应用数学和力学，第11卷第5期(1990年5月)
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

Mac-Millan方程的推广*

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摘 要

将动力学原理及Appell-Четаев定义推广到非惯性系，由此导出非惯性系中的非线性非完整系统的Mac-Millan方程。

1936年Mac-Millan推广了Lagrange方程[1]，他一开始便考虑了一阶线性非完整约束。1984年梅凤翔又将Mac-Millan方程推广到一阶非线性非完整约束系统[2]。这些都是相对惯性参加系而言的。现在我们进一步将它推广到非惯性参考系上去。

我们曾将虚位移原理与达朗倍尔原理推广到非惯性系，并将之相结合，得到了非惯性系的动力学原理。即考虑N个质点构成的力学组相对于非惯性参考系$Axyz$的运动。非惯性系A与大质量物体固连在一起，其平动加速度$d^2\mathbf{r}_A/dt^2$、角速度$\boldsymbol{\omega}$及角加速度$d\boldsymbol{\omega}/dt$都是时间的已知函数，与诸质量为m_i的质点的运动无关。则在任一时刻，一切运动学上允许的运动，只有对真实的运动来说，主动力$\mathbf{F}_i$、牵连惯性力$(-m_i\mathbf{W}_{ti})$、柯赖奥来力$(-m_i\mathbf{W}_{ci})$及相对惯性力$(-m_i\ddot{\mathbf{r}}_{Ai})$，当力学组作任何的虚位移时，所作的元功之和才等于零。就是说，对真实的运动有：

$$\sum_{i=1}^{N}(\mathbf{F}_i-m_i\mathbf{W}_{ti}-m_i\mathbf{W}_{ci}-m_i\ddot{\mathbf{r}}_{Ai})\cdot\delta\mathbf{r}_{Ai}=0 \tag{1}$$

这是对完整系和非完整系都适用的原理。式中

$$\mathbf{W}_{ti}=\frac{d^2\mathbf{r}_A}{dt^2}+\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}_{Ai}+\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_{Ai},\quad \mathbf{W}_{ci}=(2\boldsymbol{\omega}\times\mathbf{v}_{Ai}),$$

$\mathbf{r}_A$为加速系A相对于惯性系O点的矢径，$\mathbf{r}_{Ai}$为第i个质点相对加速系A的矢径，m_i为第i个质点的质量，$\mathbf{v}_{Ai}$为质点i相对A的速度。

设力学组的位形由n个广义坐标确定，系统中直角坐标可用广义坐标q_s及时间t表示为：

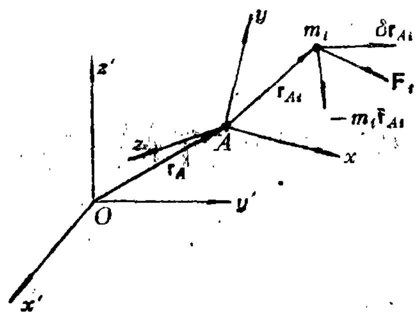
$$\mathbf{r}_{Ai}=\mathbf{r}_{Ai}(q_s;t) \tag{2}$$

力学组的运动受到g个彼此独立的一阶非线性非完整约束：

$$f_\beta(q_s,\dot{q}_s;t)=0\quad\begin{pmatrix}\beta=1,2,\cdots,g\\ s=1,2,\cdots,n\end{pmatrix} \tag{3}$$

当行列式 $$\frac{D(f_1,f_2,\cdots,f_g)}{D(\dot{q}_{s+1},\dot{q}_{s+2},\cdots,\dot{q}_n)}\neq 0$$

* 汪家訸推荐。



$Ox'y'z'$为惯性参考系，$Axyz$为非惯性参考系。
其$\ddot{r}_A, \omega, d\omega/dt$是时间的已知函数，与$m_i$的运动无关。
图 1

时，后面g个广义速度$\dot{q}_{\varepsilon+\beta}$可用前面$\varepsilon$个广义速度$\dot{q}_\sigma$表示出来，记为：

$$\dot{q}_{\varepsilon+\beta}=\dot{q}_{\varepsilon+\beta}(q_s,\dot{q}_\sigma;t) \tag{4}$$

$$(\sigma=1,2,\cdots,\varepsilon;\ \beta=1,2,\cdots,g;\ \varepsilon=n-g;\ s=1,2,\cdots,n)$$

我们假设 Appell-Четаев 关于约束加在坐标变分上的条件：

$$\delta q_{\varepsilon+\beta}=\sum_{\sigma=1}^{\varepsilon}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\delta q_\sigma \qquad (\beta=1,2,\cdots,g) \tag{5}$$

