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四阶椭圆型方程奇异摄动问题的渐近解

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摘　要

本文考虑了四阶椭圆型偏微分方程奇异摄动边值问题，建立了解及其导数的能量估计，并用Lyuternik-Vishik方法构造了形式渐近解．最后利用能量估计我们得到了渐近展开式余项的界．

一、引　言

1971年C. Comstock[1]用双变量展开的方法对矩形区域D内高阶导数项含小参数的四阶椭圆型方程的边值问题：

$$L_\varepsilon\phi\equiv-\varepsilon^2\nabla^4\phi(x,y)+a(x,y)\frac{\partial^2\phi}{\partial x^2}+b(x,y)\frac{\partial^2\phi}{\partial y^2}+c(x,y)\frac{\partial\phi}{\partial x}=0$$

$$\left(\frac{\partial\phi}{\partial n}+k\phi\right)\Big|_\Gamma=g(x,y)\qquad (k\text{是小于零的常数})$$

$$\frac{\partial^2\phi}{\partial n^2}\Big|_\Gamma=h(x,y)$$

构造了二次近似的形式渐近解，其中Γ是矩形区域D的边界，n表示Γ的内法线．但C. Comstock未作出余项估计．1978年江福汝[2]利用同样的方法改进了C. Comstock的构造过程，构造了具有任意精度的形式渐近解，然而在求形式解的过程中所需要的计算工作量却是很大的，也未给出误差估计的证明．在本文中我们用Lyusternik-Vishik方法[3]构造了形式渐近解，并且对一类边值问题利用能量方法给出了余项估计．

二、解的先验估计

我们在矩形区域$D=\{(x,y)\,|\,0\leqslant x\leqslant\alpha,\ 0\leqslant y\leqslant\beta\}$内考虑下面的边值问题：

$$-\varepsilon^2\nabla^4\phi+a(x,y)\frac{\partial^2\phi}{\partial x^2}+b(x,y)\frac{\partial^2\phi}{\partial y^2}+c(x,y)\frac{\partial\phi}{\partial x}+d(x,y)\phi$$
$$=f(x,y)\qquad((x,y)\in D)\tag{2.1}$$

$$\frac{\partial\phi}{\partial n}+k\phi=g(x,y)\qquad(k=\text{const}<0)\qquad((x,y)\in\Gamma)\tag{2.2}$$

$$\nabla^2\phi=h(x,y) \qquad ((x,y)\in\Gamma) \tag{2.3}$$

其中 Γ是D的边界，n表示Γ的内法线。假设$a(x,y)\geqslant\bar{a}>0$，$b(x,y)\geqslant\bar{b}>0$，$d(x,y)<0$。这里$\bar{a}$，$\bar{b}$均为常数，函数$a(x,y)$，$b(x,y)$，$c(x,y)$，$d(x,y)$，$f(x,y)$及$g(x,y)$，$h(x,y)$都是充分光滑函数，并且满足一定的相容性条件以保证问题(2.1)～(2.3)的解在$\bar{D}$内具有四阶连续导数。此外，假定$|d(x,y)|$适当大。

设ϕ是$\bar{D}$内具有四阶连续导数且满足边界条件(2.2)，(2.3)的任意函数，即$\phi\in C^4(\bar{D})$并满足边界条件(2.2)和(2.3)。

对方程(2.1)的两边同乘以ϕ并按区域$\bar{D}$积分，则有

$$\left(-\varepsilon^2\nabla^4\phi+a(x,y)\frac{\partial^2\phi}{\partial x^2}+b(x,y)\frac{\partial^2\phi}{\partial y^2}+c(x,y)\frac{\partial\phi}{\partial x}+d(x,y)\phi,\phi\right)=(f(x,y),\ \phi) \tag{2.4}$$

其中 $(u,v)=\int_0^\alpha\int_0^\beta uv\,dxdy$表示$u$和$v$的内积。可以证明下面的等式成立：

$$(\varepsilon^2\nabla^4,\phi)=(\varepsilon\nabla^2\phi,\varepsilon\nabla^2\phi+\varepsilon^2\int_\Gamma\phi\cdot\frac{\partial}{\partial n}\nabla^2\phi ds$$
$$-\int_0^\beta\varepsilon^2h(k\phi-g)((\alpha,y)+(0,y))dy-\int_0^\alpha\varepsilon^2h(k\phi-g)(x,\beta)+(x,0))dx \tag{2.5}$$

其中记号

$$\chi((x_1,y_1)+(x_2,y_2))=\chi(x_1,y_1)+\chi(x_2,y_2)$$

事实上，利用分部积分和边界条件(2.2)，(2.3)得到，

$$\int_0^\alpha\int_0^\beta\frac{\partial^2\nabla^2\phi}{\partial x^2}\phi dxdy=\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\phi\Big|_{x=0}^{x=\alpha}dy-\int_0^\beta\int_0^\alpha\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial\phi}{\partial x}dxdy$$
$$=\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\phi\Big|_{x=0}^{x=\alpha}dy-\int_0^\beta\nabla^2\phi\cdot\frac{\partial\phi}{\partial x}\Big|_{x=0}^{x=\alpha}dy+\int_0^\alpha\int_0^\beta\nabla^2\phi\cdot\frac{\partial^2\phi}{\partial x^2}dxdy$$
$$=\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\phi\Big|_{x=0}^{x=\alpha}dy-\int_0^\beta h\cdot k\phi((\alpha,y)+(0,y))dy+\int_0^\beta h\cdot g((\alpha,y)$$
$$+(0,y))dy+\int_0^\alpha\int_0^\beta\nabla^2\phi\cdot\frac{\partial^2\phi}{\partial x^2}dxdy$$
$$=\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\phi\Big|_{x=0}^{x=\alpha}dy+\int_0^\alpha\int_0^\beta\nabla^2\phi\cdot\frac{\partial^2\phi}{\partial x^2}dxdy$$
$$-\int_0^\beta h(k\phi-g)((\alpha,y)+(0,y))dy$$

同理

$$\int_0^\alpha\int_0^\beta\frac{\partial^2\nabla^2\phi}{\partial y^2}\cdot\phi dxdy=\int_0^\alpha\frac{\partial}{\partial y}\nabla^2\phi\cdot\phi\Big|_{y=0}^{y=\beta}dx+\int_0^\alpha\int_0^\beta\nabla^2\phi\cdot\frac{\partial^2\phi}{\partial y^2}dxdy$$
$$-\int_0^\alpha h(k\phi-g)((x,\beta)+(x,0))dx$$

