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中厚板的弹性屈曲和后屈曲

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摘要

本文采用Reissner假定考虑横向剪切变形的影响，导出弹性矩形板大挠度方程．

本文讨论考虑横向剪切变形的矩形板的弹性屈曲和后屈曲．采用文[8]提供的摄动方法，给出了完善和非完善中厚板的后屈曲平衡路径，并与经典薄板理论结果进行了比较．

一、引言

经典的弹性薄板理论忽略了横向剪切变形的影响，由此导出Kármán板大挠度方程．对于矩形板在面内压缩作用下的屈曲和后屈曲已作过诸多研究，这些研究大多基于经典薄板理论．Ziegler[1]业已指出，对于各向同性板，在发生屈曲之前，面内变形效应在数量级上是与横向剪切变形属同一量级．因此，在讨论板屈曲问题时，考虑横向剪切变形效应引起的附加挠曲，将得到更低的屈曲载荷．当板较薄时，经典薄板理论具有足够好的精度，然而，经典薄板理论所造成的误差将随着板厚的增加而增加．近20年来，对于把剪切变形效应结合到板屈曲分析中已得到了相当大的重视[4~7]，但是，对于考虑剪切变形效应的非完善矩形板的后屈曲分析尚未见到发表．

本文采用Reissner假定，考虑横向剪切变形的影响，导出矩形板大挠度方程．采用文[8]提供的摄动方法，以挠度为摄动参数，研究四边简支中厚板的弹性屈曲和后屈曲，而对于中厚板，板厚的影响是很明显的．

本文讨论了两种面内边界条件，一种为纵向边缘可移简支，一种为纵向边缘不可移简支．本文同时考虑了板初挠度（初始几何缺陷）的影响．初挠度的形式取作和矩形板小挠度解的形式一致．

二、基本方程

假定四边简支矩形板的长为a，宽为b，厚度为t，受到面内单向均匀压缩．取坐标系如图1所示．采用Reissner假定[2,3]，我们有

* 卢文达推荐．

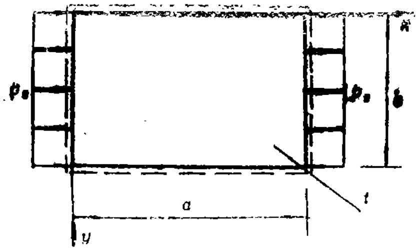

图1 矩形板受单向压缩作用

$$\sigma_x=\frac{M_x}{t^2/6}\ \frac{z}{t/2},\quad \sigma_y=\frac{M_y}{t^2/6}\ \frac{z}{t/2},\quad \tau_{xy}=\frac{M_{xy}}{t^2/6}\ \frac{z}{t/2} \tag{2.1a}$$

$$\sigma_z=-\frac{3}{4}\left(\frac{\partial N_{xz}}{\partial x}+\frac{\partial N_{yz}}{\partial y}\right)\left[\frac{z}{t/2}-\frac{1}{3}\left(\frac{z}{t/2}\right)^3\right] \tag{2.1b}$$

$$\tau_{xz}=\frac{N_{xz}}{2t/3}\left[1-\left(\frac{z}{t/2}\right)^2\right],\quad \tau_{yz}=\frac{N_{yz}}{2t/3}\left[1-\left(\frac{z}{t/2}\right)^2\right] \tag{2.1c}$$

及

$$\left.\begin{aligned}M_x&=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)+\frac{t^2}{5(1-\nu)}\left(\frac{\partial N_{xz}}{\partial x}+\nu\frac{\partial N_{yz}}{\partial y}\right)\\ M_y&=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)+\frac{t^2}{5(1-\nu)}\left(\frac{\partial N_{yz}}{\partial y}+\nu\frac{\partial N_{xz}}{\partial x}\right)\\ M_{xy}&=-(1-\nu)\,D\frac{\partial^2 W}{\partial x\partial y}+\frac{t^2}{10}\left(\frac{\partial N_{xz}}{\partial y}+\frac{\partial N_{yz}}{\partial x}\right)\end{aligned}\right\} \tag{2.2}$$

