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受弹性点支的任意形状的膜的振动*

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摘　要

本文提供了一个求解受弹性点支的任意形状的膜的振动的新方法．将弹性点支反力看作是作用于膜上的未知外力，求出了包含有未知反力的运动方程的精确解，利用弹性点支处位移和反力的线性关系导出频率方程，最后以受弹性点支的圆膜为例给出了其频率方程的具体计算公式，并数值计算了受两个对称弹性点支的圆膜的固有振动频率．

一、引　言

有许多通用方法已用于求解任意形状的膜的振动，如差分法[1]、点匹配法[2]、瑞雷—李兹法[3]、伽辽金法[4]、有限元法[5]．这些通用方法对解决任意形状的膜的振动是有效的，但由于在这些方法中均使用了运动方程的近似解和近似的边界条件，因而在计算高阶模态时并不那么得心应手，且需使用容量较大的电子计算机进行数值计算．此外，保角变换法[6]和常位移线法[7]也已用于计算一些形状较为复杂的膜的振动，但在大多数情况下，仅能求出基阶模态．早些时候，Nagaya[8]提出了一个求解任意形状膜的振动的新方法，并将之推广应用到受刚性点支的任意形状的膜的振动[9]，获得了满意的结果．到目前为止，还未曾见过对受弹性点支的任意形状的膜的振动的研究报道．本文进一步推广了Nagaya 的方法，将弹性点支反力看作是作用于膜上的外力，求出了包含有未知反力的运动方程的精确解，利用弹性点支处位移和反力的线性关系导出频率方程．本文方法对受任意多个弹性点支的任意形状的膜均是适用的．

二、微分方程及其解

受弹性点支的任意形状的膜如图1表示，用w表示膜在垂直方向的位移，则在极坐标系r，θ中，膜的运动方程为

$$T[\partial^2w/\partial r^2+(1/r)\partial w/\partial r+(1/r^2)\partial^2w/\partial\theta^2]-\rho\partial^2w/\partial t^2=q \qquad (2.1)$$

式中，T是膜的张力，ρ是单位面积膜的质量密度，t是时间，q是作用于膜上的未知的弹性点支反力．自由振动时，膜的位移和弹性点支反力可分别写成

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$$w(r,\theta,t)=W(r,\theta)\sin\omega t,\quad q(r,\theta,t)=Q(r,\theta)\sin\omega t \tag{2.2}$$

将(2.2)式代入方程(2.1)，可得均匀膜的解为

$$W(r,\theta)=(1/T)\sum_{j=1}^{2}\sum_{n=0}^{\infty}\varepsilon_n[A_{jn}J_n(\alpha r)+F_{jn}(r)]\phi_{jn} \tag{2.3}$$

式中

$$\left.\begin{aligned}&F_{jn}(r)=\frac{\pi}{2}\int_0^r Q_{jn}(\xi)\{Y_n(\alpha r)J_n(\alpha\xi)-J_n(\alpha r)Y_n(\alpha\xi)\}\xi d\xi\\&Q_{jn}=\frac{1}{\pi}\int_{-\pi}^{\pi}Q(r,\theta)\phi_{jn}d\theta\\&\alpha^2=\frac{\rho\omega^2}{T},\quad\varepsilon_0=\frac{1}{2},\quad\varepsilon_n=1\quad 当n\geqslant1时\\&\phi_{1n}=\cos n\theta,\quad\phi_{2n}=\sin n\theta\end{aligned}\right\} \tag{2.4}$$

以上两式中，ω是固有圆频率，A_{jn}是积分常数，$J_n(\alpha r)$和$Y_n(\alpha r)$分别是第一类和第二类的n阶Bessel函数，$F_{jn}(r)$是方程(2.1)的特解。

当膜在I个点受到弹性点支时，作用于膜上的支承反力可写成

$$Q(r,\theta)=-\sum_{i=1}^{I}\frac{R_i}{b_i}\delta(r-b_i)\delta(\theta-\Theta_i) \tag{2.5}$$

式中，R_i是第i个弹性点支的支承反力，$\delta(r-b_i)$和$\delta(\theta-\Theta_i)$是Dirac delta函数，b_i和Θ_i是第i个点支的极坐标，I是弹性点支的总个数。

