.5}$$

将（2.5）代入（2.4）得到

$$\frac{d^4X_m}{d\xi^4}-2\frac{\alpha_m^2}{\beta^2}\frac{d^2X_m}{d\xi^2}+\left[\frac{\alpha_m^4}{\beta^4}-\frac{\lambda^4}{\beta^4}\right]X_m=0 \qquad (m=1,2,3,\cdots,\infty) \tag{2.6}$$

式中，$\alpha_m=m\pi$，上式的解为

$$\left.\begin{aligned}X_m(\xi)&=A_m\sinh\lambda_{1m}\xi+B_m\cosh\lambda_{1m}\xi+C_m\sinh\lambda_{2m}\xi+D_m\cosh\lambda_{2m}\xi, & \lambda<\alpha_m\\ X_m(\xi)&=A_m\sin\lambda_{1m}\xi+B_m\cos\lambda_{1m}\xi+C_m\sinh\lambda_{2m}\xi+D_m\cosh\lambda_{2m}\xi, & \lambda>\alpha_m\end{aligned}\right\} \tag{2.7}$$

式中，

$$\lambda_{1m}^2=\frac{|\lambda^2-\alpha_m^2|}{\beta^2},\quad \lambda_{2m}=\frac{\lambda^2+\alpha_m^2}{\beta^2},$$

A_m，B_m，C_m，D_m为待定常数。

在ξ方向，矩形板的剪力和弯矩分别为

$$V_\xi(\xi, \eta)=-\frac{D}{a^3}\left[\frac{\partial^3 W}{\partial \xi^3}+\frac{2-\mu}{\beta^2}\frac{\partial^3 W}{\partial \eta^2\partial \xi}\right],\quad M_\xi(\xi, \eta)=-\frac{D}{a^2}\left[\frac{\partial^2 W}{\partial \xi^2}+\frac{\mu}{\beta^2}\frac{\partial^2 W}{\partial \eta^2}\right] \tag{2.8}$$

三、边 界 条 件

在自由边的点支处，点支反力和力矩是突变力，以$\xi=1/2$边为例，若边缘有r个点支，

则剪力和弯矩可分别表示为

$$V_{\xi}\left(\frac{1}{2},\ \eta\right)=\sum_{i=1}^{r}P_{i}\delta(\eta-\eta_{i}),\quad M_{\xi}\left(\frac{1}{2},\ \eta\right)=\sum_{i=1}^{r}T_{i}\delta(\eta-\eta_{i}) \tag{3.1}$$

式中，P_i，T_i是点支处的剪力和弯矩，$\delta(\eta-\eta_i)$为脉冲函数，η_i为第i个点支的纵坐标。

利用Fourier 级数，可将脉冲函数$\delta(\eta-\eta_i)$展成级数

$$\delta(\eta-\eta_i)=2\sum_{m=1}^{\infty}\sin m\pi\eta_i\sin m\pi\eta \tag{3.2}$$

将（3.1)、(3.2）代入（2.8）得到

$$\left.\begin{array}{ll}\left\{\begin{array}{l}Q_{1m}\left(\cosh\frac{1}{2}\lambda_{1m}A_m+\sinh\frac{1}{2}\lambda_{1m}B_m\right)+Q_{2m}\left(\cosh\frac{1}{2}\lambda_{2m}C_m\right.\\ \qquad\left.+\sinh\frac{1}{2}\lambda_{2m}D_m\right)+2\frac{a^3}{D}\sum_{i=1}^{r}\sin m\pi\eta_iP_i=0\\ S_{1m}\left(\sinh\frac{1}{2}\lambda_{1m}A_m+\cosh\frac{1}{2}\lambda_{1m}B_m\right)+S_{2m}\left(\sinh\frac{1}{2}\lambda_{2m}C_m\right.\\ \qquad\left.+\cosh\frac{1}{2}\lambda_{2m}D_m\right)+2\frac{a^2}{D}\sum_{i=1}^{r}\sin m\pi\eta_iT_i=0\end{array}\right. & \lambda<\alpha_m\\ \left\{\begin{array}{l}F_m\left(-\cos\frac{1}{2}\lambda_{1m}A_m+\sin\frac{1}{2}\lambda_{1m}B_m\right)+Q_{2m}\left(\cosh\frac{1}{2}\lambda_{2m}C_m\right.\\ \qquad\left.+\sinh\frac{1}{2}\lambda_{2m}D_m\right)+2\frac{a^3}{D}\sum_{i=1}^{r}\sin m\pi\eta_iP_i=0\\ -R_m\left(\sin\frac{1}{2}\lambda_{1m}A_m+\cos\frac{1}{2}\lambda_{1m}B_m\right)+S_{2m}\left(\sinh\frac{1}{2}\lambda_{2m}C_m\right.\\ \qquad\left.+\cosh\frac{1}{2}\lambda_{2m}D_m\right)+2\frac{a^2}{D}\sum_{i=1}^{r}\sin m\pi\eta_iT_i=0\end{array}\right. & \lambda>\alpha_m\end{array}\right\}$$