其中 $D=Et^3/12(1-\nu^2)$ 为抗弯刚度，E和 ν 分别为弹性模数和 Poisson 比。

当讨论屈曲问题时，我们有如下平衡方程组

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0 \tag{2.3a}$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0 \tag{2.3b}$$

$$\frac{\partial N_{xz}}{\partial x}+\frac{\partial N_{yz}}{\partial y}+N_x\frac{\partial^2 W}{\partial x^2}+2N_{xy}\frac{\partial^2 W}{\partial x\partial y}+N_y\frac{\partial^2 W}{\partial y^2}=0 \tag{2.3c}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-N_{xz}=0 \tag{2.3d}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-N_{yz}=0 \tag{2.3e}$$

引进应力函数 ϕ，且让

$$N_x=\frac{\partial^2\phi}{\partial y^2},\quad N_y=\frac{\partial^2\phi}{\partial x^2},\quad N_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \tag{2.4}$$

那么，方程(2.3a)、(2.3b)自动满足。将式(2.2)，(2.3d)，(2.3e)和(2.4)代入(2.3c)，

我们得到

$$D\nabla^4 W=\left[1-\frac{t^2}{5(1-\nu)}\nabla^2\right]\left[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right] \tag{2.5}$$

协调方程仍为

$$\nabla^4\phi=Et\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right] \tag{2.6}$$

式(2.5)和(2.6)即为考虑横向剪切变形的矩形板非线性大挠度方程。

对于非完善矩形板，以W^*和W分别表示初始的和附加的挠度，那么，方程(2.5)和(2.6)化为

$$D\nabla^4 W=\left[1-\frac{t^2}{5(1-\nu)}\nabla^2\right]\left[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}\right] \tag{2.7}$$

$$\nabla^4\phi=Et\left[\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\right] \tag{2.8}$$

假定边界支承为四边简支的，那么边界条件可表为

$$x=0,\ a;\ W=0,\ M_x=0,\ N_{xy}=0 \tag{2.9a}$$

$$\int_0^b N_x dy+p_x=0 \tag{2.9b}$$

$$y=0,\ b;\ W=0,\ M_y=0,\ N_{xy}=0 \tag{2.10a}$$

$$\int_0^a N_y dx=0 \qquad \text{(纵边可移简支)} \tag{2.10b}$$

$$V=\text{const} \qquad \text{(纵边不可移简支)} \tag{2.10c}$$

单位轴向缩短为

$$\begin{aligned}\frac{\Delta_x}{a}&=-\frac{1}{abt}\int_{-t/2}^{t/2}\int_0^b\int_0^a\frac{\partial U}{\partial x}dxdydz\\&=-\frac{1}{ab}\int_0^b\int_0^a\left[\frac{1}{Et}\left(\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2-\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\right]dxdy\end{aligned} \tag{2.11}$$

$$\frac{\Delta_y}{b}=-\frac{1}{abt}\int_{-t/2}^{t/2}\int_0^a\int_0^b\frac{\partial V}{\partial y}dydxdz$$

$$
=-\frac{1}{ab}\int_0^a\int_0^b\left[\frac{1}{Et}\left(\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)-\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2-\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\right]dydx \tag{2.12}
$$

方程(2.7)至(2.12)即为四边简支，考虑横向剪切变形的矩形板屈曲问题的控制方程。

三、分析方法与渐近解

引进

$$
\left.\begin{aligned}
&\bar{x}=\frac{\pi}{a}x,\quad \bar{y}=\frac{\pi}{b}y,\quad \beta=\frac{a}{b},\quad w=\frac{W}{t}\sqrt{12(1-\nu^2)}\\
&w^*=\frac{W^*}{t}\sqrt{12(1-\nu^2)},\quad \varphi=\frac{\phi}{D}\\
&\gamma^2=\frac{1}{5(1-\nu)}\frac{\pi^2}{\beta^2(b/t)^2},\quad m_x=\frac{M_xa^2\sqrt{12(1-\nu^2)}}{\pi^2Dt}\\
&m_y=\frac{M_ya^2\sqrt{12(1-\nu^2)}}{\pi^2Dt}\\
&\lambda_x=\frac{\sigma_xb^2t}{4\pi^2D},\quad \delta_x=\frac{12(1-\nu^2)}{4\pi^2}\frac{b^2}{t^2}\frac{\Delta_x}{a},\quad \delta_y=\frac{12(1-\nu^2)}{4\pi^2}\frac{b^2}{t^2}\frac{\Delta_y}{b}
\end{aligned}\right\} \tag{3.1}
$$

