应用数学和力学，第11卷第2期(1990年2月)
Applied Mathematics and Mechanics
应用数学和力学编委会编
重庆出版社出版

局部竖向荷载作用下圆柱形薄壳的解析解*

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摘　要

本文运用混合型级数方法，导出了局部竖向荷载作用下横向简支圆柱形薄壳的解析解，并给出了在五种局部竖向荷载作用下解的解析式．文中给出了纵边自由的圆柱形薄壳屋顶算例，计算结果表明了级数具有较好的收敛性．本文结果可以处理地下结构中的若干实际问题．

一、引　论

圆柱形薄壳屋顶及地下衬砌结构在局部荷载作用下的计算，是工程上急待解决又有一定难度的课题．前者是后者的计算基础，这里，我们先研究圆柱形薄壳的计算，其余的问题将另文研究．

本文采用了混合型级数方法，导出了在任意处受竖向作用的横向均布线荷载、纵向均布线荷载、均布于矩形域的局部面荷载、集中力及集中力偶时，横向端简支而纵向边为任意约束的圆柱形薄壳的全部内力和位移计算公式，并给出了纵向边自由时的一个数值例题．

二、基　本　方　程

在图1中，规定了本文中采用的坐标系，外力、内力及位移的正方向，以及一些参数和尺寸符号．

在圆柱形薄壳的工程理论中，求解壳体法向位移分量w及内力函数F的基本方程式为：

$$\left.\begin{aligned}&\nabla^4F+EhR\frac{\partial^2 w}{\partial^2\xi}=R^3\left[\int\frac{\partial^2 X}{\partial\varphi^2}d\xi+\int\frac{\partial^2 Y}{\partial\xi^2}d\varphi-\mu\left(\frac{\partial X}{\partial\xi}+\frac{\partial Y}{\partial\varphi}\right)\right]\\&\frac{\partial^2 F}{\partial\xi^2}-\frac{EhR}{1-\mu^2}k\nabla^4 w=R^3\left(\int Y d\varphi-Z\right)\end{aligned}\right\}\qquad(2.1)$$

或

* 国家自然科学基金和建设部科学技术发展基金联合资助项目．
潘立宙推荐．1986年5月31日第一次收到．

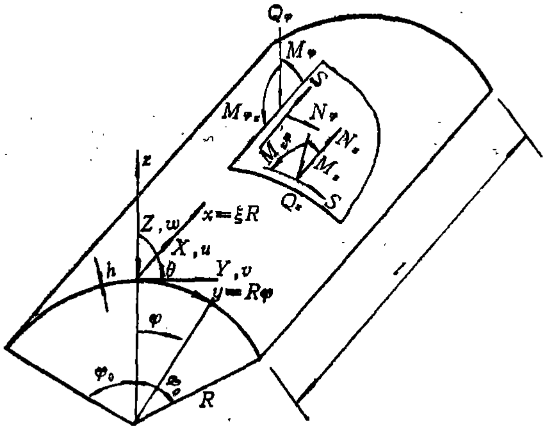


图1 坐标系及位移、内力正方向规定

$$
\left.\begin{aligned}
&\nabla^8 F+\frac{1-\mu^2}{k}\frac{\partial^4 F}{\partial\xi^4}=R^3\Big\{\nabla^4\Big[\int\frac{\partial^2 X}{\partial\varphi^2}d\xi+\int\frac{\partial^2 Y}{\partial\xi^2}d\varphi-\mu\Big(\frac{\partial X}{\partial\xi}+\frac{\partial Y}{\partial\varphi}\Big)\Big]\\
&\qquad+\frac{1-\mu^2}{k}\Big(\int\frac{\partial^2 Y}{\partial\xi^2}d\varphi-\frac{\partial^2 Z}{\partial\xi^2}\Big)\Big\}\\
&\nabla^8 w+\frac{1-\mu^2}{k}\frac{\partial^4 w}{\partial\xi^4}=\frac{(1-\mu^2)R^2}{Ehk}\Big\{\frac{\partial^3 X}{\partial\xi\partial\varphi^2}+\int\frac{\partial^4 Y}{\partial\xi^4}d\varphi-\mu\Big(\frac{\partial^3 X}{\partial\xi^3}+\frac{\partial^3 Y}{\partial\xi^2\partial\varphi}\Big)\\
&\qquad-\nabla^4\Big(\int Y d\varphi-Z\Big)\Big\}
\end{aligned}\right\}\quad(2.2)
$$