故(2.5)成立。

为了估计(2.5)右端的第二项，我们先证明以下几个引理。

引理1 若$\phi(x,y)$及其一阶偏导数在$\bar{D}$上连续，则有

$$\int_0^\beta \phi^2(t,y)dy \leqslant \frac{2}{\theta}\int_0^\alpha\int_0^\beta \phi^2(x,y)dxdy + \theta\int_0^\alpha\int_0^\beta\left(\frac{\partial\phi}{\partial x}\right)^2 dxdy$$

$$(t=0,\ \alpha;\ 0<\theta\leqslant\alpha)$$

$$\int_0^\alpha \phi^2(x,s)dx \leqslant \frac{2}{\theta}\int_0^\alpha\int_0^\beta \phi^2(x,y)dxdy + \theta\int_0^\alpha\int_0^\beta\left(\frac{\partial\phi}{\partial y}\right)^2 dxdy$$

$$(s=0,\ \beta;\ 0<\theta\leqslant\beta)$$

证明

$$\int_0^{x_1}\frac{\partial\phi}{\partial x}dx = \phi(x_1,y)-\phi(0,y)$$

这里 $0<x_1\leqslant\theta<\alpha$. 由Schwartz不等式

$$\left(\int_0^{x_1}\frac{\partial\phi}{\partial x}dx\right)^2 \leqslant x_1\int_0^x\left(\frac{\partial\phi}{\partial x}\right)^2 dx$$

又

$$\phi(0,y)=\phi(x_1,y)-\int_0^{x_1}\frac{\partial\phi}{\partial x}dx$$

将上式两边平方，利用前面不等式并关于变量y在$[0,\beta]$上积分，我们有

$$\int_0^\beta \phi^2(0,y)dx \leqslant 2\int_0^\beta \phi^2(x,y)dy + 2\int_0^\beta x_1\int_0^{x_1}\left(\frac{\partial\phi}{\partial x}\right)^2 dxdy$$

关于变量x_1将上面不等式按区间$[0,\theta]$积分得到

$$\int_0^\theta\int_0^\beta \phi^2(0,y)dydx_1 \leqslant 2\int_0^\theta\int_0^\beta \phi^2(x_1,y)dx_1dy + 2\int_0^\theta\int_0^\beta x_1\int_0^{x_1}\left(\frac{\partial\phi}{\partial x}\right)^2 dxdydx_1$$

$$\leqslant 2\int_0^\theta\int_0^\beta \phi^2(x_1,y)dx_1dy + \theta^2\int_0^\theta\int_0^\beta\left(\frac{\partial\phi}{dx}\right)^2 dxdy$$

故有

$$\int_0^\beta \phi^2(0,y)dy \leqslant \frac{2}{\theta}\int_0^\alpha\int_0^\beta \phi^2(x,y)dxdy + \theta^2\int_0^\alpha\int_0^\beta\left(\frac{\partial\phi}{\partial x}\right)^2 dxdy$$

同理可证另外三个不等式.

我们以后记

$$\|\phi\|^2_{H^0(D)}=\int_0^\alpha\int_0^\beta \phi^2 dxdy,\quad \|\phi\|^2_{H^1(D)}=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2+\phi^2\right]dxdy,$$

$$\|\phi\|^2_{H^2(D)}=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\phi^2\right]dxdy$$

引理2 设$\phi\in C^4(\bar{D})$并且是满足边界条件(2.2)和(2.3)的任意一个函数，则

$$\max\left(\left\|\frac{\partial^2}{\partial x^2}\nabla^2\phi\right\|^2_{H^0(D)},\ \left\|\frac{\partial^2}{\partial x\partial y}\nabla^2\phi\right\|^2_{H^0(D)},\ \left\|\frac{\partial^2}{\partial y^2}\nabla^2\phi\right\|^2_{H^0(D)}\right)$$

$$\leqslant\|\nabla^4\phi\|^2_{H^0(D)}+c_1\left\|\frac{\partial\nabla^2\phi}{\partial x}\right\|^2_{H^0(D)}+c_1\left\|\frac{\partial\nabla^2\phi}{\partial y}\right\|^2_{H^0(D)}$$

$$+c_2\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2\right]ds$$

这里 c_1，c_2为正的常数，$\partial/\partial s$表示沿Γ的切向偏导。

证明

$$\int_0^\alpha\int_0^\beta(\nabla^4\phi)^2dxdy=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2+\left(\frac{\partial^2}{\partial y^2}\nabla^2\phi\right)^2\right]dxdy$$

$$+\int_0^\alpha\int_0^\beta 2\frac{\partial^2\nabla^2\phi}{\partial x^2}\cdot\frac{\partial^2\nabla^2\phi}{\partial y^2}dxdy$$

$$=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2+\left(\frac{\partial^2}{\partial y^2}\nabla^2\phi\right)^2\right]dxdy+2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^2\nabla^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy$$

$$-2\int_0^\alpha\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^3\nabla^2\phi}{\partial x\partial y^2}dxdy$$

$$=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2+\left(\frac{\partial^2}{\partial y^2}\nabla^2\phi\right)^2\right]dxdy+2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^2 h}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy$$

$$-2\int_0^\alpha\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^3\nabla^2\phi}{\partial x\partial y^2}dxdy$$

$$=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2+\left(\frac{\partial^2}{\partial y^2}\nabla^2\phi\right)^2\right]dxdy+2\int_0^\alpha\int_0^\beta\left(\frac{\partial^2\nabla^2\phi}{\partial x\partial y}\right)^2dxdy$$

$$-2\int_0^\alpha\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^2\nabla^2\phi}{\partial x\partial y}\Big|_{y=0}^{y=\beta}dx+2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^2 h}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy$$

$$=\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2+\left(\frac{\partial^2}{\partial y^2}\nabla^2\phi\right)^2\right]dxdy+2\int_0^\alpha\int_0^\beta\left(\frac{\partial^2\nabla^2\phi}{\partial x\partial y}\right)^2dxdy$$

$$-2\int_0^\alpha\frac{\partial h}{\partial x}\cdot\frac{\partial^2\nabla^2\phi}{\partial x\partial y}\Big|_{y=0}^{y=\beta}dx+2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^2 h}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy$$

利用分部积分及引理 1 我们有

$$\left|\int_0^\alpha\frac{\partial h}{\partial x}\cdot\frac{\partial^2\nabla^2\phi}{\partial x\partial y}(x,\beta)dx\right|=\left|-\int_0^\alpha\frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial\nabla^2 y}{\partial y^2}(x,\beta)dx\right.$$

$$\left.+\frac{\partial h}{\partial x}\cdot\frac{\partial\nabla^2\phi}{\partial y}(x,\beta)\Big|_{x=0}^{x=\alpha}\right|$$

$$\leqslant\left|-\int_0^\alpha\frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial\nabla^2\phi}{\partial y}(x,\beta)dx\right|+\left|\frac{\partial h}{\partial x}\cdot\frac{\partial h}{\partial y}((\alpha,\beta)+(0,\beta))\right|$$

$$\leqslant\frac{1}{2}\int_0^\alpha\left(\frac{\partial^2 h}{\partial x^2}\right)^2(x,\beta)dx+\frac{1}{2}\int_0^\alpha\left(\frac{\partial\nabla^2\phi}{\partial y}\right)^2(x,\beta)dx$$

$$+\frac{1}{2}\left(\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2\right)((\alpha,\beta)+(0,\beta))$$

$$\leqslant c_1\int_0^\alpha\int_0^\beta\left(\frac{\partial^2\nabla^2\phi}{\partial y^2}\right)^2dxdy+\frac{2}{c_1}\int_0^\alpha\int_0^\beta\left(\frac{\partial\nabla^2\phi}{\partial y}\right)^2dxdy$$

$$+c_2\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2\right]ds$$

这里c_1，c_2为任意常数。

同理

$$\left|\int_0^\alpha \frac{\partial h}{\partial x}\cdot\frac{\partial^2\nabla^2\phi}{\partial x\partial y}(x,0)dx\right|\leqslant c_1\int_0^\alpha\int_0^\beta\left(\frac{\partial^2\nabla^2\phi}{\partial y^2}\right)^2dxdy+\frac{2}{c_1}\int_0^\alpha\int_0^\beta\left(\frac{\partial\nabla^2\phi}{\partial y}\right)^2dxdy+c_2\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2\right]ds$$