那么，方程(2.7)、(2.8)可表为如下无量纲形式（略去字母上标“—”）：

$$
\begin{aligned}
\bar{\nabla}^4w=\beta^2(1-\gamma^2\bar{\nabla}^2)\Big[&\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2w}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2w}{\partial y^2}\\
&+\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2w^*}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2w^*}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2w^*}{\partial y^2}\Big]
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
\bar{\nabla}^4\varphi=\beta^2\Big[&\left(\frac{\partial^2w}{\partial x\partial y}\right)^2-\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}+2\frac{\partial^2w}{\partial x\partial y}\frac{\partial^2w^*}{\partial x\partial y}\\
&-\frac{\partial^2w}{\partial x^2}\frac{\partial^2w^*}{\partial y^2}-\frac{\partial^2w}{\partial y^2}\frac{\partial^2w^*}{\partial x^2}\Big]
\end{aligned} \tag{3.3}
$$

其中

$$
\bar{\nabla}^4=\frac{\partial^4}{\partial x^4}+2\beta^2\frac{\partial^4}{\partial x^2\partial y^2}+\beta^4\frac{\partial^4}{\partial y^4} \tag{3.4a}
$$

$$
\bar{\nabla}^2=\frac{\partial^2}{\partial x^2}+\beta^2\frac{\partial^2}{\partial y^2} \tag{3.4b}
$$

边界条件化为

$$
x=0,\ \pi;\quad w=0,\quad m_x=0,\quad \varphi_{,xy}=0 \tag{3.5a}
$$

$$
\frac{1}{\pi}\int_0^\pi\beta^2\frac{\partial^2\varphi}{\partial y^2}dy+4\lambda_x\beta^2=0 \tag{3.5b}
$$

$$
y=0,\ \pi;\quad w=0,\quad m_y=0,\quad \varphi_{,xy}=0 \tag{3.6a}
$$

$$\frac{1}{\pi}\int_0^\pi \frac{\partial^2\varphi}{\partial x^2}dx=0 \qquad (\text{纵边可移简支}) \tag{3.6b}$$

$$\delta_y=0 \qquad (\text{纵边不可移简支}) \tag{3.6c}$$

单位轴向缩短化为

$$\delta_x=-\frac{1}{4\pi^2\beta^2}\int_0^\pi\int_0^\pi\left[\left(\beta^2\frac{\partial^2\varphi}{\partial y^2}-\nu\frac{\partial^2\varphi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\right]dxdy \tag{3.7}$$

$$\delta_y=-\frac{1}{4\pi^2\beta^2}\int_0^\pi\int_0^\pi\left[\left(\frac{\partial^2\varphi}{\partial x^2}-\nu\beta^2\frac{\partial^2\varphi}{\partial y^2}\right)-\frac{1}{2}\beta^2\left(\frac{\partial w}{\partial y}\right)^2-\beta^2\frac{\partial w}{\partial y}\frac{\partial w^*}{\partial y}\right]dydx \tag{3.8}$$

设方程(3.2)、(3.3)的解为如下渐近展开式

$$w(x,y,\varepsilon)=\sum_{k=1}\varepsilon^k w_k(x,y),\quad \varphi(x,y,\varepsilon)=\sum_{k=0}\varepsilon^k\varphi_k(x,y) \tag{3.9}$$

并取板的初挠度为

$$w^*(x,y,\varepsilon)=\varepsilon A_{11}^*\sin mx\sin ny=\varepsilon\mu A_{11}^{(1)}\sin mx\sin ny \tag{3.10}$$