内力式为

$$
\left.\begin{aligned}
&N_x=\frac{\partial^2 F}{R^2\partial\varphi^2}-R\int X d\xi,\quad N_\varphi=\frac{\partial^2 F}{R^2\partial\xi^2}-R\int Y d\varphi\\
&S=-\frac{\partial^2 F}{R^2\partial\xi\partial\varphi},\quad M_x=-\frac{Ehk}{1-\mu^2}\Big(\frac{\partial^2 w}{\partial\xi^2}+\mu\frac{\partial^2 w}{\partial\varphi^2}\Big)\\
&M_\varphi=-\frac{Ehk}{1-\mu^2}\Big(\frac{\partial^2 w}{\partial\varphi^2}+\mu\frac{\partial^2 w}{\partial\xi^2}\Big),\quad M_{x\varphi}=M_{\varphi x}=-\frac{Ehk}{1+\mu}\frac{\partial^2 w}{\partial\xi\partial\varphi}\\
&Q_x=-\frac{Ehk}{1-\mu^2}\frac{\partial}{R\partial\xi}\nabla^2 w,\quad Q_\varphi=-\frac{Ehk}{1-\mu^2}\frac{\partial}{R\partial\varphi}\nabla^2 w\\
&\bar{Q}_\varphi=-\frac{Ehk}{(1-\mu^2)R}\Big[\frac{\partial}{\partial\varphi}\nabla^2 w+(1-\mu)\frac{\partial^3 w}{\partial\xi^2\partial\varphi}\Big]
\end{aligned}\right\}\quad(2.3)
$$

式中 $\nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\varphi^2},\quad k=\frac{h^2}{12R^2}$

三、局部竖向荷载时的解

我们只考虑横向端简支的情形，即$x=0$和$x=l$时满足边界条件：

$$M_x=0,\ N_x=0,\ v=0,\ w=0 \quad (3.1)$$

下面的解是自动满足这些边界条件的。

1. 齐次解

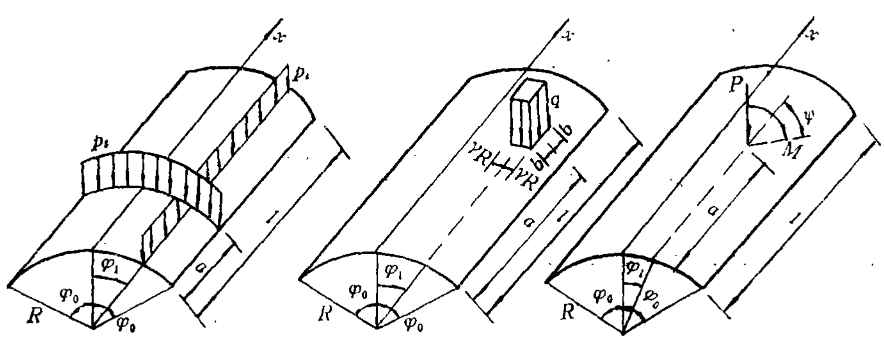
我们先由(2.2a)式求内力函数F的齐次解，然后由(2.1a)式求法向位移分量w的齐次解，进而按(2.3)式求内力，通过物理方程及几何方程可求得位移分量u，v及横向单元$Rd\varphi$的转角θ。为省篇幅，略去这些冗长的运算后，可以把这些内力及位移的齐次解写成统一的形式：

$$
\begin{aligned}
W=\sum_{n=1}^{\infty}\psi_n^W\Big\{\sum_{i=0}^{1}\big[&(C_{n,1+2i}\alpha_{n,1+i}^W-C_{n,2+2i}\beta_{n,1+i}^W)\phi_{n,1+2i}\\
&+(C_{n,1+2i}\beta_{n,1+i}^W+C_{n,2+2i}\alpha_{n,1+i}^W)\phi_{n,2+2i}\\
&+S_g(C_{n,5+2i}\alpha_{n,1+i}^W+C_{n,6+2i}\beta_{n,1+i}^W)\phi_{n,5+2i}\\
&+S_g(-C_{n,5+2i}\beta_{n,1+i}^W+C_{n,6+2i}\alpha_{n,1+i}^W)\phi_{n,6+2i}\big]\Big\}
\end{aligned}
\tag{3.2}
$$

式中W，ψ_n^W，α_{ni}^W，β_{ni}^W值列于表1内，C_{ni}为由$\varphi=\pm\varphi_0$处的纵向边界条件确定的积分常数，而