$$(m=1,\ 2,\ 3,\ \cdots,\ \infty) \tag{3.3}$$

式中

$$\left.\begin{array}{l}Q_{1m}=\lambda_{1m}^3-(2-\mu)\frac{\alpha_m^2}{\beta^2}\lambda_{1m},\quad Q_{2m}=\lambda_{2m}^3-(2-\mu)\frac{\alpha_m^2}{\beta^2}\lambda_{2m}\\ S_{1m}=\lambda_{1m}^2-\mu\frac{\alpha_m^2}{\beta^2},\quad S_{2m}=\lambda_{2m}^2-\mu\frac{\alpha_m^2}{\beta^2}\\ F_m=\lambda_{1m}^3+(2-\mu)\frac{\alpha_m^2}{\beta^2}\lambda_{1m},\quad R_m=\lambda_{1m}^2+\mu\frac{\alpha_m^2}{\beta^2}\end{array}\right\} \tag{3.4}$$

对于弹性支点，有

$$P_i=-k_i W\left(\frac{1}{2},\ \eta_i\right),\qquad T_i=\varphi_i\frac{\partial W\left(\frac{1}{2},\ \eta_i\right)}{\partial\xi}\qquad (i=1,\ 2,\ 3,\ \cdots,\ r)\tag{3.5}$$

式中，k_i，φ_i分别是$\xi=1/2$边第i个点支的横向支承刚度和扭转支承刚度。

设有常数l，当$m\leqslant l$时$\lambda_m<\alpha_m$，当$m>l$时$\lambda_m>\alpha_m$，令$K_i=k_ib^3\beta^3/D$，$\Phi_i=\varphi_ib^2\beta^2/D$，将(3.5)代入(3.3)可得到