将式(3.9)、(3.10)代入方程(3.2)、(3.3)便可获得各级摄动方程，采用文[8]类似的摄动步骤，我们可以得到大挠度渐近解

$$w=\varepsilon[A_{11}^{(1)}\sin mx\sin ny]+\varepsilon^3[A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny]+O(\varepsilon^5) \tag{3.11}$$

$$\begin{aligned}\varphi=&-B_{00}^{(0)}\frac{y^2}{2}-b_{00}^{(0)}\frac{x^2}{2}+\varepsilon^2\left[-B_{00}^{(2)}\frac{y^2}{2}-b_{00}^{(2)}\frac{x^2}{2}+B_{20}^{(2)}\cos 2mx\right.\\&\left.+B_{02}^{(2)}\cos 2ny\right]+\varepsilon^4\left[-B_{00}^{(4)}\frac{y^2}{2}-b_{00}^{(4)}\frac{x^2}{2}+B_{20}^{(4)}\cos 2mx\right.\\&+B_{02}^{(4)}\cos 2ny+B_{22}^{(4)}\cos 2mx\cos 2ny+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny\\&\left.+B_{24}^{(4)}\cos 2mx\cos 4ny+B_{42}^{(4)}\cos 4mx\cos 2ny\right]+O(\varepsilon^6)\end{aligned} \tag{3.12}$$

式中 $B_{00}^{(k)}$ 和 $b_{00}^{(k)}$ $(k=0,2,4,\cdots)$ 的关系为

$$\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}n^2\beta^2=\frac{(m^2+n^2\beta^2)^2}{(1+\mu)[1+\gamma^2(m^2+n^2\beta^2)]} \tag{3.13a}$$

$$\beta^2B_{00}^{(2)}m^2+b_{00}^{(2)}n^2\beta^2=\frac{1}{16}(m^4+n^4\beta^4)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)} \tag{3.13b}$$

$$\beta^2 B_{00}^{(4)} m^2 + b_{00}^{(4)} n^2\beta^2 = -\frac{1}{256}(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\left\langle \frac{m^8}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)] + \frac{n^8\beta^8}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)]\right\rangle$$

$$\cdot A_{11}^{(1)} A_{11}^{(1)} A_{11}^{(1)} A_{11}^{(1)} \tag{3.13c}$$

其它系数皆可表为 $A_{11}^{(1)}$ 的函数，如

$$\left.\begin{aligned}
B_{20}^{(2)} &= \frac{1}{32}\frac{n^2\beta^2}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\\
B_{02}^{(2)} &= \frac{1}{32}\frac{m^2}{n^2\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\\
A_{13}^{(3)} &= \frac{1}{16}\frac{m^4}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)](1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
A_{31}^{(3)} &= \frac{1}{16}\frac{n^4\beta^4}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)](1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{20}^{(4)} &= -\frac{1}{256}\frac{n^2\beta^2}{m^2}\frac{n^4\beta^4}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)](1+\mu)^2\\
&\quad\cdot(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{02}^{(4)} &= -\frac{1}{256}\frac{m^2}{n^2\beta^2}\frac{m^4}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)](1+\mu)^2\\
&\quad\cdot(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}
\end{aligned}\right\} \tag{3.14}$$

其中

$$\left.\begin{aligned}
g_{13} &= (m^2+9n^2\beta^2)^2 - (\beta^2 B_{00}^{(0)} m^2 + b_{00}^{(0)} 9n^2\beta^2)[1+\gamma^2(m^2+9n^2\beta^2)]\\
g_{31} &= (9m^2+n^2\beta^2)^2 - (\beta^2 B_{00}^{(0)} 9m^2 + b_{00}^{(0)} n^2\beta^2)[1+\gamma^2(9m^2+n^2\beta^2)]
\end{aligned}\right\} \tag{3.15}$$

进一步利用边界条件(3.5b)和(3.6b)或(3.6c)，我们可以得到以最大无量纲挠度为摄动参数的后屈曲平衡路径，它们是

（1） 纵向边缘可移简支

此时

$$b_{00}^{(k)} = 0 \qquad (k=0,\ 2,\ 4,\ \cdots) \tag{3.16}$$

因而导得

$$\lambda_x = \frac{1}{4\beta^2}\left\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)m^2[1+\gamma^2(m^2+n^2\beta^2)]} + \frac{1}{16}\frac{m^4+n^4\beta^4}{m^2}(1+2\mu)w_m^2\right.$$