由(2.4)式和(2.5)式可知

$$Q_{jn}(r)=-\sum_{i=1}^{I}(R_{ji}/\pi b_i)\delta(r-b_i)\psi_{jn}^{i} \tag{2.6}$$

式中

$$\psi_{1n}^{i}=\cos n\Theta_i,\quad\psi_{2n}^{i}=\sin n\Theta_i \tag{2.7}$$

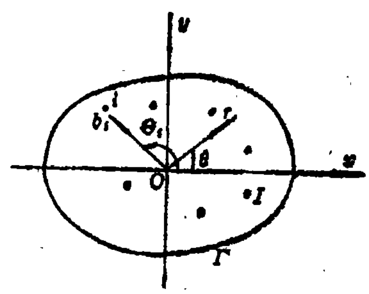

图1 受弹性点支的任意形状的膜

将(2.6)式代入(2.4)式得到

$$F_{jn}(r)=-\frac{1}{2}\sum_{i=1}^{I}R_{ji}\{Y_n(\alpha r)J_n(\alpha b_i)-J_n(\alpha r)Y_n(\alpha b_i)\}\psi_{jn}^{i}u(r-b_i) \tag{2.8}$$

式中，$u(r-b_i)$是单位阶跃函数，$u(r-b_i)=1$对$r>b_i$，$u(r-b_i)=0$，对$r<b_i$。当膜的形状关于x轴对称时，其运动可分解成对称振动和反对称振动。此时，仅需分别取$j=1$和$j=2$便可求得对称振动和反对称振动的模态。如令$W=\sum_{j=1}^{2}W_j$，并将(2.8)式代入(2.3)式，位移函数可写成

$$W_j=\frac{1}{T}\sum_{n=0}^{\infty}\varepsilon_n[A_{jn}J_n(\alpha r)-\frac{1}{2}\sum_{i=1}^{I}R_{ji}\{Y_n(\alpha r)J_n(\alpha b_i)-J_n(\alpha r)Y_n(\alpha b_i)\}\cdot u(r-b_i)\psi_{jn}^{i}]\phi_{jn} \tag{2.9}$$

式中的求和$\sum\limits_{n}$对于反对称振动应取$n=1,2,3,\cdots,\infty$。

边缘固支的膜的边界条件为

$$(w)_\Gamma=0 \tag{2.10}$$

将上式沿膜的外边界展成Fourier级数，有

$$W_j=\frac{1}{T}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon_m\left(A_{jn}S^{j}_{nm}-\sum_{i=1}^{I}R_{ji}T^{ij}_{nm}\right)\phi_{jm}=0 \qquad 对\ j=1,2 \tag{2.11}$$

式中

$$\left.\begin{aligned}S^{j}_{nm}&=\frac{2\varepsilon_n}{\pi}\int_0^{\pi}J_n(ar_\Gamma)\phi_{jn}\phi_{jm}d\theta\\ T^{ij}_{nm}&=\frac{\varepsilon_n}{\pi}\int_0^{\pi}\{Y_n(ar_\Gamma)J_n(ab_i)-J_n(ar_\Gamma)Y_n(ab_i)\}\psi^{i}_{jn}\phi_{jn}\phi_{jm}d\theta\end{aligned}\right\} \tag{2.12}$$

这里，求和$\sum\limits_{m}$与前面求和$\sum\limits_{n}$是一样的，r_Γ是膜边界的坐标值，对任意形状的膜，均可表示为θ的函数。当n,m均截断到$N+1$时，(2.11)式可写成下面矩阵形式

$$\begin{bmatrix} & \\ & \Delta^1 & \\ & \end{bmatrix}\begin{bmatrix}A_{10}\\ A_{11}\\ \vdots\\ A_{1N}\end{bmatrix}=\begin{bmatrix}\sum\limits_{i=1}^{I}R_{1i}\sum\limits_{n'=0}^{N}T^{i1}_{n'0}\\ \sum\limits_{i=1}^{I}R_{1i}\sum\limits_{n'=0}^{N}T^{i1}_{n'1}\\ \vdots\\ \sum\limits_{i=1}^{I}R_{1i}\sum\limits_{n'=0}^{N}T^{i1}_{n'N}\end{bmatrix} \tag{2.13}$$