（c_1，$c_2>0$，任意常数）

$$\left|\int_0^\beta \frac{\partial\nabla^2\phi}{\partial x}\cdot\frac{\partial^2 h}{\partial y^2}(\alpha,y)dy\right|\leqslant c_1\int_0^\alpha\int_0^\beta\left(\frac{\partial^2\nabla^2\phi}{\partial x^2}\right)^2dxdy+\frac{2}{c_1}\int_0^\alpha\int_0^\beta\left(\frac{\partial\nabla^2\phi}{\partial x}\right)^2dxdy+c_2\int_\Gamma\left(\frac{\partial^2 h}{\partial s^2}\right)^2ds\qquad(c_1,c_2>0,\text{任意常数})$$

综合上述各式可以得到引理 2 的结论。

引理3 不等式

$$\int_0^\alpha\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2dxdy\leqslant\frac{\|\nabla^2\phi\|^2_{H^0(D)}}{\varepsilon}+\varepsilon\left\|\frac{\partial^2}{\partial x^2}\nabla^2\phi\right\|^2_{H^0(D)}+\frac{\int_\Gamma h^2ds}{\varepsilon}$$

$$\int_0^\alpha\int_0^\beta\left(\frac{\partial}{\partial y}\nabla^2\phi\right)^2dxdy\leqslant\frac{\|\nabla^2\phi\|^2_{H^0(D)}}{\varepsilon}+\varepsilon\left\|\frac{\partial^2}{\partial y^2}\nabla^2\phi\right\|^2_{H^0(D)}+\frac{\int_\Gamma h^2ds}{\varepsilon}$$

对满足边界条件(2.2)和(2.3)的一切$\phi\in C^4(\bar D)$成立。

证明

$$\int_0^\alpha\int_0^\beta\frac{\partial^2}{\partial x^2}\nabla^2\phi\cdot\nabla^2\phi dxdy=\int_0^\beta\frac{\partial}{\partial x}\nabla^2\phi\cdot\nabla^2\phi\Big|_{x=0}^{x=\alpha}dy-\int_0^\alpha\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2dxdy$$

$$=\int_0^\beta\frac{\partial}{\partial x}\nabla^2\phi\cdot h\Big]_{x=0}^{x=\alpha}dy-\int_0^\alpha\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2dxdy$$

由引理 1 我们有

$$\left|\int_0^\beta\frac{\partial}{\partial x}\nabla^2\phi\cdot h(\alpha,y)dy\right|\leqslant c_1\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2(\alpha,y)dy+\frac{1}{c_1}\int_0^\beta h^2(\alpha,y)dy$$

$$\leqslant c_1\left(c_0\int_0^\alpha\int_0^\beta\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2dxdy+\frac{2}{c_0}\int_0^\alpha\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2dxdy\right)+\frac{1}{c_1}\int_0^\beta h^2(\alpha,y)dy$$

这里 c_0，c_1为任意正常数。由不等式

$$\int_0^\alpha\int_0^\beta P\cdot Qdxdy\leqslant\omega\int_0^\alpha\int_0^\beta P^2dxdy+\frac{1}{\omega}\int_0^\alpha\int_0^\beta Q^2dxdy\qquad(\omega>0)\tag{2.6}$$

我们有

$$\left|\int_0^\alpha\int_0^\beta\frac{\partial^2}{\partial x^2}\nabla^2\phi\cdot\nabla^2\phi dxdy\right|\leqslant\varepsilon\int_0^\alpha\int_0^\beta\left(\frac{\partial^2}{\partial x^2}\nabla^2\phi\right)^2dxdy+\frac{1}{\varepsilon}\int_0^\alpha\int_0^\beta(\nabla^2\phi)^2dxdy$$

若我们取$c_1=\varepsilon$，综合以上各式可证得引理3。

引理4 满足方程(2.1)及边界条件(2.2)，(2.3)的ϕ有如下的估计

$$\left(\varepsilon\frac{\partial}{\partial x}\nabla^2\phi,\ \varepsilon\frac{\partial}{\partial x}\nabla^2\phi\right)+\left(\varepsilon\frac{\partial}{\partial y}\nabla^2\phi,\ \varepsilon\frac{\partial}{\partial y}\nabla^2\phi\right)+\left(\frac{\partial^2\phi}{\partial x^2},\ \frac{\partial^2\phi}{\partial x^2}\right)$$

$$+\left(\frac{\partial^2\phi}{\partial x\partial y},\ \frac{\partial^2\phi}{\partial x\partial y}\right)+\left(\frac{\partial^2\phi}{\partial y^2},\ \frac{\partial^2\phi}{\partial y^2}\right)\leqslant d_1\|f\|^2_{H^0(D)}+d_2\|f\|^2_{H^1(D)}$$

$$+d_3\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2+\left(\frac{\partial g}{\partial s}\right)^2+g^2\right]ds$$

这里 d_1，d_2，d_3 为正的常数。

证明 对方程(2.1)的两边同乘以$\nabla^2\phi$并按区域D积分，则有

$$\left(-\varepsilon^2\nabla^4\phi+a(x,y)\frac{\partial^2\phi}{\partial x^2}+b(x,y)\frac{\partial^2\phi}{\partial y^2}+c(x,y)\frac{\partial\phi}{\partial x}+d(x,y)\phi,\nabla^2\phi\right)$$
$$=(f(x,y),\nabla^2\phi) \tag{2.7}$$

对上式左端的第一项利用边界条件(2.2)和(2.3)进行分部积分，得到

$$(-\varepsilon^2\nabla^4\phi,\nabla^2\phi)=\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial x},\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right)+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial y},\varepsilon\frac{\partial\nabla^2\phi}{\partial y}\right)$$
$$-\varepsilon^2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot h\Big|_{x=0}^{x=\alpha}dy-\varepsilon^2\int_0^\alpha\frac{\partial}{\partial y}\nabla^2\phi\cdot h\Big|_{y=0}^{y=\beta}dx \tag{2.8}$$

因为

$$\left|\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot h(\alpha,y)dy\right|\leqslant\int_0^\beta\varepsilon\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2(\alpha,y)dy+\frac{1}{\varepsilon}\int_0^\beta h^2(\alpha,y)dy$$

$$\left|\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2(\alpha,y)dy\right|\leqslant c_1\int_0^\alpha\int_0^\beta\left(\frac{\partial^2\nabla^2\phi}{\partial x^2}\right)^2dxdy$$
$$+\frac{1}{c_1}\int_0^\alpha\int_0^\beta\left(\frac{\partial}{\partial x}\nabla^2\phi\right)^2dxdy \qquad (c_1>0,\ \text{常数})$$

