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具有初始跳跃的双曲型方程奇异摄动问题的数值解法

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摘　要

本文考虑具有初始跳跃的二阶双曲型方程初边值问题．首先给出解的导数估计．然后在一非均匀网格上构造了一个差分格式．最后在能量范数意义下证明了差分格式解的一致收敛性．

一、引　言

我们在区域$\bar{D}=\{0\leqslant x\leqslant l,\ 0\leqslant t\leqslant T\}$内讨论二阶双曲型方程的初边值问题:

$$L_\varepsilon u\equiv\varepsilon\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right)+A(x,t)\frac{\partial u}{\partial t}+B(x,t)u=F(x,t)\qquad(x,t)\in D\tag{1.1}$$

$$u(x,0)=\varphi(x),\ \varepsilon\partial u(x,0)/\partial t=\psi(x)\qquad 0\leqslant x\leqslant l\tag{1.2}$$

$$u(0,t)=0,\ u(l,t)=0\qquad 0\leqslant t\leqslant T\tag{1.3}$$

这是一个具有初始跳跃的奇异摄动问题．这方面问题的渐近解最早是由Lyusternik和Vishik[1,2]讨论的．此后Kasymov对常微分方程和偏微分方程作出了一系列的工作，例如[3～6]．Hsiao和Weinacht[7,8]对线性和半线性双曲-抛物方程Cauchy问题的解构造了一致有效渐近展开式．1980年，Pečenkina[9]对二阶线性常微分方程周期问题研究了差分解法．本文对二阶双曲型方程的初边值问题(1.1)～(1.3)研究了关于小参数ε一致收敛的差分方法．

我们作如下假定

H1　函数$A(x,t)$，$B(x,t)$，$F(x,t)$，$\varphi(x)$和$\psi(x)$在区域$\bar{D}$内充分光滑，且$A(x,t)\geqslant\alpha>0$，$B(x,t)\geqslant\beta>0$对一切$x\in\bar{D}$成立．函数$\psi(x)\neq0(0<x<l)$．

H2　上述函数满足一定的相容性条件使得问题(1.1)～(1.3)的解$u(x,t)\in C^4(\bar{D})$．这些条件至少包括

$$\varphi(0)=0,\ \psi(0)=0;\ \varphi(l)=0,\ \psi(l)=0;$$
$$\varphi''(0)=0,F(0,0)=0;\ \varphi''(l)=0,\ F(l,0)=0$$

本文在第二节构造了问题(1.1)～(1.3)的渐近解；第三节给出渐近解的余项估计及解的导数估计；第四节在非均匀网格上建立数值解问题(1.1)～(1.3)的一个差分格式；第五节和第六节讨论了离散的能量不等式并证明在离散能量范数意义下差分解一致收敛于精确解．

二、渐近解的构造

由于(1.2)中的第二个初始条件含有小参数ε，因此与通常的情形不同，当$\psi\neq0$时，摄动问题(1.1)～(1.3)不能直接退化为

$$L_0w\equiv A(x,t)\partial w/\partial t+B(x,t)w=F(x,t) \tag{2.1}$$

$$w(x,0)=\varphi(x) \tag{2.2}$$

在这里应存在初始跳跃，即退化问题的初始条件应为

$$w(x,0)=\varphi(x)+\Delta(x) \tag{2.3}$$

其中$\Delta(x)$为初始跳跃，我们确定于下。

众所周知，方程(1.1)在$t=0$附近的奇异部分是$\varepsilon\partial^2u/\partial t^2+A(x,0)\partial u/\partial t=0$。在初始条件(1.2)下解此方程，则得到$u=(x,t,\varepsilon)=\varphi(x)+[1-\exp[-A(x,0)t/\varepsilon]]\psi(x)/A(x,0)$。从而有$\partial u/\partial t=\psi(x)\exp[-A(x,0)t/\varepsilon]/\varepsilon$。当$t=t_\varepsilon=-\varepsilon\ln\varepsilon/\alpha$时有$|\partial u(x,t_\varepsilon,\varepsilon)/\partial t|\leqslant M=\text{const}$。假定$\lim\limits_{\varepsilon\to0}u(x,t_\varepsilon,\varepsilon)=\varphi(x)+\Delta(x)$。对方程$\varepsilon\partial^2u/\partial t^2+A(x,0)\partial u/\partial t=0$关于$t$从0到$t_\varepsilon$积分，然后对所得方程取极限$\varepsilon\to0$，即得到$\Delta(x)=\psi(x)/A(x,0)$。因此退化问题的初始条件为