$$+\frac{1}{256}[1+\gamma^2(m^2+n^2\beta^2)]\left\langle 2(1+\mu)^2(1+2\mu)^2\left[\frac{m^4(m^4+n^4\beta^4)}{g_{13}}\right.\right.$$

$$\cdot [1+\gamma^2(m^2+9n^2\beta^2)]+\frac{n^4\beta^4(m^4+n^4\beta^4)}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)]\Big]$$

$$-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]\Big[\frac{m^8}{g_{13}}[1+\gamma^2(m^2$$

$$+9n^2\beta^2)]+\frac{n^8\beta^8}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)]\Big]\Big\rangle w_m^4+\cdots\Big\} \tag{3.17}$$

$$\delta_x=\lambda_x+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\frac{m^2}{\beta^2}[1+\gamma^2(m^2+n^2\beta^2)](1+\mu)^2$$

$$\cdot(1+2\mu)^2\Big\langle\frac{m^4}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)]+\frac{n^4\beta^4}{g_{31}}[1+\gamma^2(9m^2$$

$$+n^2\beta^2)]\Big\rangle w_m^4+\cdots \tag{3.18}$$

其中

$$\left.\begin{aligned}g_{13}=&(m^2+9n^2\beta^2)^2m^2[1+\gamma^2(m^2+n^2\beta^2)](1+\mu)\\&-(m^2+n^2\beta^2)^2m^2[1+\gamma^2(m^2+9n^2\beta^2)]\\g_{31}=&(9m^2+n^2\beta^2)^2m^2[1+\gamma^2(m^2+n^2\beta^2)](1+\mu)\\&-9(m^2+n^2\beta^2)^2m^2[1+\gamma^2(9m^2+n^2\beta^2)]\end{aligned}\right\} \tag{3.19}$$

（2） 纵向边缘不可移简支

此时

$$b_{00}^{(0)}=\nu\beta^2B_{00}^{(0)} \tag{3.20}$$

因而导得

$$\lambda_x=\frac{1}{4\beta^2}\Big\{\frac{(m^2+n^2\beta^2)^2}{(1+\mu)(m^2+\nu n^2\beta^2)[1+\gamma^2(m^2+n^2\beta^2)]}+\frac{1}{16}\frac{m^4+3n^4\beta^4}{m^2+\nu n^2\beta^2}$$

$$\cdot(1+2\mu)w_m^2+\frac{1}{256}[1+\gamma^2(m^2+n^2\beta^2)]\Big\langle 2(1+\mu)^2(1+2\mu)^2$$

$$\cdot\Big[\frac{m^4(m^4+3n^4\beta^4)}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)]+\frac{n^4\beta^4(m^4+3n^4\beta^4)}{g_{31}}$$

$$\cdot[1+\gamma^2(9m^2+n^2\beta^2)]\Big]-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\Big[\frac{m^8}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)]+\frac{n^8\beta^8}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)]\Big]\Big\rangle w_m^4+\cdots\Big\} \tag{3.21}$$

$$\delta_x=\lambda_x(1-\nu^2)+\frac{1}{32}\frac{m^2+\nu n^2\beta^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\frac{(m^2+\nu n^2\beta^2)^2}{\beta^2}$$

$$\cdot[1+\gamma^2(m^2+n^2\beta^2)](1+\mu)^2(1+2\mu)^2\Big\langle\frac{m^4}{g_{13}}[1+\gamma^2(m^2+9n^2\beta^2)]$$

$$+\frac{n^4\beta^4}{g_{31}}[1+\gamma^2(9m^2+n^2\beta^2)]\Big\rangle w_m^4+\cdots \tag{3.22}$$

其中

$$\left.\begin{aligned} g_{13}=&(m^2+9n^2\beta^2)^2(m^2+\nu n^2\beta^2)[1+\gamma^2(m^2+n^2\beta^2)](1+\mu)\\ &-(m^2+n^2\beta^2)^2(m^2+9\nu n^2\beta^2)[1+\gamma^2(m^2+9n^2\beta^2)]\\ g_{31}=&(9m^2+n^2\beta^2)^2(m^2+\nu n^2\beta^2)[1+\gamma^2(m^2+n^2\beta^2)](1+\mu)\\ &-(m^2+n^2\beta^2)^2(9m^2+\nu n^2\beta^2)[1+\gamma^2(9m^2+n^2\beta^2)] \end{aligned}\right\} \tag{3.23}$$