所以，利用引理2和引理3我们可得

$$\left|\varepsilon^2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot h(\alpha,y)dy\right|\leqslant\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2\right]ds$$
$$+\varepsilon\|f\|^2_{H^0(D)}+\varepsilon\|\phi\|^2_{H^2(D)}+\varepsilon\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial x},\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right) \tag{2.9}$$

同理可得

$$\left|\varepsilon^2\int_0^\beta\frac{\partial\nabla^2\phi}{\partial x}\cdot h(0,y)dy\right|\leqslant\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2\right]ds$$
$$+\varepsilon\|f\|^2_{H^0(D)}+\varepsilon\|\phi\|^2_{H^2(D)}+\varepsilon\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial x},\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right) \tag{2.10}$$

$$\left|\varepsilon^2\int_0^\alpha\frac{\partial}{\partial y}\nabla^2\phi\cdot h(x,s)dx\right|\leqslant\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2\right]ds$$
$$+\varepsilon\|f\|^2_{H^0(D)}+\varepsilon\|\phi\|^2_{H^2(D)}+\varepsilon\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial y},\varepsilon\frac{\partial\nabla^2\phi}{\partial y}\right) \qquad (s=0,\beta)$$

因此由(2.8)，(2.9)和(2.10)得到(2.7)左端第一项的估计

$$(\varepsilon^2\nabla^4\phi,\nabla^2\phi)+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial x},\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right)+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial y},\varepsilon\frac{\partial\nabla^2\phi}{\partial y}\right)$$
$$\leqslant\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2\right]ds+\varepsilon\|f\|^2_{H^0(D)}+\varepsilon\|\phi\|^2_{H^0(D)} \tag{2.10$'$}$$

等式(2.7)左端的第二、三两项可以改写为

$$\left(a(x,y)\frac{\partial^2\phi}{\partial x^2}+b(x,y)\frac{\partial^2\phi}{\partial y^2},\ \nabla^2\phi\right)=\left(a(x,y)\frac{\partial^2\phi}{\partial x^2},\frac{\partial^2\phi}{\partial x^2}\right)$$

$$+\left(b(x,y)\frac{\partial^2\phi}{\partial y^2},\frac{\partial^2\phi}{\partial y^2}\right)+\left((a(x,y)+b(x,y)\frac{\partial^2\phi}{\partial x^2},\frac{\partial^2\phi}{\partial y^2}\right) \tag{2.11}$$

而

$$\int_0^\alpha\int_0^\beta(a+b)\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial^2\phi}{\partial y^2}dxdy=\int_0^\beta(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy$$

$$-\int_0^\alpha\int_0^\beta\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial x}\left((a+b)\frac{\partial^2\phi}{\partial y^2}\right)dxdy$$

$$=\int_0^\beta(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy-\int_0^\alpha\int_0^\beta\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x}(a+b)\frac{\partial^2\phi}{\partial y^2}dxdy$$

$$-\int_0^\alpha\int_0^\beta(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^3\phi}{\partial x\partial y^2}dxdy$$

$$=\int_0^\beta(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy-\int_0^\alpha\int_0^\beta\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x}(a+b)\frac{\partial^2\phi}{\partial y^2}dxdy$$

$$-\int_0^\alpha(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big|_{y=0}^{y=\beta}dx+\int_0^\alpha\int_0^\beta\frac{\partial}{\partial y}(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}dxdy$$

$$+\int_0^\alpha\int_0^\beta(a+b)\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2dxdy \tag{2.12}$$

由边界条件得到

$$\int_0^\beta(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy=\int_0^\beta(a+b)k\cdot\phi\cdot\frac{\partial^2\phi}{\partial y^2}((\alpha,y)+(0,y))dy$$

$$-\int_0^\beta(a+b)g\cdot\frac{\partial^2\phi}{\partial y^2}((\alpha,y)+(0,y))dy$$

$$=(a+b)\cdot k\cdot\phi\frac{\partial\phi}{\partial y}((\alpha,y)+(0,y))\Big|_{y=0}^{y=\beta}-\int_0^\beta\frac{\partial}{\partial y}((a+b)\cdot k\phi)\frac{\partial\phi}{\partial y}((\alpha,y)$$

$$+(0,y))dy-\int_0^\beta(a+b)g\cdot\frac{\partial^2\phi}{\partial y^2}((\alpha,y)+(0,y))dy$$

$$=(a+b)\cdot k^2\cdot\phi^2((\alpha,\beta)+(\alpha,0)+(0,\beta)+(0,0))$$

$$-(a+b)\cdot k\phi g((\alpha,\beta)+(\alpha,0)+(0,\beta)+(0,0))$$

$$-\int_0^\beta\frac{\partial}{\partial y}((a+b)k\phi)\cdot\frac{\partial\phi}{\partial y}((\alpha,y)+(0,y))dy$$

$$-(a+b)g\cdot\frac{\partial\phi}{\partial y}((\alpha,y)+(0,y))\Big|_{y=0}^{y=\beta}$$

$$+\int_0^\beta\frac{\partial}{\partial y}((a+b)g)\cdot\frac{\partial\phi}{\partial y}((\alpha,y)+(0,y))dy$$

$$=(a+b)\cdot k^2\phi^2((\alpha,\beta)+(\alpha,0)+(0,\beta)+(0,0))-2(a+b)\cdot k\phi g((\alpha,\beta)$$

$$+(\alpha,0)+(0,\beta)+(0,0))-\int_0^\beta\frac{\partial}{\partial y}((a+b)k\phi)\frac{\partial\phi}{\partial y}((\alpha,y)+(0,y))dy$$

$$+(a+b)g^2((\alpha,\beta)+(\alpha,0)+(0,\beta)+(0,0))$$

$$+\int_0^\beta \frac{\partial}{\partial y}((a+b)g)\cdot\frac{\partial\phi}{\partial y}((a,y)+(0,y))dy \tag{2.13}$$

因为

$$\int_0^\beta \frac{\partial}{\partial y}((a+b)k\phi)\cdot\frac{\partial\phi}{\partial y}((a,y)+(0,y))dy$$

$$=\int_0^\beta k\frac{\partial}{\partial y}(a+b)\cdot\phi\frac{\partial\phi}{\partial y}((a,y)+(0,y))dy$$

$$+\int_0^\beta k(a+b)\left(\frac{\partial\phi}{\partial y}\right)^2((a,y)+(0,y))dy$$

所以

$$\int_0^\beta \frac{\partial}{\partial y}((a+b)k\phi)\frac{\partial\phi}{\partial y}((a,y)+(0,y))dy$$

$$-\int_0^\beta k(a+b)\left(\frac{\partial\phi}{\partial y}\right)^2((a,y)+(0,y))dy$$

$$=\int_0^\beta k\frac{\partial}{\partial y}(a+b)\cdot\phi\cdot\frac{\partial\phi}{\partial y}((a,y)+(0,y))dy$$

$$\leqslant\frac{1}{\omega_1}\int_0^\beta \phi^2((a,y)+(0,y)dy$$

$$+\omega_1\int_0^\beta\left(k\frac{\partial}{\partial y}(a+b)\cdot\frac{\partial\phi}{\partial y}\right)^2((a,y)+(0,y))dy$$

适当地选取 ω_1 并利用引理 1 得到

$$\int_0^\beta \frac{\partial}{\partial y}((a+b)k\phi)\cdot\frac{\partial\phi}{\partial y}((a,y)+(0,y))dy$$

$$+c_1\int_0^\beta\left(\frac{\partial\phi}{\partial y}\right)^2((a,y)+(0,y))dy$$

$$\leqslant c_2\int_0^\beta \phi^2((a,y)+(0,y))dy\leqslant c_3\int_0^a\int_0^\beta\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\phi^2\right)dxdy \tag{2.13a}$$

这里 c_1, c_2, c_3 为正常数.