$$w(x,0)=\varphi(x)+\psi(x)/A(x,0) \tag{2.3'}$$

从而我们可构造问题(1.1)～(1.3)的渐近解为

$$\begin{aligned}\bar u_0(x,t,\varepsilon)=&w(x,t)+v_0^{(0)}(x,t/\varepsilon)+\varepsilon v_1^{(0)}(x,t/\varepsilon)+v_0^{(1)}(x/\sqrt{\varepsilon},t)\\&+\sqrt{\varepsilon}v_1^{(1)}(x/\sqrt{\varepsilon},t)+v_0^{(2)}((l-x)/\sqrt{\varepsilon},t)+\sqrt{\varepsilon}v_1^{(2)}((l-x)/\sqrt{\varepsilon},t)\end{aligned} \tag{2.4}$$

其中$w(x,t)$是问题(2.1)，(2.3)的解，$v_0^{(0)},v_1^{(0)},v_0^{(1)},v_1^{(1)},v_0^{(2)},v_1^{(2)}$分别满足如下的方程：

$$\left.\begin{aligned}&\frac{\partial^2v_0^{(0)}}{\partial\tau^2}+A(x,0)\frac{\partial v_0^{(0)}}{\partial\tau}=0\\&v_0^{(0)}(x,0)=\varphi(x)-w(x,0)=-\psi(x)/A(x,0),\ \partial v_0^{(0)}/\partial\tau|_{\tau=0}=\psi(x)\end{aligned}\right\} \tag{2.5}$$

其中$\tau=t/\varepsilon$;

$$\left.\begin{aligned}&\frac{\partial^2v_1^{(0)}}{\partial\tau^2}+A(x,0)\frac{\partial v_1^{(0)}}{\partial\tau}=\Psi_1(x,\tau)\\&v_1^{(0)}(x,0)=0,\ \frac{\partial v_1^{(0)}}{\partial\tau}(x,0)=-\frac{\partial w}{\partial t}(x,0)\end{aligned}\right\} \tag{2.6}$$

其中 $\Psi_1(x,\tau)=-\tau\dfrac{\partial A}{\partial t}(x,0)\dfrac{\partial v_0^{(0)}}{\partial\tau}-B(x,0)v_0^{(0)}$

$$\left.\begin{aligned}&A(0,t)\frac{\partial v_0^{(1)}}{\partial t}-\frac{\partial^2v_0^{(1)}}{\partial\xi^2}+B(0,t)v_0^{(1)}=0\\&v_0^{(1)}(\xi,0)=0,\ v_0^{(1)}(0,t)=-w(0,t)\end{aligned}\right\} \tag{2.7}$$

其中$\xi=x/\sqrt{\varepsilon}$;

$$\left.\begin{aligned}&A(0,t)\frac{\partial v_1^{(1)}}{\partial t}-\frac{\partial^2v_1^{(1)}}{\partial\xi^2}+B(0,t)v_1^{(1)}=-\xi\frac{\partial A}{\partial x}(0,t)\frac{\partial v_0^{(1)}}{\partial t}-\xi\frac{\partial B}{\partial x}(0,t)v_0^{(1)}\\&v_1^{(1)}(\xi,0)=0,\ v_1^{(1)}(0,t)=0\end{aligned}\right\} \tag{2.8}$$

$$\left.\begin{aligned}&A(l,t)\frac{\partial v_0^{(2)}}{\partial t}-\frac{\partial^2 v_0^{(2)}}{\partial \eta^2}+B(l,t)v_0^{(2)}=0\\&v_0^{(2)}(\eta,0)=0,\ v_0^{(2)}(0,t)=-w(l,t)\end{aligned}\right\}\tag{2.9}$$

其中$\eta=(l-x)/\sqrt{\varepsilon}$；

$$\left.\begin{aligned}&A(l,t)\frac{\partial v_1^{(2)}}{\partial t}-\frac{\partial^2 v_1^{(2)}}{\partial \eta^2}+B(l,t)v_1^{(2)}=\eta\frac{\partial A}{\partial x}(l,t)\frac{\partial v_0^{(2)}}{\partial t}+\eta\frac{\partial B}{\partial x}(l,t)v_0^{(2)}\\&v_1^{(2)}(\eta,0)=0,\ v_1^{(2)}(0,t)=0\end{aligned}\right\}\tag{2.10}$$