对非惯性参考系的力学组也适用，因此可用这个定义将非线性非完整约束(3)线性化。

现在我们利用非惯性系的基本动力学原理(1)及约束加在变分上的条件(5)来推导推广的Mac-Millan方程。为此，我们先求出下列关系：

1) 由(2)式可得

$$\delta r_{Ai}=\sum_{s=1}^{n}\frac{\partial r_{Ai}}{\partial q_s}\delta q_s=\sum_{\sigma=1}^{\varepsilon}\frac{\partial r_{Ai}}{\partial q_\sigma}\delta q_\sigma+\sum_{\beta=1}^{g}\frac{\partial r_{Ai}}{\partial q_{\varepsilon+\beta}}\delta q_{\varepsilon+\beta}$$

利用(5)式将上式写成

$$\delta r_{Ai}=\sum_{\sigma=1}^{\varepsilon}\left(\frac{\partial r_{Ai}}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial r_{Ai}}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\right)\delta q_\sigma \tag{6}$$

将(2)式对时间微商，得

$$\dot{r}_{Ai}=\sum_{\sigma=1}^{\varepsilon}\frac{\partial r_{Ai}}{\partial q_\sigma}\dot{q}_\sigma+\sum_{\beta=1}^{g}\frac{\partial r_{Ai}}{\partial q_{\varepsilon+\beta}}\dot{q}_{\varepsilon+\beta}+\frac{\partial r_{Ai}}{\partial t}$$

$\dot{r}_{Ai}$对$\dot{q}_\sigma$求偏微商得：

$$\frac{\partial(\dot{r}_{Ai})}{\partial\dot{q}_\sigma}=\frac{\partial r_{Ai}}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial r_{Ai}}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}$$

将上式乘以δq_σ，然后对σ求和，便得：

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial(\dot{r}_{Ai})}{\partial\dot{q}_\sigma}\delta q_\sigma=\sum_{\sigma=1}^{\varepsilon}\left(\frac{\partial r_{Ai}}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial r_{Ai}}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma}\right)\delta q_\sigma \tag{7}$$

由(6)和(7)式得：

$$\delta r_{Ai}=\sum_{s=1}^{n}\frac{\partial r_{Ai}}{\partial q_s}\delta q_s=\sum_{\sigma=1}^{\varepsilon}\frac{\partial(\dot{r}_{Ai})}{\partial\dot{q}_\sigma}\delta q_\sigma \tag{8}$$

2） 系统的动能为：

$$T=\frac{1}{2}\sum_{i=1}^{N}m_i\dot{\mathbf{r}}_{Ai}\cdot\dot{\mathbf{r}}_{Ai} \tag{9}$$

设动能$\tilde{T}$是已利用(4)式消去$\dot{q}_{\epsilon+\beta}$而得的表达式，则

$$\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}=\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma},$$

$$\frac{\partial\tilde{T}}{\partial q_\sigma}=\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}$$

而

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}=\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_{Ai}\cdot\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}+\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\left(\frac{d}{dt}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}\right)$$

所以

$$\sum_{\sigma=1}^{\epsilon}\left(\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}\right)\delta q_\sigma=\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_{Ai}\cdot\sum_{\sigma=1}^{\epsilon}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}\delta q_\sigma$$

$$+\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\sum_{\sigma=1}^{\epsilon}\left(\frac{d}{dt}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}\right)\delta q_\sigma \tag{10}$$

将(8)式代入(10)式得：

$$\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_{Ai}\cdot\delta\mathbf{r}_{Ai}=\sum_{\sigma=1}^{\epsilon}\left(\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}\right)\delta q_\sigma$$

$$-\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\sum_{\sigma=1}^{\epsilon}\left(\frac{d}{dt}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}\right)\delta q_\sigma \tag{11}$$

3） 利用(7)和(8)式得：

$$\sum_{i=1}^{N}(\mathbf{F}_i-m_i\mathbf{W}_{ti}-m_i\mathbf{W}_{ci})\cdot\delta\mathbf{r}_{Ai}$$

$$=\sum_{i=1}^{N}\left\{\mathbf{F}_i-m_i\mathbf{W}_{ti}-m_i\mathbf{W}_{ci})\cdot\sum_{\sigma=1}^{\epsilon}\left(\frac{\partial\mathbf{r}_{Ai}}{\partial q^\sigma}+\sum_{\beta=1}^{g}\frac{\partial\mathbf{r}_{Ai}}{\partial q_{\epsilon+\beta}}\frac{\partial\dot{q}_{\epsilon+\beta}}{\partial\dot{q}_\sigma}\right)\right\}\delta q_\sigma,$$