由不等式(2.6)得到

$$\left|\int_0^\beta \frac{\partial}{\partial y}((a+b)g)\cdot\frac{\partial\phi}{\partial x}((a,y)+(0,y))dy\right|$$

$$\leqslant\omega_1\int_0^\beta\left(\frac{\partial}{\partial y}((a+b)g)\right)^2((a,y)+(0,y))dy$$

$$+\frac{1}{\omega_1}\int_0^\beta\left(\frac{\partial\phi}{\partial y}\right)^2((a,y)+(0,y))dy \tag{2.13b}$$

ω_1 为任意正常数．由不等式 $|uv|\leqslant\omega_1|u|^2+\frac{1}{\omega_1}|v|^2$ 得到

$$|(a+b)k\phi g((a,\beta)+(a,0)+(0,\beta)+(0,0))|$$

$$\leqslant\omega_2((a+b)kg)^2((a,\beta)+(a,0)+(0,\beta)+(0,0))$$

$$+\frac{1}{\omega_2}\phi^2((a,\beta)+(a,0)+(0,\beta)+(0,0)) \tag{2.13c}$$

其中 ω_2 为任意正常数.

因此适当地选取ω_1, ω_2, 由(2.13), (2.13a), (2.13b)及(2.13c) 可以得到估计式

$$
\begin{aligned}
&-\int_0^\beta (a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy+c_1(a+b)\cdot k^2\phi^2((\alpha,\beta)+(\alpha,0)\\
&\quad+(0,\beta)+(0,0))+c_2\int_0^\beta\left(\frac{\partial\phi}{\partial y}\right)^2((\alpha,y)+(0,y))dy\\
&\leqslant c_3\int_0^\alpha\int_0^\beta\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\phi^2\right)dxdy\\
&\quad+c_4\int_0^\beta\left(\frac{\partial}{\partial y}((a+b)g)\right)^2((\alpha,y)+(0,y))dy\\
&\quad+c_5((a+b)\cdot kg)^2((\alpha,\beta)+(\alpha,0)+(0,\beta)+(0,0))\\
&\leqslant c_3\int_0^\alpha\int_0^\beta\left(\left(\frac{\partial\phi}{\partial x}\right)^2+\phi^2\right)dxdy+c_4\int_\Gamma\left(\left(\frac{\partial g}{\partial s}\right)^2+g^2\right)ds
\end{aligned}
\tag{2.14}
$$

而

$$
\begin{aligned}
&-\int_0^\alpha(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big|_{y=0}^{y=\beta}dx\\
&\quad=-\int_0^\alpha(a+b)\frac{\partial\phi}{\partial x}\cdot\left(k\frac{\partial\phi}{\partial x}-\frac{\partial g}{\partial x}\right)((x,\beta)+(x,0))dx\\
&\quad=-\int_0^\alpha k(a+b)\left(\frac{\partial\phi}{\partial x}\right)^2((x,\beta)+(x,0))dx\\
&\quad\quad+\int_0^\alpha(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial g}{\partial x}((x,\beta)+(x,0))dx
\end{aligned}
$$

由此易得

$$
\begin{aligned}
&\int_0^\alpha(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big|_{y=0}^{y=\beta}dx+c_1\int_0^\alpha\left(\frac{\partial\phi}{\partial x}\right)^2((x,\beta)+(x,0))dx\\
&\quad\leqslant c_2\int_\Gamma\left(\frac{\partial g}{\partial s}\right)^2ds
\end{aligned}
\tag{2.15}
$$

这里 c_1, c_2 为正常数.

利用不等式(2.6)得到

$$
\begin{aligned}
&\left|-\int_0^\alpha\int_0^\beta\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x}(a+b)\frac{\partial^2\phi}{\partial y^2}dxdy+\int_0^\alpha\int_0^\beta\frac{\partial}{\partial y}(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}dxdy\right|\\
&\leqslant c_1\|\phi\|^2_{H^2(D)}+c_2\|\phi\|^2_{H^1(D)}
\end{aligned}
\tag{2.16}
$$

这里 c_1, c_2 是任意正常数.

由(2.12), 我们有

$$
\begin{aligned}
&-\int_0^\alpha\int_0^\beta(a+b)\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial^2\phi}{2y^2}dxdy+\int_0^\alpha\int_0^\beta(a+b)\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2dxdy\\
&=-\int_0^\beta(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y^2}\Big|_{x=0}^{x=\alpha}dy+\int_0^\alpha\int_0^\beta\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x}(a+b)\frac{\partial^2\phi}{\partial y^2}dxdy\\
&\quad+\int_0^\alpha(a+b)\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big|_{y=0}^{y=\beta}dx-\int_0^\alpha\int_0^\beta\frac{\partial}{\partial x}(a+b)\frac{\partial\phi}{\phi x}\cdot\frac{\partial^2\phi}{\partial x\partial y}dxdy
\end{aligned}
$$

从而由(2.14)~(2.16)以及不等式(2.6)得到下面估计式:

$$-\int_0^\alpha\int_0^\beta (a+b)\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial^2\phi}{\partial y^2}dxdy+\int_0^\alpha\int_0^\beta (a+b)\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 dxdy$$

$$+c_1(a+b)\cdot k^2\phi^2((\alpha,\beta)+(\alpha,0)+(0,\beta)+(0,0))$$

$$+c_2\left(\int_0^\beta\left(\frac{\partial\phi}{\partial y}\right)^2((\alpha,y)+(0,y))dy+\int_0^\alpha\left(\frac{\partial\phi}{\phi x}\right)^2((x,\beta)+(x,0))dx\right)$$

$$\leqslant c_3\|\phi\|^2_{H^2(D)}+c_4\|\phi\|^2_{H^1(D)}+c_5\|\phi\|^2_{H^0(D)}+c_6\int_\Gamma\left(\left(\frac{\partial g}{\partial s}\right)^2+g^2\right)ds \quad (2.17)$$

这里 c_1, c_1, c_3, c_4, c_5, c_6 为任意正常数。

于是有

$$-\int_0^\alpha\int_0^\beta (a+b)\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial^2\phi}{\partial y^2}dxdy+\int_0^\alpha\int_0^\beta (a+b)\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 dxdy$$

$$\leqslant c_3\|\phi\|^2_{H^2(D)}+c_4\|\phi\|^2_{H^1(D)}+c_5\|\phi\|^2_{H^0(D)}+c_6\int_\Gamma\left[\left(\frac{\partial g}{\partial s}\right)^2+g^2\right]ds \quad (2.17)'$$