不难算出 $v_0^{(0)}(x,\tau)=-\exp[-A(x,0)\tau]\psi(x)/A(x,0)$，并且渐近解中的各项分别有如下的估计：

$$\left|\frac{\partial^k w}{\partial x^j\partial t^{k-j}}\right|\leqslant C\tag{2.11a}$$

$$\left|\frac{\partial^k v_0^{(0)}}{\partial x^j\partial t^{k-j}}\right|\leqslant C\varepsilon^{-(k-j)}\exp[-\alpha_1 t/\varepsilon]\tag{2.11b}$$

$$\left|\frac{\partial^k v_1^{(0)}}{\partial x^j\partial t^{k-j}}\right|\leqslant C[1+\varepsilon^{-(k-j)}\exp[-\alpha_1 t/\varepsilon]]\tag{2.11c}$$

$$\left|\frac{\partial^k v_i^{(1)}}{\partial x^j\partial t^{k-j}}\right|\leqslant C\varepsilon^{-j/2}\exp[-\beta_1 x/\sqrt{\varepsilon}]\qquad i=0,1\tag{2.11d}$$

$$\left|\frac{\partial^k v_i^{(2)}}{\partial x^j\partial t^{k-j}}\right|\leqslant C\varepsilon^{-j/2}\exp[-\beta_1(l-x)/\sqrt{\varepsilon}]\qquad i=0,1\tag{2.11e}$$

其中α_1，β_1为正常数．

三、渐近解的余项估计及精确解的导数估计

设余项$R(x,t,\varepsilon)=u(x,t,\varepsilon)-\bar{u}_0(x,t,\varepsilon)$，其中$u$表示问题 (1.1)～(1.3) 的精确解．则$R$满足的初边值条件为

$$R(x,0,\varepsilon)=0,\ \partial R(x,0,\varepsilon)/\partial t=0$$

$$R(0,t,\varepsilon)=v_0^{(2)}(l/\sqrt{\varepsilon},t)+\sqrt{\varepsilon}\,v_1^{(2)}(l/\sqrt{\varepsilon},t)=O(\varepsilon^n)$$

$$R(l,t,\varepsilon)=v_0^{(1)}(l/\sqrt{\varepsilon},t)+\sqrt{\varepsilon}\,v_1^{(1)}(l/\sqrt{\varepsilon},t)=O(\varepsilon^n)$$

其中，n是任意的正整数．为以后讨论的方便，我们作变换

$$z(x,t,\varepsilon)=R(x,t,\varepsilon)-((l-x)/l)R(0,t,\varepsilon)-(x/l)R(l,t,\varepsilon)\tag{3.1}$$

则$z(x,t,\varepsilon)$满足双曲型方程初边值问题：

$$Lz=\bar{F}(x,t,\varepsilon)\tag{3.2}$$

$$z(x,0,\varepsilon)=0,\ \partial z(x,0,\varepsilon)/\partial t=0\tag{3.3}$$

$$z(0,t,\varepsilon)=0,\ z(l,t,\varepsilon)=0\tag{3.4}$$

其中$\bar{F}$有估计（根据(2.11)）：

$$\left|\frac{\partial^k\bar{F}}{\partial x^j\partial t^{k-j}}\right|\leqslant C\varepsilon\left[1+\varepsilon^{-j/2}\left(\exp[-\beta_1 x/\sqrt{\varepsilon}]+\exp\left[-\beta_1\frac{l-x}{\sqrt{\varepsilon}}\right]\right)+\varepsilon^{-(k-j)}\exp[-\alpha_1 t/\varepsilon]\right]\tag{3.5}$$