$$
\left.
\begin{aligned}
&\phi_{n,1+2i}=\exp\left[-\kappa_{n,1+i}\frac{\varphi}{\omega_n}\right]\sin\mu_{n,1+i}\frac{\varphi}{\omega_n}\\
&\phi_{n,5+2i}=\exp\left[\kappa_{n,1+i}\frac{\varphi}{\omega_n}\right]\sin\mu_{n,1+i}\frac{\varphi}{\omega_n}\\
&\phi_{n,2+2i}=\phi_{n,1+2i}\cot\mu_{n,1+i}\frac{\varphi}{\omega_n}\\
&\phi_{n,6+2i}=\phi_{n,5+2i}\cot\mu_{n,1+i}\frac{\varphi}{\omega_n}\\
&\kappa_{n,1+i}=\sqrt{\{\sqrt{[1+(-1)^i\varepsilon_n]^2+1}+(-1)^i[1+(-1)^i\varepsilon_n]\}/2}\\
&\mu_{n,1+i}=\sqrt{\{\sqrt{[1+(-1)^i\varepsilon_n]^2+1}-(-1)^i[1+(-1)^i\varepsilon_n]\}/2}\\
&\omega_n=\frac{\sqrt{\varepsilon_n}}{\alpha_n},\quad \varepsilon_n^4=\frac{4k\alpha_n^4}{1-\mu^2}\quad \alpha_n=\frac{n\pi R}{l},\quad i=0,1
\end{aligned}
\right\}
\tag{3.3}
$$

2. 特 解



(a) (b) (c)

图2 荷载类型

表1 齐次解的系数值

W	ψ_i^W	S_q	α_{n1}^W	β_{n1}^W	α_{n2}^W	β_{n2}^W
N_x	$\frac{1}{\varepsilon_n}\sin\alpha_n\xi$	+	$1+\varepsilon_n$	-1	$-(1-\varepsilon_n)$	-1
N_φ	$\sin\alpha_n\xi$	+	-1	0	-1	0
S	$\frac{1}{\sqrt{\varepsilon_n}}\cos\alpha_n\xi$	−	$+\kappa_{n1}$	$-\mu_{n1}$	$+\kappa_{n2}$	$-\mu_{n2}$
M_x	$\frac{\omega_n^2R}{2}\sin\alpha_n\xi$	+	$+\mu$	$-\varepsilon_n+\mu(1+\varepsilon_n)$	$-\mu$	$\varepsilon_n+\mu(1-\varepsilon_n)$
M_φ	$\frac{\omega_n^2R}{2}\sin\alpha_n\xi$	+	$+1$	$1+\varepsilon_n(1-\mu)$	-1	$1-\varepsilon_n(1-\mu)$
$M_{x\varphi}$	$\frac{(1-\mu)\omega_n^2R\sqrt{\varepsilon_n}}{2}\cos\alpha_n\xi$	−	$-\mu_{n1}$	$-\kappa_{n1}$	$+\mu_{n2}$	$+\kappa_{n2}$
Q_x	$\frac{\omega_n\sqrt{\varepsilon_n}}{2}\cos\alpha_n\xi$	+	$+1$	$+1$	-1	$+1$
Q_φ	$\frac{\omega_n}{2}\sin\alpha_n\xi$	−	$-(\kappa_{n1}+\mu_{n1})$	$-\kappa_{n1}+\mu_{n1}$	$+\kappa_{n2}-\mu_{n2}$	$-(\kappa_{n2}+\mu_{n2})$
$\overline{Q}_\varphi$	$\frac{\omega_n}{2}\sin\alpha_n\xi$	−	$-\kappa_{n1}-\mu_{n1}[1-\varepsilon_n(1-\mu)]$	$+\mu_{n1}-\kappa_{n1}[1-\varepsilon_n(1-\mu)]$	$+\kappa_{n2}-\mu_{n2}[1+\varepsilon_n(1-\mu)]$	$-\mu_{n2}-\kappa_{n2}[1+\varepsilon_n(1-\mu)]$
u	$\frac{R}{Eh\alpha_n\varepsilon_n}\cos\alpha_n\xi$	+	$-[1+\varepsilon_n(1+\mu)]$	$+1$	$1-\varepsilon_n(1+\mu)$	$+1$
v	$\frac{\omega_nR}{Eh\varepsilon_n^2}\sin\alpha_n\xi$	−	$-\kappa_{n1}[1-\varepsilon_n(1+\mu)]+\mu_{n1}$	$+\mu_{n1}[1-\varepsilon_n(1+\mu)]+\kappa_{n1}$	$+\kappa_{n2}[1+\varepsilon_n(1+\mu)]+\mu_{n2}$	$-\mu_{n2}[1+\varepsilon_n(1+\mu)]+\kappa_{n2}$
w	$\frac{2R}{Eh\varepsilon_n^2}\sin\alpha_n\xi$	+	0	-1	0	$+1$
θ	$\frac{\omega_n}{Eh\varepsilon_n^2}\sin\alpha_n\xi$	−	$-\kappa_{n1}[1-\varepsilon_n(1+\mu)]+\mu_{n1}\cdot\left(1+\frac{2}{\omega_n^2}\right)$	$+\mu_{n1}[1-\varepsilon_n(1+\mu)]+\kappa_{n1}\cdot\left(1+\frac{2}{\omega_n^2}\right)$	$+\kappa_{n2}[1+\varepsilon_n(1+\mu)]+\mu_{n2}\cdot\left(1-\frac{2}{\omega_n^2}\right)$	$-\mu_{n2}[1+\varepsilon_n(1+\mu)]+\kappa_{n2}\cdot\left(1-\frac{2}{\omega_n^2}\right)$