从而由(2.7), (2.10)', (2.11), (2.12)及(2.17)', 得到

$$\left(\varepsilon\frac{\phi\nabla^2\phi}{\partial x},\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right)+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial y},\varepsilon\frac{\partial\nabla^2\phi}{\partial y}\right)+\left(a(x,y)\frac{\partial^2\phi}{\partial x^2},\frac{\partial^2\phi}{\partial x^2}\right)$$

$$+\left(b(x,y)\frac{\partial^2\phi}{\partial y^2},\frac{\partial^2\phi}{\partial y^2}\right)+\left((a+b)\frac{\partial^2\phi}{\partial x\partial y},\frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$\leqslant\left|\left(c(x,y)\frac{\partial\phi}{\partial x},\nabla^2\phi\right)\right|+|(d(x,y)\phi,\nabla^2\phi)|+|(f(x,y),\nabla^2\phi)|$$

$$+\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2\right]ds+\varepsilon\|f\|^2_{H^0(D)}+\varepsilon\|\phi\|^2_{H^0(D)}$$

$$+c_3\|\phi\|^2_{H^2(D)}+c_4\|\phi\|^2_{H^1(D)}+c_5\|\phi\|^2_{H^0(D)}+c_6\int_\Gamma\left[\left(\frac{\partial g}{\partial s}\right)^2+g^2\right]ds$$

适当选取 c_3 即可得到引理 4 的结论。

现在来估计(2.5)右端的第二项。由引理1我们有

$$\left|\varepsilon^2\int_\Gamma\phi\cdot\frac{\partial\nabla^2\phi}{\partial n}ds\right|\leqslant c_1\left(\int_\Gamma\varepsilon^2\left(\phi\cdot\frac{\partial\nabla^2\phi}{\partial n}\right)^2ds\right)^{\frac12}$$

$$\leqslant\left\{\frac{c_1}{\delta}\int_0^\alpha\int_0^\beta\left[\left(\varepsilon^2\phi\cdot\frac{\partial\nabla^2\phi}{\partial x}\right)^2+\left(\varepsilon^2\phi\cdot\frac{\partial\nabla^2\phi}{\partial y}\right)\right]dxdy\right.$$

$$\left.+c_2\delta\int_0^\alpha\int_0^\beta\left[\left(\frac{\partial}{\partial x}\left(\varepsilon^2\phi\cdot\frac{\partial\nabla^2\phi}{\partial x}\right)\right)^2+\left(\frac{\partial}{\partial y}\left(\varepsilon^2\phi\cdot\frac{\partial\nabla^2\phi}{\partial y}\right)\right)^2\right]dxdy\right\}^{\frac12}$$

取 $\delta=\varepsilon$, 并利用引理2, 引理3及引理4, 可得到(2.5)右端第二项的如下估计:

$$\left|\varepsilon^2\int_\Gamma\phi\cdot\frac{\partial\nabla^2\phi}{\partial n}ds\right|\leqslant\left\{c_1\frac{\varepsilon^2}{\varepsilon}\int_0^\alpha\int_0^\beta\left[\left(\varepsilon\phi\cdot\frac{\partial\nabla^2\phi}{\partial x}\right)^2+\left(\varepsilon\phi\cdot\frac{\partial\nabla^2\phi}{\partial y}\right)^2\right]dxdy\right.$$

$$+c_2\varepsilon\int_0^\alpha\int_0^\beta\left[\left(\varepsilon^2\frac{\partial\phi}{\partial x}\cdot\frac{\partial\nabla^2\phi}{\partial x}+\varepsilon^2\phi\cdot\frac{\partial^2\nabla^2\phi}{\partial x^2}\right)^2\right.$$

$$\left.\left.+\left(\varepsilon^2\frac{\partial\phi}{\partial y}\cdot\frac{\partial\nabla^2\phi}{\partial y}+\varepsilon^2\phi\cdot\frac{\partial^2\nabla^2\phi}{2y^2}\right)^2\right]dxdy\right\}^{\frac12}$$

$$\leqslant c_3\varepsilon^{\frac12}\left\{\int_0^\alpha\int_0^\beta\left[\phi^4+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right)^2+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial y}\right)^2\right]dxdy\right.$$

$$+\int_0^\alpha\int_0^\beta\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^4+\Big(\frac{\partial\phi}{\partial y}\Big)^4+\phi^4+\Big(\varepsilon^2\frac{\partial\nabla^2\phi}{\partial x}\Big)^2$$

$$+\Big(\varepsilon^2\frac{\partial\nabla^2\phi}{\partial y}\Big)^2+\Big(\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial x^2}\Big)^2+\Big(\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial y^2}\Big)\Big]dxdy\Big\}^{\frac{1}{2}}$$

$$\leqslant c_3\varepsilon^{\frac{1}{2}}\int_0^\alpha\int_0^\beta\Big(f^2+\phi^2+\Big(\frac{\partial\phi}{\partial x}\Big)^2+\Big(\frac{\partial\phi}{\partial y}\Big)^2\Big)dxdy$$

$$+\varepsilon\int_\Gamma\Big(\Big(\frac{\partial^2h}{\partial s^2}\Big)^2+\Big(\frac{\partial h}{\partial s}\Big)^2+h^2+\Big(\frac{\partial g}{\partial s}\Big)^2+g^2\Big)ds \tag{2.18}$$

此外，在(2.4)中

$$-\Big(a(x,y)\frac{\partial^2\phi}{\partial x^2}+b(x,y)\frac{\partial^2\phi}{\partial y^2}+c(x,y)\frac{\partial\phi}{\partial x}+d(x,y)\phi,\phi\Big)$$

$$=\int_0^\beta\int_0^\alpha\Big[a(x,y)\Big(\frac{\partial\phi}{\partial x}\Big)^2+b(x,y)\Big(\frac{\partial\phi}{\partial y}\Big)^2\Big]dxdy$$

$$-\int_0^\beta a(x,y)\frac{\partial\phi}{\partial x}\cdot\phi\Big|_{x=0}^{x=\alpha}dy-\int_0^\alpha b(x,y)\frac{\partial\phi}{\partial y}\cdot\phi\Big|_{y=0}^{y=\beta}dx$$

$$+\int_0^\alpha\int_0^\beta\Big[\Big(\frac{\partial a}{\partial x}-c\Big)\frac{\partial\phi}{\partial x}+\frac{\partial b}{\partial y}\cdot\frac{\partial\phi}{\partial y}\Big]\cdot\phi dxdy$$

$$-\int_0^\beta\int_0^\alpha d(x,y)\phi^2dxdy \tag{2.19}$$

利用边界条件(2.2)及引理1有下面的估计式

$$\Big|\int_0^\beta a\frac{\partial\phi}{\partial x}\cdot\phi\Big|_{x=0}dy\Big|\leqslant\Big|\int_0^\beta ak\phi^2(0,y)dy\Big|+\Big|\int_0^\beta ag\phi(0,y)dy\Big|$$

$$\leqslant c_1\int_0^\beta\phi^2(0,y)dy+\bar{c}_1\int_0^\beta a^2g^2(0,y)dy$$

$$\leqslant c_1\int_0^\alpha\int_0^\beta\Big(\frac{\partial\phi}{\partial x}\Big)^2dxdy+c_2\int_0^\alpha\int_0^\beta\phi^2dxdy+c_3\int_0^\alpha\int_0^\beta g^2(0,y)dy \tag{2.20}$$