$$\left\{\begin{aligned}
&\left.\begin{aligned}
&Q_{1m}\left(\cosh\frac{1}{2}\lambda_{1m}A_m+\sinh\frac{1}{2}\lambda_{1m}B_m\right)+Q_{2m}\left(\cosh\frac{1}{2}\lambda_{2m}C_m\right.\\
&\quad\left.+\sinh\frac{1}{2}\lambda_{2m}D_m\right)-2\sum_{i=1}^{r}\left\{\sin m\pi\eta_iK_i\left[\sum_{n=1}^{l}\left(\sinh\frac{1}{2}\lambda_{1n}A_n\right.\right.\right.\\
&\quad\left.+\cosh\frac{1}{2}\lambda_{1n}B_n\right)\sin n\pi\eta_i+\sum_{n=l+1}^{\infty}\left(\sin\frac{1}{2}\lambda_{1n}A_n+\cos\frac{1}{2}\lambda_{1n}B_n\right)\sin n\pi\eta_i\\
&\quad\left.\left.+\sum_{n=1}^{\infty}\left(\sinh\frac{1}{2}\lambda_{2n}C_n+\cosh\frac{1}{2}\lambda_{2n}D_n\right)\sin n\pi\eta_i\right]\right\}=0\\
&S_{1m}\left(\sinh\frac{1}{2}\lambda_{1m}A_m+\cosh\frac{1}{2}\lambda_{1m}B_m\right)+S_{2m}\left(\sinh\frac{1}{2}\lambda_{2m}C_m\right.\\
&\quad\left.+\cosh\frac{1}{2}\lambda_{2m}D_m\right)+2\sum_{i=1}^{r}\left\{\sin m\pi\eta_i\Phi_i\left[\sum_{n=1}^{l}\lambda_{1n}\left(\cosh\frac{1}{2}\lambda_{1n}A_n\right.\right.\right.\\
&\quad\left.+\sinh\frac{1}{2}\lambda_{1n}B_n\right)\sin n\pi\eta_i+\sum_{n=l+1}^{\infty}\lambda_{1n}\left(\cos\frac{1}{2}\lambda_{1n}A_n-\sin\frac{1}{2}\lambda_{1n}B_n\right)\sin n\pi\eta_i\\
&\quad\left.\left.+\sum_{n=1}^{\infty}\lambda_{2n}\left(\cosh\frac{1}{2}\lambda_{2n}C_n+\sinh\frac{1}{2}\lambda_{2n}D_n\right)\sin n\pi\eta_i\right]\right\}=0\\
&\quad(m=1,\ 2,\ 3,\ \cdots,\ l)
\end{aligned}\right.\\
&\left.\begin{aligned}
&F_m\left(-\cos\frac{1}{2}\lambda_{1m}A_m+\sin\frac{1}{2}\lambda_{1m}B_m\right)+Q_{2m}\left(\cosh\frac{1}{2}\lambda_{2m}C_m+\sinh\frac{1}{2}\lambda_{2m}D_m\right)\\
&\quad-2\sum_{i=1}^{r}\left\{\sin m\pi\eta_iK_i\left[\sum_{n=1}^{l}\left(\sinh\frac{1}{2}\lambda_{1n}A_n+\cosh\frac{1}{2}\lambda_{1n}B_n\right)\sin n\pi\eta_i\right.\right.\\
&\quad+\sum_{n=l+1}^{\infty}\left(\sin\frac{1}{2}\lambda_{1n}A_n+\cos\frac{1}{2}\lambda_{1n}B_n\right)\sin n\pi\eta_i+\sum_{n=1}^{\infty}\left(\sinh\frac{1}{2}\lambda_{2n}C_n\right.\\
&\quad\left.\left.\left.+\cosh\frac{1}{2}\lambda_{2n}D_n\right)\sin n\pi\eta_i\right]\right\}=0\\
&-R_m\left(\sin\frac{1}{2}\lambda_{1m}A_m+\cos\frac{1}{2}\lambda_{1m}B_m\right)+S_{2m}\left(\sinh\frac{1}{2}\lambda_{2m}C_m\right.\\
&\quad\left.+\cosh\frac{1}{2}\lambda_{2m}D_m\right)+2\sum_{i=1}^{r}\left\{\sin m\pi\eta_i\Phi_i\left[\sum_{n=1}^{l}\lambda_{1n}\left(\cosh\frac{1}{2}\lambda_{1n}A_n\right.\right.\right.
\end{aligned}\right.
\end{aligned}\right.$$

$$
\left.\begin{aligned}
&+\sinh\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i+\sum_{n=l+1}^{\infty}\lambda_{1n}\Big(\cos\frac{1}{2}\lambda_{1n}A_n-\sin\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i\\
&+\sum_{n=1}^{\infty}\lambda_{2n}\Big(\cosh\frac{1}{2}\lambda_{2n}C_n+\sinh\frac{1}{2}\lambda_{2n}D_n\Big)\sin n\pi\eta_i\Big]\Big\}=0\\
&(m=l+1,\ l+2,\ l+3,\ \cdots,\ \infty)
\end{aligned}\right\}
\tag{3.6}
$$