注： $\theta=\frac{1}{R}\left(\frac{\partial w}{\partial\varphi}+v\right)$

1）线荷载p_t(图2a)

我们先考虑在$x=a$弧上有沿曲线均布线荷载p_t的情形。将p_t展为三角级数，得荷载分量为：

$$\left.\begin{aligned}&X=0,\quad Y=\sum_{n=1}^{\infty}p_n\sin\varphi\sin\alpha_n\xi\\&Z=\sum_{n=1}^{\infty}p_n\cos\varphi\sin\alpha_n\xi,\quad p_n=\frac{2p_t}{l}\sin\frac{n\pi a}{l}\end{aligned}\right\}\tag{3.4}$$

取内力函数F及法向位移分量的特解为

$$F=R^2\sum_{n=1}^{\infty}F_n\cos\varphi\sin\alpha_n\xi,\quad Ehkw=\sum_{n=1}^{\infty}w_n\cos\varphi\sin\alpha_n\xi\tag{3.5}$$

将(3.4)、(3.5)或代入(2.2)式，求得

$$\left.\begin{aligned}F_n&=\frac{Rp_n}{\alpha_n K_n}\left[\frac{k\alpha_n}{1-\mu^2}(1+\alpha_n^2)^2(\alpha_n^2-\mu)+2\alpha_n^3\right]\\w_n&=\frac{R^2kp_n}{K_n}[2+(4+\mu)\alpha_n^2+\alpha_n^4]\\K_n&=(1+\alpha_n^2)^4\frac{k}{1-\mu^2}+\alpha_n^4\end{aligned}\right\}\tag{3.6}$$

然后，按齐次解相同的步骤，得内力及位移分量的特解值为：

$$\left.\begin{aligned}N_x&=-\sum_{n=1}^{\infty}F_n\cos\varphi\sin\alpha_n\xi\\N_\varphi&=-\sum_{n=1}^{\infty}(\alpha_n^2F_n-Rp_n)\cos\varphi\sin\alpha_n\xi\\S&=\sum_{n=1}^{\infty}\alpha_nF_n\sin\varphi\cos\alpha_n\xi\\M_x&=\frac{1}{1-\mu^2}\sum_{n=1}^{\infty}w_n(\alpha_n^2+\mu)\cos\varphi\sin\alpha_n\xi\\M_\varphi&=\frac{1}{1-\mu^2}\sum_{n=1}^{\infty}w_n(1+\mu\alpha_n^2)\cos\varphi\sin\alpha_n\xi\\M_{x\varphi}&=M_{\varphi x}=\frac{1}{1+\mu}\sum_{n=1}^{\infty}\alpha_nw_n\sin\varphi\cos\alpha_n\xi\\Q_x&=\frac{1}{(1-\mu^2)R}\sum_{n=1}^{\infty}w_n\alpha_n(1+\alpha_n^2)\cos\varphi\cos\alpha_n\xi\\Q_\varphi&=-\frac{1}{(1-\mu^2)R}\sum_{n=1}^{\infty}w_n(1+\alpha_n^2)\sin\varphi\sin\alpha_n\xi\\\bar{Q}_\varphi&=-\frac{1}{(1-\mu^2)R}\sum_{n=1}^{\infty}w_n[1+(2-\mu)\alpha_n^2]\sin\varphi\sin\alpha_n\xi\\u&=\frac{R}{Eh}\sum_{n=1}^{\infty}\frac{1}{\alpha_n}[(1-\mu\alpha_n^2)F_n+\mu p_nR]\cos\varphi\cos\alpha_n\xi\\v&=\frac{R}{Eh}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\{[1+(2+\mu)\alpha_n^2]F_n+\mu Rp_n\}\sin\varphi\sin\alpha_n\xi\\w&=\frac{1}{Ehk}\sum_{n=1}^{\infty}w_n\cos\varphi\sin\alpha_n\xi\end{aligned}\right\}\tag{3.7}$$

$$\theta=\frac{1}{Eh}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^3}\left\{-\frac{w_n\alpha_n^2}{Rk}+[(2+\mu)\alpha_n^2+1]F_n+\mu p_nR\right\}\sin\varphi\sin\alpha_n\xi$$

2) 局部均布荷载（图2b）

我们只考虑局部竖向面荷载作用在图2(b)所示矩形域内的情形。把荷载分解为对称及反对称于x轴的两部分，并将其荷载分量展为三角级数，迭加后得荷载分量为：

$$\left.\begin{aligned}&X=0\\&Y=\sum_{n=1}^{\infty}\left[\frac{y_{0n}^A}{2}+\sum_{m=1}^{\infty}(y_{mn}^A\cos\beta_m\varphi+y_{mn}^S\sin\beta_m\varphi)\right]\sin\alpha_n\xi\\&Z=\sum_{n=1}^{\infty}\left[\frac{z_{0n}^S}{2}+\sum_{m=1}^{\infty}(z_{mn}^S\cos\beta_m\varphi+z_{mn}^A\sin\beta_m\varphi)\right]\sin\alpha_n\xi\end{aligned}\right\}\tag{3.8}$$