$$\begin{bmatrix} & \\ & \Delta^2 & \\ & \end{bmatrix}\begin{bmatrix}A_{21}\\ A_{22}\\ \vdots\\ A_{2N}\end{bmatrix}=\begin{bmatrix}\sum\limits_{i=1}^{I}R_{2i}\sum\limits_{n'=1}^{N}T^{i2}_{n'1}\\ \sum\limits_{i=1}^{I}R_{2i}\sum\limits_{n'=1}^{N}T^{i2}_{n'2}\\ \vdots\\ \sum\limits_{i=1}^{I}R_{2i}\sum\limits_{n'=1}^{N}T^{i2}_{n'N}\end{bmatrix} \tag{2.14}$$

式中

$$[\Delta^1]=\begin{bmatrix} S^1_{00} & S^1_{10} & \cdots & S^1_{N0} \\ S^1_{01} & & \ddots & \vdots \\ \vdots & & & \vdots \\ S^1_{0N} & \cdots\cdots & & S^1_{NN} \end{bmatrix},\quad [\Delta^2]=\begin{bmatrix} S^2_{11} & S^2_{21} & \cdots & S^2_{N1} \\ S^2_{12} & & \ddots & \vdots \\ \vdots & & & \vdots \\ S^2_{1N} & \cdots\cdots & & S^2_{NN} \end{bmatrix} \tag{2.15}$$

由(2.13)式和(2.14)式可求得常数A_{jn}为

$$A_{jn}=\sum_{i=1}^{I}R_{ji}E^{i}_{jn} \tag{2.16}$$

式中

$$E^{i}_{jn}=\sum_{m=0}^{N}\Big(\sum_{n'=0}^{N}T^{i\,j}_{n'm}\Big)|D^{j}_{mn}|\Big/|\Delta^{j}| \tag{2.17}$$

这里，$|D^{j}_{mn}|=(-1)^{m+n}|\Delta^{j}_{mn}|$是行列式$|\Delta^{j}|$的代数余子式，$|\Delta^{j}_{mn}|$是行列式$|\Delta^{j}|$去掉第$m$行第$n$列的子行列式，由(2.9)式和(2.16)式可得

$$W_j=\frac{1}{T}\sum_{i=1}^{I}R_{ji}\sum_{n=0}^{N}\varepsilon_n[E^{i}_{jn}J_n(ar)-\frac{1}{2}\{Y_n(ar)J_n(ab_i)$$
$$-J_n(ar)Y_n(ab_i)\}\cdot u(r-b_i)\psi^{i}_{jn}]\phi_{jn} \tag{2.18}$$

在弹性点支处，支承反力满足

$$R_{js}=-K_s(W)_{r=b_s,\theta=\Theta_s}\qquad 对s=1,2,\cdots,I \tag{2.19}$$

式中，K_s是第s个弹性点支的支承刚度。

将(2.18)式代入(2.19)式可得

$$R_{js}=-\frac{K_s}{T}\sum_{i=1}^{I}R_{ji}\sum_{n=0}^{N}\varepsilon_n[E^{i}_{jn}J_n(ab_s)-\frac{1}{2}\{Y_n(ab_s)J_n(ab_i)$$
$$-J_n(ab_s)Y_n(ab_i)\}u(b_s-b_i)\psi^{i}_{jn}]\psi^{s}_{jn}\qquad 对s=1,2,\cdots,I \tag{2.20}$$

因而可得到受弹性点支的膜的频率方程为

$$\begin{vmatrix} B^{j}_{11}+T/K_1 & B^{j}_{12} & \cdots & B^{j}_{1I} \\ B^{j}_{21} & & \ddots & \vdots \\ \vdots & & & \vdots \\ B^{j}_{I1} & \cdots\cdots & & B^{j}_{II}+T/K_I \end{vmatrix}=0 \tag{2.21}$$

式中

$$B^{j}_{si}=\sum_{n=0}^{N}\varepsilon_n[E^{i}_{jn}J_n(ab_s)-\frac{1}{2}\{Y_n(ab_s)J_n(ab_i)$$
$$-J_n(ab_s)Y_n(ab_i)\}u(b_s-b_i)\psi^{i}_{jn}]\psi^{s}_{jn} \tag{2.22}$$