在$\xi=-1/2$边，若有s个弹性点支，类似地有

$$
\left.\begin{aligned}
&\left\{\begin{aligned}
&Q_{1m}\Big(\cosh\frac{1}{2}\lambda_{1m}A_m-\sinh\frac{1}{2}\lambda_{1m}B_m\Big)+Q_{2m}\Big(\cosh\frac{1}{2}\lambda_{2m}C_m\\
&-\sinh\frac{1}{2}\lambda_{2m}D_m\Big)+2\sum_{i=r+1}^{r+s}\Big\{\sin m\pi\eta_iK_i\Big[\sum_{n=1}^{l}\Big(-\sinh\frac{1}{2}\lambda_{1n}A_n\\
&+\cosh\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i+\sum_{n=l+1}^{\infty}\Big(-\sin\frac{1}{2}\lambda_{1n}A_n+\cos\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i\\
&+\sum_{n=1}^{\infty}\Big(-\sinh\frac{1}{2}\lambda_{2n}C_n+\cosh\frac{1}{2}\lambda_{2n}D_n\Big)\sin n\pi\eta_i\Big]\Big\}=0\\
&S_{1m}\Big(-\sinh\frac{1}{2}\lambda_{1m}A_m+\cosh\frac{1}{2}\lambda_{1m}B_m\Big)+S_{2m}\Big(-\sinh\frac{1}{2}\lambda_{2m}C_m\\
&+\cosh\frac{1}{2}\lambda_{2m}D_m\Big)-2\sum_{i=r+1}^{r+s}\Big\{\sin m\pi\eta_i\Phi_i\Big[\sum_{n=1}^{l}\lambda_{1n}\Big(\cosh\frac{1}{2}\lambda_{1n}A_n\\
&-\sinh\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i+\sum_{n=l+1}^{\infty}\lambda_{1n}\Big(\cos\frac{1}{2}\lambda_{1n}A_n+\sin\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i\\
&+\sum_{n=1}^{\infty}\lambda_{2n}\Big(\cosh\frac{1}{2}\lambda_{2n}C_n-\sinh\frac{1}{2}\lambda_{2n}D_n\Big)\sin n\pi\eta_i\Big]\Big\}=0\\
&(m=1,\ 2,\ 3,\ \cdots,\ l)
\end{aligned}\right.\\
&\left\{\begin{aligned}
&-F_m\Big(\cos\frac{1}{2}\lambda_{1m}A_m+\sin\frac{1}{2}\lambda_{1m}B_m\Big)+Q_{2m}\Big(\cosh\frac{1}{2}\lambda_{2m}C_m-\sinh\frac{1}{2}\lambda_{2m}D_m\Big)\\
&+2\sum_{i=r+1}^{r+s}\Big\{\sin m\pi\eta_iK_i\Big[\sum_{n=1}^{l}\Big(-\sinh\frac{1}{2}\lambda_{1n}A_n+\cosh\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i\\
&+\sum_{n=l+1}^{\infty}\Big(-\sin\frac{1}{2}\lambda_{1n}A_n+\cos\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i+\sum_{n=1}^{\infty}\Big(-\sinh\frac{1}{2}\lambda_{2n}C_n\\
&+\cosh\frac{1}{2}\lambda_{2n}D_n\Big)\sin n\pi\eta_i\Big]\Big\}=0\\
&R_m\Big(\sin\frac{1}{2}\lambda_{1m}A_m-\cos\frac{1}{2}\lambda_{1m}B_m\Big)-S_{2m}\Big(\sinh\frac{1}{2}\lambda_{2m}C_m-\cosh\frac{1}{2}\lambda_{2m}D_m\Big)
\end{aligned}\right.
\end{aligned}\right.
$$

$$
\left.\begin{aligned}
&-2\sum_{i=r+1}^{r+s}\Big\{\sin m\pi\eta_i\Phi_i\Big[\sum_{n=1}^{l}\lambda_{1n}\Big(\cosh\frac{1}{2}\lambda_{1n}A_n-\sinh\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i\\
&+\sum_{n=l+1}^{\infty}\lambda_{1n}\Big(\cos\frac{1}{2}\lambda_{1n}A_n+\sin\frac{1}{2}\lambda_{1n}B_n\Big)\sin n\pi\eta_i+\sum_{n=1}^{\infty}\lambda_{2n}\Big(\cosh\frac{1}{2}\lambda_{2n}C_n\\
&-\sinh\frac{1}{2}\lambda_{2n}D_n\Big)\sin n\pi\eta_i\Big]\Big\}=0\\
&(m=l+1,\ l+2,\ l+3,\ \cdots,\ \infty)
\end{aligned}\right\}
\tag{3.7}
$$