式中

$$\left.\begin{aligned}&y_{mn}^S=q_n(B_m^{-1}\cos B_m\varphi_1\sin B_m\gamma-A_m^{-1}\cos A_m\varphi_1\sin A_m\gamma)\sin\alpha_n\xi_1\sin\alpha_n\xi_2\\&z_{mn}^S=q_n(A_m^{-1}\cos A_m\varphi_1\sin A_m\gamma+B_m^{-1}\cos B_m\varphi_1\sin B_m\gamma)\sin\alpha_n\xi_1\sin\alpha_n\xi_2\\&y_{mn}^A=q_n(A_m^{-1}\sin A_m\varphi_1\sin A_m\gamma+B_m^{-1}\sin B_m\varphi_1\sin B_m\gamma)\sin\alpha_n\xi_1\sin\alpha_n\xi_2\\&z_{mn}^A=q_n(A_m^{-1}\sin A_m\varphi_1\sin A_m\gamma-B_m^{-1}\sin B_m\varphi_1\sin B_m\gamma)\sin\alpha_n\xi_1\sin\alpha_n\xi_2\\&q_n=4q/n\pi\varphi_0,\ A_m=1+\beta_m,\ B_m=1-\beta_m\\&\beta_m=m\pi/\varphi_0,\ \xi_1=a/R,\ \xi_2=b/R\\&n=1,2,3,\cdots;\ m=\begin{cases}1,2,3\cdots & (\text{用于}y_{mn}^S,\ z_{mn}^A)\\0,1,2,3,\cdots & (\text{用于}y_{nm}^A,\ z_{mn}^S)\end{cases}\end{aligned}\right\}\tag{3.9}$$

代(3.8)式入(2.2)式，便可以求得F及w的特解，进而得内力及位移分量的特解为：

$$N_x=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_m^2(F_{mn}^S\cos\beta_m\varphi+F_{mn}^A\sin\beta_m\varphi)\sin\alpha_n\xi$$

$$N_\varphi=-\sum_{n=1}^{\infty}\left\{\alpha_n^2F_{0n}+\sum_{m=1}^{\infty}\left[\left(\alpha_n^2F_{mn}^S-\frac{R}{\beta_m}y_{mn}^S\right)\cos\beta_m\varphi+\left(\alpha_n^2F_{mn}^A+\frac{R}{\beta_m}y_{mn}^A\right)\sin\beta_m\varphi\right]\right\}\sin\alpha_n\xi$$

$$S=\sum_{n=1}^{\infty}\left[\frac{R}{2\alpha_n}y_{0n}^A+\alpha_n\sum_{m=1}^{\infty}\beta_m(F_{mn}^S\sin\beta_m\varphi-F_{mn}^A\cos\beta_m\varphi)\right]\cos\alpha_n\xi$$

$$M_x=\frac{1}{1-\mu^2}\sum_{n=1}^{\infty}\left\{\alpha_n^2w_{0n}+\sum_{m=1}^{\infty}(\alpha_n^2+\mu\beta_m^2)\cdot(w_{mn}^S\cos\beta_m\varphi+w_{mn}^A\sin\beta_m\varphi)\right\}\sin\alpha_n\xi$$