一般情况下，(2.12)式中S^{j}_{nm}和T^{ij}_{nm}的解析值是难以求得的，只能数值解出。此外，对

于边界用分段函数表示的膜，仅需将(2.12)式分段积分即可。

三、算　例

现以受弹性点支的圆膜为例，说明本文方法的使用。对于半径为R的圆膜，(2.12)式可解析求出为

$$\left.\begin{aligned}&S^{i}_{nm}=\varepsilon_n J_n(\alpha R)\quad \text{对}n=m,\ S^{i}_{mn}=0\quad \text{对}n\neq m\\&T^{ij}_{nm}=(\varepsilon_n/2)\{Y_n(\alpha R)J_n(\alpha b_i)-J_n(\alpha R)Y_n(\alpha b_i)\}\psi^{i}_{jn}\quad \text{对}n=m\\&T^{ij}_{nm}=0\quad \text{对}n\neq m\end{aligned}\right\}\tag{3.1}$$

常数A_{jn}可求得为

$$A_{jn}=\frac{1}{2}\sum_{i=1}^{I}R_{ji}\{Y_n(\alpha R)J_n(\alpha b_i)-J_n(\alpha R)Y_n(\alpha b_i)\}\psi^{i}_{jn}/J_n(\alpha R)\tag{3.2}$$

由上式可知

$$E^{i}_{jn}=\frac{1}{2}\{Y_n(\alpha R)J_n(\alpha b_i)-J_n(\alpha R)Y_n(\alpha b_i)\}\psi^{i}_{jn}/J_n(\alpha R)\tag{3.3}$$

将上式代入(2.22)式可得

$$\begin{aligned}B^{i}_{si}=\sum_{n=0}^{N}\frac{\varepsilon_n}{2}[&\{Y_n(\alpha R)J_n(\alpha b_i)-J_n(\alpha R)Y_n(\alpha b_i)\}J_n(\alpha b_s)/J_n(\alpha R)\\&-\{Y_n(\alpha b_s)J_n(\alpha b_i)-J_n(\alpha b_s)Y_n(\alpha b_i)\}u(b_s-b_i)]\psi^{i}_{jn}\psi^{s}_{jn}\end{aligned}\tag{3.4}$$

将上式代入(2.21)式可直接解出固有频率。对于仅受一个弹性点支的膜，$\Theta_1=0$，$b_1=b$，$K_1=K$，其对称振动的频率方程为

$$\sum_{n=0}^{N}\frac{\varepsilon_n}{2}\{Y_n(\alpha R)J_n(\alpha b)-J_n(\alpha R)Y_n(\alpha b)\}\cdot\frac{J_n(\alpha b)}{J_n(\alpha R)}+\frac{T}{K}=0\tag{3.5}$$

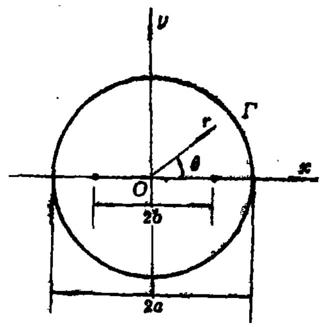

图2 受两个对称弹性点支的圆膜

对于受二个对称弹性点支的膜，$\Theta_1=0$，$\Theta_2=\pi$，$b_1=b_2=b$，$K_1=K_2=K$，其对称振动的频率方程为

$$\left(B^{1}_{11}+\frac{T}{K}\right)\left(B^{1}_{22}+\frac{T}{K}\right)-B^{1}_{12}B^{1}_{21}=0\tag{3.6}$$

式中

$$\left.\begin{aligned}&B^{1}_{11}=B^{1}_{22}=\sum_{n=0}^{N}\frac{\varepsilon_n}{2}\{Y_n(\alpha R)J_n(\alpha b)-J_n(\alpha R)Y_n(\alpha b)\}\frac{J_n(\alpha b)}{J_n(\alpha R)}\\&B^{1}_{12}=B^{1}_{21}=\sum_{n=0}^{N}(-1)^n\frac{\varepsilon_n}{2}\{Y_n(\alpha R)J_n(\alpha b)-J_n(\alpha R)Y_n(\alpha b)\}\frac{J_n(\alpha b)}{J_n(\alpha R)}\end{aligned}\right\}\tag{3.7}$$