若$\xi=-1/2$边自由无点支，只需令上式中的$K_i=\Phi_i=0$ $(i=r+1,\ r+2,\ \cdots,\ r+s)$即可。

若$\xi=-1/2$边固支，有

$$W(-1/2,\ \eta)=0,\ \partial W(-1/2,\ \eta)/\partial\xi=0 \tag{3.8}$$

将(2.5)代入上式得到

$$
\left.\begin{aligned}
&\left\{\begin{aligned}
&-\sinh\frac{1}{2}\lambda_{1m}A_m+\cosh\frac{1}{2}\lambda_{1m}B_m-\sinh\frac{1}{2}\lambda_{2m}C_m\\
&\qquad+\cosh\frac{1}{2}\lambda_{2m}D_m=0\\
&\lambda_{1m}\Big(\cosh\frac{1}{2}\lambda_{1m}A_m-\sinh\frac{1}{2}\lambda_{1m}B_m\Big)+\lambda_{2m}\Big(\cosh\frac{1}{2}\lambda_{2m}C_m\\
&\qquad-\sinh\frac{1}{2}\lambda_{1m}D_m\Big)=0
\end{aligned}\right. \qquad \lambda<\alpha_m\\
&\left\{\begin{aligned}
&-\sin\frac{1}{2}\lambda_{1m}A_m+\cos\frac{1}{2}\lambda_{1m}B_m-\sinh\frac{1}{2}\lambda_{2m}C_m\\
&\qquad+\cosh\frac{1}{2}\lambda_{2m}D_m=0\\
&\lambda_{1m}\Big(\cos\frac{1}{2}\lambda_{1m}A_m+\sin\frac{1}{2}\lambda_{1m}B_m\Big)+\lambda_{2m}\Big(\cosh\frac{1}{2}\lambda_{2m}C_m\\
&\qquad-\sinh\frac{1}{2}\lambda_{1m}D_m\Big)=0
\end{aligned}\right. \qquad \lambda>\alpha_m\\
&\qquad(m=1,\ 2,\ 3,\ \cdots,\ \infty)
\end{aligned}\right\}
\tag{3.9}
$$

若$\xi=-1/2$边简支，有

$$W(-1/2,\ \eta)=0,\ M_\xi(-1/2,\ \eta)=0 \tag{3.10}$$

将(2.5)代入上式得到

$$
\left\{\begin{aligned}
&-\sinh\frac{1}{2}\lambda_{1m}A_m+\cosh\frac{1}{2}\lambda_{1m}B_m-\sinh\frac{1}{2}\lambda_{2m}C_m\\
&\qquad+\cosh\frac{1}{2}\lambda_{2m}D_m=0\\
&S_{1m}\Big(-\sinh\frac{1}{2}\lambda_{1m}A_m+\cosh\frac{1}{2}\lambda_{1m}B_m\Big)\\
&\qquad+S_{2m}\Big(-\sinh\frac{1}{2}\lambda_{2m}C_m+\cosh\frac{1}{2}\lambda_{2m}D_m\Big)=0
\end{aligned}\right. \qquad \lambda<\alpha_m
$$

$$\left.\begin{cases} -\sin\frac{1}{2}\lambda_{1m}A_m+\cos\frac{1}{2}\lambda_{1m}B_m-\sinh\frac{1}{2}\lambda_{2m}C_m \\ \qquad +\cosh\frac{1}{2}\lambda_{2m}D_m=0 \\ R_m\left(\sin\frac{1}{2}\lambda_{1m}A_m-\cos\frac{1}{2}\lambda_{1m}B_m\right)+S_{2m}\left(-\sinh\frac{1}{2}\lambda_{2m}C_m\right. \\ \qquad \left.+\cosh\frac{1}{2}\lambda_{2m}D_m\right)=0 \\ \qquad (m=1,\ 2,\ 3,\ \cdots,\ \infty) \end{cases}\quad \lambda>\alpha_m\right\} \tag{3.11}$$

在以上诸式中，截断令$m=n=1$，2，3，…，q，则可得到$4q$个方程，共有$4q$个待定系数A_i，B_i，C_i，D_i（$i=1$，2，3，…，q），构成完全解。令待定系数的行列式值为零，即得到特征方程，解之可得到各阶固有频率及A_i，B_i，C_i，D_i（$i=1$，2，3，…，q）各常数的比值。将其代回（2.5）则可得到各阶振型。在以上分析中，当K_i或Φ_i取0或∞时，则可得到各种特殊情况。

四、算　　例

考虑一对边简支、另一对边自由中点各有一横向弹性点支（$K\neq0$，$\Phi=0$）的矩形板。由于纵边支承条件对称，因而在ξ方向的振动可分解为对称振动和反对称振动。对称振动的振型为

$$\left.\begin{aligned} X_m(\xi)&=B_m\cosh\lambda_{1m}\xi+D_m\cosh\lambda_{2m}\xi \qquad \lambda<\alpha_m \\ X_m(\xi)&=B_m\cos\lambda_{1m}\xi+D_m\cosh\lambda_{2m}\xi \qquad \lambda>\alpha_m \end{aligned}\right\} \tag{4.1}$$