$$
\left.
\begin{aligned}
M_{\varphi} &= \frac{1}{1-\mu^{2}}\sum_{n=1}^{\infty}\Big[\mu\alpha_n^{2}w_{0n}+\sum_{m=1}^{\infty}(\beta_m^{2}+\mu\alpha_n^{2}) \\
&\qquad\cdot(w_{mn}^{S}\cos\beta_m\varphi+w_{mn}^{A}\sin\beta_m\varphi)\Big]\sin\alpha_n\xi \\
M_{x\varphi} &= M_{\varphi x}=\frac{1}{1+\mu}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\beta_m(w_{mn}^{S}\sin\beta_m\varphi-w_{mn}^{A}\cos\beta_m\varphi)\cos\alpha_n\xi \\
Q_x &= \frac{1}{(1-\mu^{2})R}\sum_{n=1}^{\infty}\Big[\alpha_n^{3}w_{0n}+\sum_{m=1}^{\infty}\alpha_n(\alpha_n^{2}+\beta_m^{2}) \\
&\qquad\cdot(w_{mn}^{S}\cos\beta_m\varphi+w_{mn}^{A}\sin\beta_m\varphi)\Big]\cos\alpha_n\xi \\
Q_{\varphi} &= \frac{1}{(1-\mu^{2})R}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_m(\alpha_n^{2}+\beta_m^{2}) \\
&\qquad\cdot(-w_{mn}^{S}\sin\beta_m\varphi+w_{mn}^{A}\cos\beta_m\varphi)\sin\alpha_n\xi \\
\bar{Q}_{\varphi} &= \frac{1}{(1-\mu^{2})R}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_m[(2-\mu)\alpha_n^{2}+\beta_m^{2}] \\
&\qquad\cdot(-w_{mn}^{S}\sin\beta_m\varphi+w_{mn}^{A}\cos\beta_m\varphi)\sin\alpha_n\xi \\
u &= \frac{R}{Eh}\sum_{n=1}^{\infty}\Big\{-\mu\alpha_nF_{0n}+\frac{1}{\alpha_n}\sum_{m=1}^{\infty}\Big[\Big([\beta_m^{2}-\mu\alpha_n^{2}]F_{mn}^{S} \\
&\qquad+\frac{\mu R}{\beta_m}y_{mn}^{S}\Big)\cos\beta_m\varphi+\Big([\beta_m^{2}-\mu\alpha_m^{2}]F_{mn}^{A} \\
&\qquad-\frac{\mu R}{\beta_m}y_{mn}^{A}\Big)\sin\beta_m\varphi\Big]\Big\}\cos\alpha_n\xi \\
v &= \frac{R}{Eh}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^{2}}\Big\{(1+\mu)Ry_{0n}^{A}+\sum_{m=1}^{\infty}[(\beta_m[(2+\mu)\alpha_n^{2} \\
&\qquad+\beta_m^{2}]F_{mn}^{S}+\mu Ry_{mn}^{S})\sin\beta_m\varphi-(\beta_m[(2+\mu)\alpha_n^{2} \\
&\qquad+\beta_m^{2}]F_{mn}^{A}-\mu Ry_{mn}^{A})\cos\beta_m y]\Big\}\sin\alpha_n\xi \\
w &= \frac{1}{Ehk}\sum_{n=1}^{\infty}\Big[w_{0n}+\sum_{m=1}^{\infty}(w_{mn}^{S}\cos\beta_m\varphi+w_{mn}^{A}\sin\beta_m\varphi)\Big]\sin\alpha_n\xi \\
\theta &= \frac{1}{Eh}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^{2}}\Big\{(1+\mu)Ry_{0n}^{A}+\sum_{m=1}^{\infty}\Big[\Big(-\frac{\alpha_n^{2}\beta_m}{kR}w_{mn}^{S} \\
&\qquad+[(2+\mu)\alpha_n^{2}+\beta_m^{2}]\beta_mF_{mn}^{S}+\mu Ry_{mn}^{S}\Big)\sin\beta_m\varphi \\
&\qquad+\Big(\frac{\alpha_n^{2}\beta_m}{kR}w_{mn}^{A}-\beta_m[(2+\mu)\alpha_n^{2}+\beta_m^{2}]F_{mn}^{A} \\
&\qquad+\mu Ry_{mn}^{A}\Big)\cos\beta_m\varphi\Big]\Big\}\sin\alpha_n\xi
\end{aligned}
\right\}
\tag{3.10}
$$

式中

$$
\left.\begin{aligned}
&F_{0n}=\frac{1-\mu^2}{\alpha_n^4+(1-\mu^2)/k}\frac{Rz_{0n}^S}{2\alpha_n^2 k}\text{ , } w_{0n}=Rk\alpha_n^2F_{0n}\\
&K_n=(\alpha_n^2+\beta_m^2)^4\frac{k}{1+\mu^2}+\alpha_n^4\\
&F_{mn}^S=\frac{R}{\alpha_n\beta_m K_n}\left[\frac{k\alpha_n}{1-\mu^2}(\alpha_n^2+\beta_m^2)^2(\alpha_n^2-\mu\beta_m^2)y_{mn}^S+\alpha_n^2(y_{mn}^S+\beta_m z_{mn}^S)\right]\\
&w_{mn}^S=\frac{R^2k}{K_n}\{\beta_m[(2+\mu)\alpha_n^2+\beta_m^2]y_{mn}^S+(\alpha_n^2+\beta_m^2)^2z_{mn}^S\}\\
&F_{mn}^A=\frac{R}{\alpha_n\beta_m K_n}\left\{\frac{k\alpha_n}{1-\mu^2}(\alpha_n^2+\beta_m^2)^2(\mu\beta_m^2-\alpha_n^2)y_{mn}^A-\alpha_n^2(y_{mn}^A-\beta_m z_{mn}^A)\right\}\\
&w_{mn}^A=\frac{R^2k}{K_n}\{-\beta_m[(2+\mu)\alpha_n^2+\beta_m^2]y_{mn}^A+(\alpha_n^2+\beta_m^2)^2z_{mn}^A\}
\end{aligned}\right\}\quad(3.11)
$$