令

$$Q_\sigma=\sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial\mathbf{r}_{Ai}}{\partial q_\sigma},\quad Q_{\epsilon+\beta}=\sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial\mathbf{r}_{Ai}}{\partial q_{\epsilon+\beta}}$$

$$(Q_\sigma)_t=\sum_{i=1}^{N}(-m_i\mathbf{W}_{ti})\cdot\frac{\partial\mathbf{r}_{Ai}}{\partial q_\sigma},\quad (Q_{\epsilon+\beta})_t=\sum_{i=1}^{N}(-m_i\mathbf{W}_{ti})\cdot\frac{\partial\mathbf{r}_{Ai}}{\partial q_{\epsilon+\beta}},$$

$$(Q_\sigma)_c=\sum_{i=1}^{N}(-m_i\mathbf{W}_{ci})\cdot\frac{\partial\mathbf{r}_{Ai}}{\partial q_\sigma},\quad (Q_{\epsilon+\beta})_c=\sum_{i=1}^{N}(-m_i\mathbf{W}_{ci})\cdot\frac{\partial\mathbf{r}_{Ai}}{\partial q_{\epsilon+\beta}},$$

$$\tilde{Q}_\sigma=Q_\sigma+\sum_{\beta=1}^{g}Q_{\varepsilon+\beta}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma},$$

$$(\tilde{Q}_\sigma)_t=(Q_\sigma)_t+\sum_{\beta=1}^{g}(Q_{\varepsilon+\beta})_t\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma},$$

$$(\tilde{Q}_\sigma)_c=(Q_\sigma)_c+\sum_{\beta=1}^{g}(Q_{\varepsilon+\beta})_c\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_\sigma},$$

于是

$$\sum_{i=1}^{N}(\mathbf{F}_i-m_i\mathbf{W}_{ti}-m_i\mathbf{W}_{ci})\cdot\delta\mathbf{r}_{Ai}=\sum_{\sigma=1}^{\varepsilon}[\tilde{Q}_\sigma+(\tilde{Q}_\sigma)_t+(\tilde{Q}_\sigma)_c]\delta q_\sigma \tag{12}$$

将(11)和(12)式代入非惯性参考系的动力学方程(1)得：

$$\sum_{\sigma=1}^{\varepsilon}\left\{-\left(\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}\right)+\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\left(\frac{d}{dt}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}\right)\right.$$
$$\left.+\tilde{Q}_\sigma+(\tilde{Q}_\sigma)_t+(Q_\sigma)_c\right\}\delta q_\sigma=0$$

由于δq_σ彼此独立($\sigma=1,2,\cdots,\varepsilon$)，所以得：

$$\frac{d}{dt}\frac{\partial\tilde{T}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}}{\partial q_\sigma}=(\tilde{Q}_\sigma+\tilde{Q}_{\sigma t}+\tilde{Q}_{\sigma c})+\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}_{Ai})\cdot\left(\frac{d}{dt}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}\right) \tag{13}$$

(13)式即非惯性参考系中非线性非完整系统的推广了的 Mac-Millan 方程。如为完整系则可证有

$$\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}=\frac{\partial\mathbf{r}_{Ai}}{\partial q_\sigma},\quad\frac{d}{dt}\frac{\partial\mathbf{r}_{Ai}}{\partial q_\sigma}=\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma},$$
$$(i=1,2,\cdots,N;\ \sigma=1,2,\cdots,\varepsilon)$$

于是(13)左右端的

$$\frac{d}{dt}\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial\dot{q}_\sigma}-\frac{\partial(\dot{\mathbf{r}}_{Ai})}{\partial q_\sigma}=0,$$

(13)式便成为非惯性参考系中的第二类拉格朗日方程。当$(\tilde{Q}_\sigma)_t=(\tilde{Q}_\sigma)_c=0$时，便得惯性系中的一阶非线性非完整系统的Mac-Millan方程[2]。

感谢汪家訸教授的热情帮助。

参 考 文 献

[1] Mac-Millan, *Dynamics of Rigid Bodies*, New York (1936).

[2] 梅凤翔，Mac-Millan 方程对非线性非完整系统的推广，应用数学和力学，5, 5 (1984), 665—672。

Extended Mac-Millan's Equation

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Abstract

The principle of classical dynamics and Appell-Четаев assumption are extended to non-inertial frame, from which extended Mac-Millan's equation is derived for non-holonomic system in non-inertial system.