由式(3.17)、(3.21)，当 $w_m=0$ 时，我们得到临界屈曲载荷。由式(3.1)可以看出，当板相当薄时（即当 t/b 很小时），γ^2 趋于零，此时式(3.16)至(3.23)回到Kármán板结果[8]。

四、结果和讨论

利用渐近分析导出的公式，我们容易计算得到考虑横向剪切变形的完善($W^*/t=0.0$)矩形板的屈曲载荷，并与以往所得结果[4~7]进行了比较。计算结果如表1和表2所示。表中本文结果按纵边可移简支给出。计算结果表明

（1） 本文计算所得屈曲载荷的精度与Reddy理论相当。

（2） 当板宽厚比 $b/t\geqslant50$时，经典薄板理论具有良好的近似，而当宽厚比$b/t<50$时，则必须考虑横向剪切变形对屈曲载荷的影响。

表1 单向压缩简支方板($\nu=0.3$)，屈曲载荷比较

$\beta=1.0$	$\sigma_x b^2t/\pi^2D$						
b/t	精确解[4]	有限元解[7]	本文	Reddy & Phan [5]		Senthilnathan, Lim, Lee & Chow [6]	薄板理论
				FSDPT	HSDPT		
20	3.924	3.941	3.9443	3.9443	3.9443	3.9443	4.0
10	3.741	3.745	3.7864	3.7864	3.7865	3.7865	4.0
5	3.150	3.162	3.2637	3.2636	3.2653	3.2653	4.0

表2 单向压缩简支矩形板($\nu=0.3$)屈曲载荷

	$\sigma_x b^2t/\pi^2D$											
β	0.2(27.040)**				0.4(8.410)				1.0(4.000)			
b/t	本文	文[5]		文[6]	本文	文[5]		文[6]	本文	文[5]		文[6]
		FSDPT	HSDPT			FSDPT	HSDPT			FSDPT	HSDPT	
2	1.3989	1.3988	1.6851	1.6851	1.3761	1.3761	1.4455	1.4455	1.6597	1.6597	1.6759	1.6759
5	6.8757	8.8753	7.0529	7.0529	4.6265	4.6264	4.6466	4.6466	3.2637	3.2636	3.2653	3.2653
10	15.6014	15.601	15.658	15.658	6.9824	6.9824	6.9853	6.9853	3.7864	3.7864	3.7865	3.7965
20	22.8514	22.851	22.859	22.859	8.0010	8.0010	8.0012	8.0012	3.9443	3.9443	3.9443	3.9443
50	26.2695	26.269	26.270	26.270	8.3417	8.3417	8.3417	8.3417	3.9909	3.9909	3.9909	3.9909
100	26.8431	26.843	26.843	26.840	8.3928	8.3928	8.3928	8.3928	3.9977	3.9977	3.9977	3.9977

** 括号中表示薄板理论解.

图 2 为本文理论计算结果与文[4]按三维弹性理论计算结果的比较。可以看出，即使对

于中厚板，本文计算结果亦具有良好的近似.

图 3 为完善($W^*/t=0.0$)和非完善($W^*/t=0.1$)矩形板对应两种面内边界条件的后屈曲载荷-挠度曲线，并与经典薄板理论结果进行了比较.图示表明，对于中等厚度矩形板，在挠

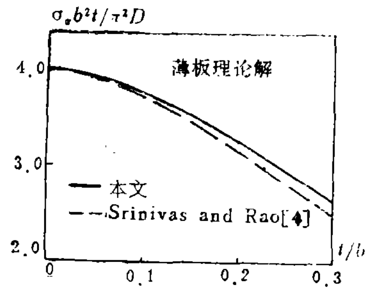

图 2 屈曲载荷随板厚宽比 t/b 变化曲线比较($\nu=0.3$)