无论是受一个弹性点支的圆膜还是受两个对称弹性点支的圆膜，其反对称振动的频率方程均为

$$J_n(\alpha R)=0 \tag{3.8}$$

表1给出了受两个对称弹性点支的圆膜对称振动时的无量纲基频，计算时仅取 $N=5$ 便可得到很好的结果。

表1 受两个对称弹性点支的圆膜对称振动时的无量纲基频 $\alpha R(b_1=b_2=b, K_1=K_2=K)$

K/T	b/R							
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
1.0	2.64	2.66	2.63	2.59	2.54	2.49	2.46	2.42
10.0	2.87	2.96	2.97	2.89	2.78	2.65	2.56	2.47
100.0	2.91	3.02	3.06	2.98	2.86	2.71	2.58	2.48
1000.0	2.92	3.03	3.07	2.99	2.87	2.72	2.58	2.48
∞	(3.00)	(3.16)	(3.19)	(3.10)	(3.93)	(2.77)	(2.63)	(2.53)

* （ ）中的数值取自文献[9].

四、结 束 语

本文给出了一个分析受任意多个弹性点支的任意形状的膜的振动的新方法。将弹性点支反力看作是作用于膜上的未知外力，求出了包含有未知反力的运动方程的精确解；应用Nagaya方法将位移沿膜的边界Fourier级数展开，利用弹性点支处支承反力与位移的线性关系导出了频率方程。算例表明，只要将求和项数取得适当的多，可求得精度很好的高阶模态。

参 考 文 献

[1] Steele, C. W., Numerical computation of electric and magnetic field in a uniform waveguide of arbitrary cross-section, *Journal of Computational Physics*, 3 (1968).

[2] Bates, R. H. T., The theory of point-matching method for perfectly conducting waveguides and transmission lines, *Institution of Electrical and Electronic Engineers Transactions on Microwave Theory and Techniques*, MTT-17 (1969).

[3] Bulley, R. M. and J. B. Davies, Computation of approximate polynomial solutions to TE modes in an arbitrary shaped waveguide, *Institution of Electrical and Electronic Engineers Transactions on Microwave Theory and Techniques*, MTT-17 (1969).

[4] Laura, P. A. A., A simple method for the calculation of cut-off frequencies of waveguide with arbitrary cross-section, *Proceedings of the Institution of Electrical and Electronic Engineers* (1966), 54.

[5] Arlett, P. L., A. K. Bahrani and O. C. Zienkiewicz, Application of finite elements to the solution of Helmholtz's equation, *Proceedings of the Institution of Electrical Engineers* (1968), 115.

[6] Laura, P. A. A., E. Romanelli and M. J. Maurizi, On the analysis of waveguides of doubly-connected cross-section by the method of conformal mapping, *Journal of Sound and Vibration*, **20** (1972).

[7] Mazumdar, J., Transverse vibration of membranes of arbitrary shape by the method of constant-deflection contours, *Journal of Sound and Vibration*, **27** (1973).

[8] Nagaya, K., Vibrations and dynamic response of membranes with arbitrary shape, *American Society of Mechanical Engineers Journal of Applied Mechanics*, **45** (1978).

[9] Nagaya, K., Vibration of an arbitrarily shaped membrane with point supports, *Journal of Sound and Vibration*, **65** (1979).

Vibration of Arbitrarily Shaped Membranes with Elastical Supports at Points

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Abstract

This paper presents a new method for solving the vibration of arbitrarily shaped memebranes with elastical supports at points. The reaction forces of elastical supports at points are regarded as unknown external forces acting on the membranes. The exact solution of the equation of motion is given which includes terms representing the unknown reaction forces. The frequency equation is derived by the use of the linear relationship of the displacements with the reaction forces of elastical supports at points. Finally the calculating formulae of the frequency equation of circular membranes are analytically performed as examples and the inherent frequencies of circular membranes with symmetric elastical supports at two points are numerically calculated.