将上式代入（2.8）得到

$$\left.\begin{aligned} &\begin{cases} Q_{1m}\sinh\frac{1}{2}\lambda_{1m}B_m+Q_{2m}\sinh\frac{1}{2}\lambda_{2m}D_m-2\sin\frac{m\pi}{2}K\left[\sum\limits_{n=1}^{l}\cosh\frac{1}{2}\lambda_{1n}B_n\sin\frac{n\pi}{2}\right. \\ \qquad \left.+\sum\limits_{n=l+1}^{\infty}\cos\frac{1}{2}\lambda_{1n}B_n\sin\frac{n\pi}{2}+\sum\limits_{n=1}^{\infty}\cosh\frac{1}{2}\lambda_{2n}D_n\sin\frac{n\pi}{2}\right]=0 \\ S_{1m}\cosh\frac{1}{2}\lambda_{1m}B_m+S_{2m}\cosh\frac{1}{2}\lambda_{2m}D_m=0 \\ \qquad (m=1,\ 2,\ 3,\ \cdots,\ l) \end{cases} \\ &\begin{cases} F_m\sin\frac{1}{2}\lambda_{1m}B_m+Q_{2m}\sinh\frac{1}{2}\lambda_{2m}D_m-2\sin\frac{m\pi}{2}K\left[\sum\limits_{n=1}^{l}\cosh\frac{1}{2}\lambda_{1n}B_n\sin\frac{n\pi}{2}\right. \\ \qquad \left.+\sum\limits_{n=l+1}^{\infty}\cos\lambda_{1n}B_n\sin\frac{n\pi}{2}\sum\limits_{n=1}^{\infty}\cosh\frac{1}{2}\lambda_{2n}D_n\sin\frac{n\pi}{2}\right]=0 \\ R_m\cos\frac{1}{2}\lambda_{1m}B_m-S_{2m}\cosh\frac{1}{2}\lambda_{2m}D_m=0 \\ \qquad (m=l+1,\ l+2,\ l+3,\ \cdots,\ \infty) \end{cases} \end{aligned}\right\} \tag{4.2}$$

反对称振动的振型为

$$\left.\begin{aligned} X_m(\xi)&=A_m\sinh\lambda_{1m}\xi+C_m\sinh\lambda_{2m}\xi \qquad \lambda<\alpha_m \\ X_m(\xi)&=A_m\sin\lambda_{1m}\xi+C_m\sinh\lambda_{2m}\xi \qquad \lambda>\alpha_m \end{aligned}\right\} \tag{4.3}$$

将上式代入（2.8）得到

$$\left\{\begin{aligned}&\left\{\begin{aligned}&Q_{1m}\cosh\frac{1}{2}\lambda_{1m}A_m+Q_{2m}\cosh\frac{1}{2}\lambda_{2m}C_m-2\sin\frac{m\pi}{2}K\Big[\sum_{n=1}^{l}\sinh\frac{1}{2}\lambda_{1n}A_n\\ &\qquad\cdot\sin\frac{n\pi}{2}+\sum_{n=l+1}^{\infty}\sin\frac{1}{2}\lambda_{1n}A_n\sin\frac{n\pi}{2}+\sum_{n=1}^{\infty}\sinh\frac{1}{2}\lambda_{2n}C_n\sin\frac{n\pi}{2}\Big]=0\\ &S_{1m}\sinh\frac{1}{2}\lambda_{1m}A_m+S_{2m}\sinh\frac{1}{2}\lambda_{2m}C_m=0\end{aligned}\right.\\ &\qquad\qquad(m=1,\ 2,\ 3,\ \cdots,\ l)\\ &\left\{\begin{aligned}&-F_m\cos\frac{1}{2}\lambda_{1m}A_m+Q_{2m}\cosh\frac{1}{2}\lambda_{2m}C_m-2\sin\frac{m\pi}{2}K\Big[\sum_{n=1}^{l}\sinh\frac{1}{2}\lambda_{1n}A_n\\ &\qquad\cdot\sin\frac{n\pi}{2}+\sum_{n=l+1}^{\infty}\sin\frac{1}{2}\lambda_{1n}A_n\sin\frac{n\pi}{2}+\sum_{n=1}^{\infty}\sinh\frac{1}{2}\lambda_{2n}C_n\sin\frac{n\pi}{2}\Big]=0\\ &R_m\sin\frac{1}{2}\lambda_{1m}A_m-S_{2m}\sinh\frac{1}{2}\lambda_{2m}C_m=0\end{aligned}\right.\\ &\qquad\qquad(m=l+1,\ l+2,\ l+3,\ \cdots,\ \infty)\end{aligned}\right. \tag{4.4}$$