（3）集中力P（图2c）

我们可以把这种情形看作为图2(b)示局部均布荷载在 $4\gamma Rbq\to P$，而$b\to 0$，$\gamma\to 0$的极限情形。因此，把(3.8)～(3.11)式作上述极限运算后，便可以得到集中荷载P作用在(a,φ_1)点处时的特解。经过运算后，公式(3.8)、(3.10)及(3.11)的形式不变，唯(3.9)式变为：

$$
\left.\begin{aligned}
&y_{mn}^S=\frac{2P}{\varphi_0 Rl}\sin\varphi_1\sin\beta_m\varphi_1\sin\alpha_n\xi_1\\
&z_{mn}^S=\frac{2P}{\varphi_0 Rl}\cos\varphi_1\cos\beta_m\varphi_1\sin\alpha_n\xi_1\\
&y_{mn}^A=\frac{2P}{\varphi_0 Rl}\sin\varphi_1\cos\beta_m\varphi_1\sin\alpha_n\xi_1\\
&z_{mn}^A=\frac{2P}{\varphi_0 Rl}\cos\varphi_1\sin\beta_m\varphi_1\sin\alpha_n\xi_1\\
&n=1,2,3,\cdots\\
&m=\begin{cases}1,2,3,\cdots & (\text{用于}y_{mn}^S,z_{mn}^A)\\ 0,1,2,3,\cdots, & (\text{用于}y_{mn}^A,z_{mn}^S)\end{cases}
\end{aligned}\right\}\quad(3.12)
$$

（4）集中力偶M(图2c)

设集中力偶M的作用点为(a,φ_1)，作用面与x轴成ψ角。我们可以把这个力偶看作为相距Δs，大小为$M\Delta s$ 而方向相反的两个集中力，在 $\Delta s\to 0$ 时的极限情形。这样，利用集中力的结果，作极限运算后，不难求得特解。这时公式 (3.8)、(3.10) 及 (3.11)的形式不 变，而(3.12)式变为：

$$
\begin{aligned}
&y_{mn}^S=\frac{2M}{\varphi_0 R^2 l}[\alpha_n\sin\varphi_1\sin\beta_m\varphi_1\cos\alpha_n\xi_1\cos\psi\\
&\qquad+(\cos\varphi_1\sin\beta_m\varphi_1+\beta_m\sin\varphi_1\cos\beta_m\varphi_1)\sin\alpha_n\xi_1\sin\psi]\\
&z_{mn}^S=\frac{2M}{\varphi_0 R^2 l}[\alpha_n\cos\varphi_1\cos\beta_m\varphi_1\cos\alpha_n\xi_1\cos\psi\\
&\qquad-(\sin\varphi_1\cos\beta_m\varphi_1+\beta_m\cos\varphi_1\sin\beta_m\varphi_1)\sin\alpha_n\xi_1\sin\psi]
\end{aligned}
$$

$$
\left.\begin{aligned}
y^A_{mn}=&\frac{2M}{\varphi_0 R^2 l}[\alpha_n\sin\varphi_1\cos\beta_m\varphi_1\cos\alpha_n\xi_1\cos\psi\\
&+(\cos\varphi_1\cos\beta_m\varphi_1-\beta_m\sin\varphi_1\sin\beta_m\varphi_1)\sin\alpha_n\xi_1\sin\psi]\\
z^A_{mn}=&\frac{2M}{\varphi_0 R^2 l}[\alpha_n\cos\varphi_1\sin\beta_m\varphi_1\cos\alpha_n\xi_1\cos\psi\\
&-(\sin\varphi_1\sin\beta_m\varphi_1-\beta_m\cos\varphi_1\cos\beta_m\varphi_1)\sin\alpha_n\xi_1\sin\psi]\\
n=&1,2,3,\cdots;\\
m=&\begin{cases}1,2,3\cdots & (\text{用于}y^S_{mn},z^A_{mn})\\ 0,1,2,3,\cdots & (\text{用于}y^A_{mn},z^S_{mn})\end{cases}
\end{aligned}\right\} \quad (3.13)
$$

（5）线荷载p_l(图2a)