这里 c_1，c_2，c_3 为任意正常数。

同理可得

$$\Big|\int_0^\beta a\frac{\partial\phi}{\partial x}\cdot\phi\Big|_{x=\alpha}dy\Big|\leqslant c_1\int_0^\alpha\int_0^\beta\Big(\frac{\partial\phi}{\partial x}\Big)^2dxdy+c_2\int_0^\alpha\int_0^\beta\phi^2dxdy$$

$$+c_3\int_0^\beta g^2(\beta,y)dy$$

$$\Big|\int_0^\alpha b\frac{\partial\phi}{\partial y}\cdot\phi\Big|_{y=0}^{y=\beta}dx\Big|\leqslant c_1\int_0^\alpha\int_0^\beta\Big(\frac{\partial\phi}{\partial y}\Big)^2dxdy+c_2\int_0^\alpha\int_0^\beta dxdy+c_3\int_\Gamma g^2ds \tag{2.21}$$

这里 c_1，c_2，c_3 为任意正常数。

将(2.5)，(2.18)，(2.19)，(2.20)，(2.21)代入(2.4)，则得下面的定理。

定理1 在对方程(2.1)的系数和右端函数所作的假设条件下，问题(2.1)～(2.3)的解有如下先验估计：

$$(\varepsilon\nabla^2\phi,\ \varepsilon\nabla^2\phi)+\Big(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial x}\Big)+\Big(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\Big)+(\phi,\phi)$$

$$\leqslant c\|f\|^2_{H^0(D)}+\bar{c}\int_\Gamma g^2ds+\bar{c}\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+h^2+\left(\frac{\partial g}{\partial s}\right)^2+g^2\right]ds$$

这里 c，$\bar{c}$ 为正的常数。

从引理2，3，4及定理1，我们可以直接得出下面的推论。

推论1

$$\left(\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial x^2},\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial x^2}\right)+\left(\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial y^2},\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial y^2}\right)+\left(\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial x\partial y},\varepsilon^2\frac{\partial^2\nabla^2\phi}{\partial x\partial y}\right)$$

$$+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial x},\varepsilon\frac{\partial\nabla^2\phi}{\partial x}\right)+\left(\varepsilon\frac{\partial\nabla^2\phi}{\partial y},\varepsilon\frac{\partial\nabla^2\phi}{\partial y}\right)+\|\phi\|^2_{H^2(D)}$$

$$\leqslant M\|f\|^2_{H^0(D)}+N\int_\Gamma g^2ds+\varepsilon\int_\Gamma\left[\left(\frac{\partial^2 h}{\partial s^2}\right)^2+\left(\frac{\partial h}{\partial s}\right)^2+\left(\frac{\partial g}{\partial s}\right)^2+g^2+h^2\right]ds$$

这里 M，N是正的常数。

三、渐 近 解 的 构 造

我们看出，当$\varepsilon=0$时问题(2.1)～(2.3)退化为问题：

$$a(x,y)\frac{\partial^2 w}{\partial x^2}+b(x,y)\frac{\partial^2 w}{\partial y^2}+c(x,y)\frac{\partial w}{\partial x}+d(x,y)w=f(x,y) \tag{3.1}$$

$$\left(\frac{\partial w}{\partial n}+kw\right)_\Gamma=g(x,y) \qquad (k<0，常数) \tag{3.2}$$

因此在区域D的四条边上都要产生边界层．记

$$\Gamma_1=\{(x,y):\ 0\leqslant x\leqslant\alpha,\ y=0\},\qquad \Gamma_2=\{x,y\}:\ x=\alpha,\ 0\leqslant y\leqslant\beta\}$$
$$\Gamma_3=\{(x,y):\ 0\leqslant x\leqslant\alpha,\ y=\beta\},\qquad \Gamma_4=(x,y):\ x=0,\ 0\leqslant y\leqslant\beta\}$$

并用$v^{(\alpha)}$，$v^{(\beta)}$，$v^{(\gamma)}$，$v^{(\delta)}$分别表示Γ_4，Γ_2，Γ_1和Γ_3附近的边界层函数：

$$\left.\begin{aligned}
v^{(\alpha)}&=\sum_{n=0}^{\infty}\varepsilon^{n+2}v_n^{(\alpha)}\left(\frac{x}{\varepsilon},y\right)\\
v^{(\beta)}&=\sum_{n=0}^{\infty}\varepsilon^{n+2}v_n^{(\beta)}\left(\frac{\alpha-x}{\varepsilon},y\right)\\
v^{(\gamma)}&=\sum_{n=0}^{\infty}\varepsilon^{n+2}v_n^{(\gamma)}\left(x,\frac{y}{\varepsilon}\right)\\
v^{(\delta)}&=\sum_{n=0}^{\infty}\varepsilon^{n+2}v_n^{(\delta)}\left(x,\frac{\beta-y}{\varepsilon}\right)
\end{aligned}\right\} \tag{3.3}$$

我们构造(2.1)～(2.3)的渐近解为以下形式：

$$\phi_\varepsilon=\sum_{n=0}^{\infty}\varepsilon^n w_n(x,y)+v^{(\alpha)}+v^{(\beta)}+v^{(\gamma)}+v^{(\delta)} \tag{3.4}$$

这里的$\sum_{n=0}^{\infty}\varepsilon^n w_n(x,y)$为$\phi_\varepsilon$的外展开式。

将表达式(3.4)式代入(2.1)～(2.3)我们有

$$-\varepsilon^2\sum_{n=0}^{\infty}\varepsilon^n\nabla^4 w_n+\sum_{n=0}^{\infty}\varepsilon^n\Big[a(x,y)\frac{\partial^2}{\partial x^2}+b(x,y)\frac{\partial^2}{\partial y^2}+c(x,y)\frac{\partial}{\partial x}$$

$$+d(x,y)\Big]w_n+\sum_{n=0}^{\infty}(-\varepsilon^2)\varepsilon^{n+2}\nabla^4(v_n^{(\alpha)}+v_n^{(\beta)}+v_n^{(\gamma)}+v_n^{(\delta)})$$

$$+\sum_{n=0}^{\infty}\varepsilon^{n+2}\Big(c(x,y)\frac{\partial}{\partial x}+d(x,y)\Big)(v_n^{(\alpha)}+v_n^{(\beta)}+v_n^{(\gamma)}+v_n^{(\delta)})$$

$$=f(x,y) \tag{3.5}$$

$$\left\{\sum_{n=0}^{\infty}\varepsilon^n\Big(\frac{\partial w_n}{\partial n}+kw_n\Big)+\sum_{n=0}^{\infty}\varepsilon^{n+2}\Big(\frac{\partial}{\partial n}+k\Big)(v_n^{(\alpha)}+v_n^{(\beta)}+v_n^{(\gamma)}+v_n^{(\delta)})\right\}\Big|_\Gamma$$

$$=g(x,y) \tag{3.6}$$

$$\left\{\sum_{n=0}^{\infty}\varepsilon^n\nabla^2 w_n+\sum_{n=0}^{\infty}\varepsilon^{n+2}\nabla^2(v_n^{(\alpha)}+v_n^{(\beta)}+v_n^{(\gamma)}+v_n^{(\delta)})\right\}\Big|_\Gamma=0 \tag{3.7}$$