从（4.2）、（4.4）两式中可看出，当$m=n=2$，4，6，…时，在$\xi=\pm 1/2$边，均有

$$W\left(-\frac{1}{2},\ \eta\right)=W\left(\frac{1}{2},\ \eta\right)=\frac{\partial W(-1/2,\ \eta)}{\partial\xi}=\frac{\partial W(1/2,\ \eta)}{\partial\xi}=0$$

这时，自由边中点的点支对振动无影响，此时的振动与自由边无点支时的情况完全一样，固有频率及振型的计算已有结果[7]，本文为考虑点支对矩形板振动的影响，因而只给出$m=n=1$，3，5，…时不同点支刚度下方板的前几阶固有频率，如表1所示。

表 1　两对边简支、另两对边自由中点各有一弹性点支方板的前 8 阶对称和反对称振动的频率参数λ^2

振动形式	$K=0.5$	$K=5$	$K=10$	$K=25$	$K=50$	$K=100$	$K=500$	$K=10^6$
对称	9.70	10.71	11.60	13.41	15.02	16.46	18.22	18.79(18.76)
	36.52	37.41	38.36	41.00	44.69	49.82	61.30	66.58(66.22)
	87.76	87.92	88.09	88.59	89.34	90.58	94.55	97.27(97.05)
	121.33	121.56	121.80	122.45	123.32	124.44	126.28	126.92(126.88)
	133.52	133.80	134.11	135.12	136.97	140.93	162.24	183.56(181.88)
	223.78	223.93	224.10	224.58	225.35	226.74	232.56	236.76(236.48)
	244.78	244.84	244.92	245.15	245.53	246.33	251.92	260.94(260.15)
	279.92	280.01	280.12	280.44	280.96	281.97	287.71	294.80(294.23)

续表

振动形式	$K=0.5$	$K=5$	$K=10$	$K=25$	$K=50$	$K=100$	$K=500$	$K=10^6$
	16.09	17.60	19.08	22.54	26.42	30.91	38.79	42.09(41.86)
反	75.07	75.53	76.02	77.36	79.16	81.53	85.43	86.66(86.58)
	95.51	95.81	96.14	97.20	99.08	102.97	120.12	132.84(131.92)
对	163.81	164.00	164.21	164.83	165.84	167.73	176.71	185.90(185.15)
	211.95	212.12	212.31	212.87	213.79	215.54	224.64	234.05(233.34)
称	252.24	252.36	252.49	252.89	253.54	254.85	263.43	276.96(275.78)
	302.29	302.41	302.54	302.92	303.54	304.74	310.85	316.05(315.71)
	325.24	325.33	325.43	325.73	326.24	327.42	339.09	379.63(375.06)

$\mu=0.333$,( )中的数据取自文献[8].

从计算结果可看出，K越大，固有频率越高，当K很大(10^6)时，其计算结果与文献[8]的刚性点支结果是一致的。

五、结 束 语

本文应用脉冲函数的Fourier级数展开法求解两对边简支、另两对边任意支承自由边有弹性点支矩形板的固有频率和振型，不但方法简单，计算结果还表明其有很高的精度。

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Transverse Vibration of Rectangular Plates Elastically Supported at Points on Edges

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Abstract

This paper studies transverse vibration of rectangular plates with two opposite edges simply supported, other two edges arbitrarily supported and free edges elastically supported at points. A highly accurate solution is presented for calculating inherent frequencies and mode shape of rectangular plates elastically supported at points. The number and location of these points on free edges may be completely arbitrary. This paper uses impulse function to represent reaction and moment at points. Fourier series is used to expand the impulse function along the edges. Characteristic equations satisfying all boundary conditions are given. Inherent frequencies and mode shape with any accuracy can be gained.