我们可以把这种情形看作为图2(b)所示局部均布荷载在$a=b=l/2$，$2\gamma Rq\to p_l$，$\gamma\to 0$的极限情形。运算后，公式(3.8)、(3.10)及(3.11) 的形式不变，唯 (3.9) 式变为：

$$
\left.\begin{aligned}
&y^S_{mn}=\frac{4p_l}{n\pi R\varphi_0}\sin\varphi_1\sin\beta_m\varphi_1,\quad z^S_{mn}=\frac{4p_l}{n\pi R\varphi_0}\cos\varphi_1\cos\beta_m\varphi_1\\
&y^A_{mn}=\frac{4p_l}{n\pi R\varphi_0}\sin\varphi_1\cos\beta_m\varphi_1,\quad z^A_{mn}=\frac{4p_l}{n\pi R\varphi_0}\cos\varphi_1\sin\beta_m\varphi_1\\
&n=1,3,5,\cdots;\\
&m=\begin{cases}1,2,3,\cdots & (\text{用于}y^S_{mn},z^A_{mn})\\ 0,1,2,3,\cdots & (\text{用于}y^A_{mn},z^S_{mn})\end{cases}
\end{aligned}\right| \quad (3.14)
$$

由于(3.14)式中$n=1,3,5,\cdots$，这时(3.8)、(3.10)及(3.11)式，虽然形式不变，但各式中的n亦应取$n=1,3,5,\cdots$.

四、数 值 例 题

下面我们研究图3所示横向端简支，纵向边自由，在中央处受集中荷载$P=1$t时的圆柱形薄壳屋顶的计算。这是应用本文结果的一个最简单例题。基本参数为跨度$l=20$m，半径$R=10$m，厚度$h=0.08$m，圆心角$2\varphi_0=100°$。材料弹性模量$E=3\times10^5$t/m^2，泊松比$\mu=1/6$。这时的荷载作用点坐标为$a=10$m及$\varphi_1=0°$。

主要计算结果如图4所示。从这些结果可以看出，在集中荷载作用点及其附近的内力很大，而且弯矩衰减很快，迅速减少。计算表明，在本例中，一般只取级数项$n=15$，$m=30$便能得到满意的结果，但是在集中力作用点及其附近的内力例外，级数收敛性要慢一些，须取$n=70$及$m=50$才能得到满意的结果。

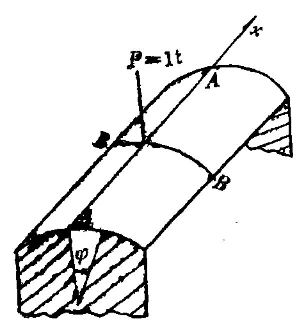
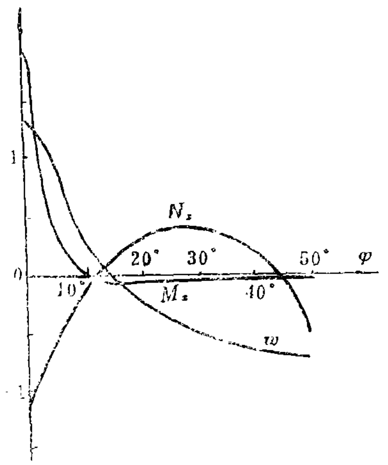


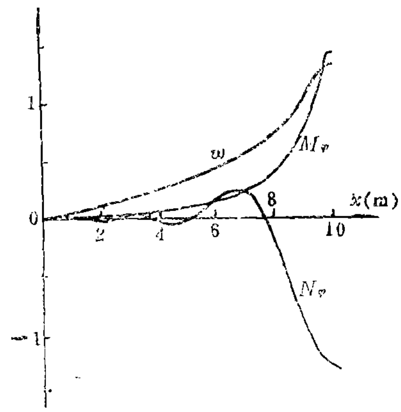
图3 数值例题图

五、结 束 语

本文用解析法所导得的横向端简支而纵向边为任意约束的圆柱形薄壳，在任意处受五种局部竖向荷载作用时的内力及位移式，形式简洁，处理严密，基本上消除了通常集中荷载作用时易于出现的奇性。所得公式易于实现计算程序，在一般微机上即可计算，收敛良好。



(a)　$x=10$m截面



(b)　$\varphi=0°$截面

图4　$x=10$m及$\varphi=0°$截面的内力及法向位移分量变化图(N_x, N_φ—2000kg/m, M_x, M_φ—200kg-m/m, w—mm)

本文既解决了竖向局部荷载作用下圆柱形薄壳屋顶结构的计算问题，又为处理各种组合圆柱形薄壳的空间分析，提供了必要的理论依据。因此，本文在理论上是有一定价值的，在工程上是有实际意义的。

感谢卢平同志为本文完成了部分数值计算工作。

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Calculation for Cylindrical Shell under Local Vertical Loadings

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Abstract

In this paper, we use the method of mixed-type series to derive the analytical solutions of cylindrical shell, which is simply supported along the transverse edges and subjected to the local vertical loads, and give the analytical expressions of the solutions for this kind of shell under five types of local vertical loading. A numerical example for a cylindrical shell roof, which is simply supported along the transverse edges and is free along the longitudinal edges, is given in this paper and from the calculated results it may be seen that the convergence of the solutions is considerably satisfactory. Using the solutions of this paper, we can deal with some practical problems of underground structure.