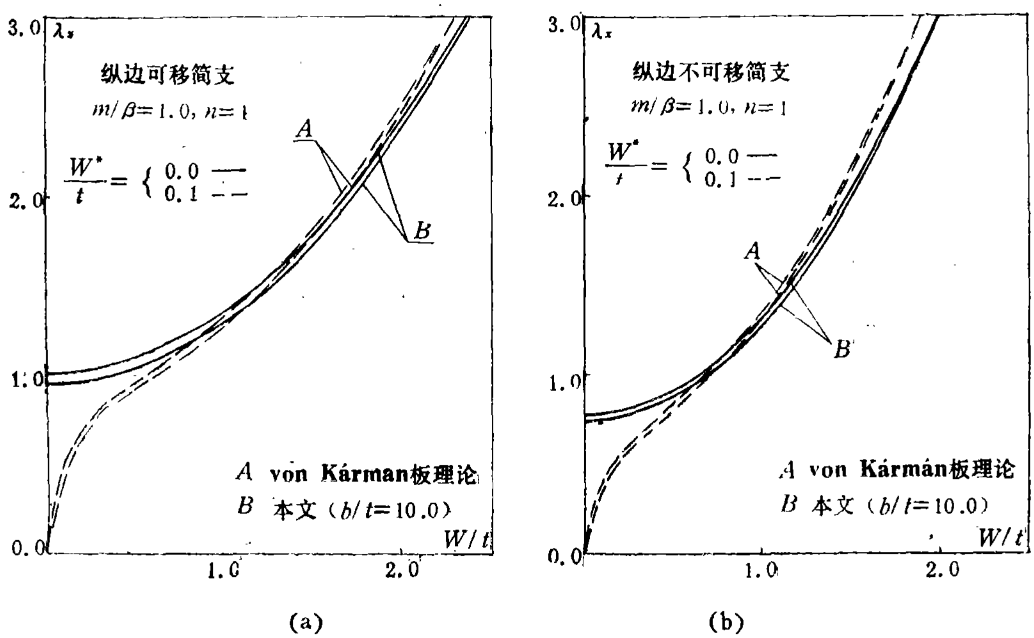

图 3 后屈曲载荷-挠度曲线比较

度不大时 ($W/t<1.0$)，横向剪切变形对后屈曲平衡路径的影响是明显的. 事实上，当挠度较大时，由于板塑性变形的影响，板后屈曲载荷-挠度曲线呈下降趋势[9].

参 考 文 献

[1] Ziegler, H., The influence of inplane deformations on the buckling loads of isotropic elastic plates, *Ing.-Arch.*, 53 (1983), 61—72.

[2] Reissner, E., On the theory of bending of elastic plates, *J. Math. Phys.*, 23 (1944), 184—191.

[3] Reissner, E., On the effect of transverse shear deformation on the bending of elastic plates, *J. Appl. Mech.*, 12 (1945), 69—77.

[4] Srinivas, S. and A. K. Rao, Buckling of thick rectangular plates, *AIAA J.*, 7 (1969), 1645—1646.

[5] Reddy, J. N. and N. D. Phan, Stability and vibration of isotropic orthotropic and laminated plates according to a high-order shear deformation theory, *J. Sound Vibration*, **98** (1985), 157—170.

[6] Senthilnathan, N. R., S. P. Lim, K. H. Lee and S. T. Chow, Buckling of shear-deformable plate, *AIAA J.*, **25** (1987), 1268—1271.

[7] Singh, G. and Y. V. K. Sadasiva Rao, Stability of thick angle-ply composite plates, *Computers and Structures*, **29** (1988), 317—322.

[8] 沈惠申、张建武，单向压缩简支矩形板后屈曲摄动分析，应用数学和力学，**9**，8 (1988)，741—752.

[9] 沈惠申，矩形板屈曲和后屈曲的弹塑性分析，应用数学和力学.（待发表）

Buckling and Postbuckling of Moderately Thick Plates

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Abstract

This paper gives the basic differential equations for finite deflections of elastic plates according to Reissner's approximate stress distributions. The buckling and post-buckling problems of elastic rectangular plates, including the effect of transverse shear deformation, are solved and discussed, by using perturbation method suggested in ref. [8]. The postbuckling equilibrium paths of perfect and imperfect moderately thick rectangular plates are presented and compared with the results based on thin plate theory.