按 Lyusterik-Vishik 方法，利用算子 L_ε 的原来分解以及 L_ε 在$\Gamma_i(i=1,2,3,4)$ 附近的第二次分解，由(3.5)，(3.6)，(3.7)得到确定w_i和 $v_i^{(s)}(s=\alpha,\beta,\gamma,\delta)$ 方程和定解条件：

$$\left.\begin{aligned}&a(x,y)\frac{\partial^2 w_0}{\partial x^2}+b(x,y)\frac{\partial^2 w_0}{\partial y^2}+c(x,y)\frac{\partial w_0}{\partial x}+d(x,y)w_0\\&=f(x,y)\\&\Big(\frac{\partial w_0}{\partial n}+kw_0\Big)\Big|_\Gamma=g(x,y)\end{aligned}\right\} \tag{3.8}$$

$$\left.\begin{aligned}&\frac{\partial^4 v_0^{(\alpha)}}{\partial t_1^4}-a(0,y)\frac{\partial^2 v_0^{(\alpha)}}{\partial t_1^2}=0\\&\frac{\partial^2 v_0^{(\alpha)}}{\partial t_1^2}\Big|_{t_1=0}=-\nabla^2 w_0+h(x,y)\Big]_{x=0}\end{aligned}\quad (t_1=x/\varepsilon)\right\} \tag{3.9}$$

由(3.9)可以求得 $v_0^{(\alpha)}=g^{(\alpha)}(y)\exp(-\sqrt{a(0,y)}t_1)$，这里 $g^{(\alpha)}(y)$ 是关于 y 有二阶连续导数的函数。

$$\left.\begin{aligned}&\frac{\partial^4 v_0^{(\beta)}}{\partial t_2^4}-b(x,0)\frac{\partial^2 v_0^{(\beta)}}{\partial t_2^2}=0\\&\frac{\partial^2 v_0^{(\beta)}}{\partial t_2^2}\Big|_{t_2=0}=-\nabla^2 w_0+h(x,y)\Big]_{y=0}\end{aligned}\quad (t_2=y/\varepsilon)\right\} \tag{3.10}$$

由 (3.10) 可以求得$v_0^{(\beta)}=g^{(\beta)}(x)\exp(-\sqrt{b(x,0)}\cdot t_2)$.

同理可得到在$x=\alpha$和$y=\beta$附近边界层函数的首项 $v_0^{(\gamma)}$ 和 $v_0^{(\delta)}$。

一般地，函数w_i及$v_i^{(\delta)}$，$v_i^{(\beta)}$，$v_i^{(\gamma)}$，$v_i^{(\delta)}$由下面的递推方程确定 $(i=2,3,4,\cdots)$

$$\left.\begin{aligned}&a(x,y)\frac{\partial^2 w_i}{\partial x^2}+b(x,y)\frac{\partial^2 w_i}{\partial y^2}+c(x,y)\frac{\partial w_i}{\partial x}+d(x,y)w_i=\nabla^4 w_{i-2}\\&\Big(\frac{\partial w_i}{\partial n}+kw_i\Big)\Big|_\Gamma=\Big\{-\varepsilon\frac{\partial}{\partial n}(v_{i-1}^{(\alpha)}+v_{i-1}^{(\beta)}+v_{i-1}^{(\gamma)}+v_{i-1}^{(\delta)})\\&\qquad-k(v_{i-2}^{(\alpha)}+v_{i-2}^{(\beta)}+v_{i-2}^{(\gamma)}+v_{i-2}^{(\delta)})\Big\}\Big|_\Gamma\end{aligned}\right\} \tag{3.11}$$

$$
\left.
\begin{aligned}
&-\frac{\partial^4 v_i^{(a)}}{\partial t_1^4}+a(0,y)\frac{\partial^2 v_i^{(a)}}{\partial t_1^2}=\sum_{j=0}^{i-1}Q_j(t_1,y)\frac{\partial^2 v_j^{(a)}}{\partial t_1^2}\\
&\frac{\partial^2}{\partial t_1^2}v_i^{(a)}\Big|_{t_1=0}=-\nabla^2 w_i\Big|_{x=0}-\frac{\partial^2}{\partial y^2}v_{i-2}^{(a)}\Big|_{t_1=0}
\end{aligned}
\right\}
\quad (3.12)
$$

其中 $Q_j(t_1,y)$为关于t_1的多项式.

类似于(3.12)，我们可以求得$v_i^{(\beta)}$，$v_i^{(\gamma)}$，$v_i^{(\delta)}$满足的方程和定解条件.

从而我们可以得到问题(2.1)～(2.3)解的渐近展开式的定理.

定理2 在$\bar{D}$内问题(2.1)～(2.3)的解有如下的渐近展开式:

$$
\phi_\varepsilon=\sum_{n=0}^{N}\varepsilon^n w_n(x,y)+\sum_{n=0}^{N}\varepsilon^{n+2}(\bar{v}_n^{(a)}+\bar{v}_n^{(\beta)}+\bar{v}_n^{(\gamma)}+v_n^{(\delta)})+R_N(x,y,\varepsilon)
$$

$$
\|R_N\|=O(\varepsilon^{N+1})
$$

其中 $\bar{v}_n^{(a)}$，$\bar{v}_n^{(\beta)}$，$\bar{v}_n^{(\gamma)}$，$\bar{v}_n^{(\delta)}$分别由$v_n^{(a)}$，$v_n^{(\beta)}$，$v_n^{(\gamma)}$，$v_n^{(\delta)}$乘上平滑函数以后得到的.

证明 由(2.1)～(2.3)，(3.5)～(3.12)得到余项$R_N(x,y,\varepsilon)$所满足的方程和定解条件：

$$
\mathscr{L}_\varepsilon R_N=O(\varepsilon^{N+1}),\ \left(\frac{\partial R_N}{\partial n}+kR_N\right)\Big|_\Gamma=O(\varepsilon^{N+1}),\ \nabla^2 R_N\Big|_\Gamma=O(\varepsilon^{N+1})
$$

利用定理1中的能量不等式即得到所需的结论.

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Asymptotic Solution of Singular Perturbation Problems for the Fourth-Order Elliptic Differential Equations

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Abstract

In this paper we consider the singularly perturbed boundary value problem for the fourth-order elliptic differential equation, establish the energy estimates of the solution and its derivatives and construct the formal asymptotic solution by Lyuternik-Vishik's method. Finally, by means of the energy estimates we obtain the bound of the remainder of the asymptotic solution.