下面我们来估计z，首先证明下述引理．

引理3.1 设$z(x,t,\varepsilon)$是问题(3.2)～(3.4)的解，则下面的能量不等式成立：

$$E_1\equiv\varepsilon\left\|\frac{\partial z}{\partial t}\right\|_t^2+\varepsilon\left\|\frac{\partial z}{\partial x}\right\|_t^2+\|z\|_t^2+\int_0^l\int_0^t\left(\frac{\partial z}{\partial t}\right)^2dtdx$$

$$\leqslant C\left(\int_0^l\int_0^t\bar{F}^2dtdx\right)$$

$$E_2\equiv\varepsilon\left\|\frac{\partial^2 z}{\partial t^2}\right\|_t^2+\varepsilon\left\|\frac{\partial^2 z}{\partial t\partial x}\right\|_t^2+\left\|\frac{\partial z}{\partial t}\right\|_t^2+\int_0^l\int_0^t\left(\frac{\partial^2 z}{\partial t^2}\right)^2dtdx$$

$$\leqslant C\left[\int_0^l\int_0^t\left(\left(\frac{\partial\bar{F}}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2+z^2\right)dtdx+\varepsilon\|\psi_0\|^2\right]$$

其中$\psi_0(x)=\bar{F}(x,0)/\varepsilon$,

$$E_3\equiv\varepsilon\left\|\frac{\partial^2\bar{z}}{\partial t\partial x}\right\|_t^2+\varepsilon\left\|\frac{\partial^2\bar{z}}{\partial x^2}\right\|_t^2+\left\|\frac{\partial\bar{z}}{\partial x}\right\|_t^2+\int_0^l\int_0^t\left(\frac{\partial^2\bar{z}}{\partial t\partial x}\right)^2dtdx$$

$$\leqslant C\left[\int_0^l\int_0^t\left(\left(\frac{\partial F_1}{\partial x}\right)^2+\left(\frac{\partial\bar{z}}{\partial t}\right)^2+\bar{z}^2\right)dtdx+\varepsilon\|\psi_1'\|^2+\varepsilon\|\varphi_1''\|^2+\|\varphi_1'\|^2\right]$$

其中$\bar{z}(x,t,\varepsilon)=z(x,t,\varepsilon)-z_0(x,t,\varepsilon)$，$z_0(x,t,\varepsilon)$满足

$$-\varepsilon\frac{\partial^2 z_0}{\partial x^2}+B(x,t)z_0=\bar{F}(x,t,\varepsilon),\quad z_0(0,t,\varepsilon)=z_0(l,t,\varepsilon)=0 \tag{3.6}$$

$$F_1=\varepsilon\frac{\partial^2 z_0}{\partial t^2}+A(x,t)\frac{\partial z_0}{\partial t},\quad \varphi_1=\bar{z}(x,0,\varepsilon)=-z_0(x,0,\varepsilon)$$

$$\psi_1=\frac{\partial\bar{z}}{\partial t}(x,0,\varepsilon)=-\frac{\partial z_0}{\partial t}(x,0,\varepsilon)$$

$$E_4\equiv\varepsilon\left\|\frac{\partial^3\bar{z}}{\partial t^2\partial x}\right\|_t^2+\varepsilon\left\|\frac{\partial^3\bar{z}}{\partial t\partial x^2}\right\|_t^2+\left\|\frac{\partial^2\bar{z}}{\partial t\partial x}\right\|_t^2+\int_0^l\int_0^t\left(\frac{\partial^3\bar{z}}{\partial t^2\partial x}\right)^2dtdx$$

$$\leqslant C\left[\int_0^l\int_0^t\left(\left(\frac{\partial F_2}{\partial x}\right)^2+\left(\frac{\partial^2\bar{z}}{\partial t^2}\right)^2+\left(\frac{\partial\bar{z}}{\partial t}\right)^2\right)dtdx+\varepsilon\|\psi_2'\|^2+\varepsilon\|\psi_1''\|^2+\|\psi_1'\|^2\right]$$

其中

$$F_2=\frac{\partial F_1}{\partial t}-\frac{\partial A}{\partial t}\frac{\partial\bar{z}}{\partial t}-\frac{\partial B}{\partial t}\bar{z}$$

$$\psi_2=\frac{\partial^2\bar{z}}{\partial t^2}(x,0,\varepsilon)=\varphi_1''-\frac{1}{\varepsilon}A(x,0)\psi_1-\frac{1}{\varepsilon}B(x,0)\varphi_1+\frac{1}{\varepsilon}F_1(x,0,\varepsilon)$$

证明 以$\partial z/\partial t$乘方程(3.2)的两边，并对变元x，t在$[0,l]\times[0,t]$上积分，利用Gronwall不等式可得第一个能量不等式。

方程(3.2)对t求导，得$\partial z/\partial t$满足

$$L\frac{\partial z}{\partial t}=\frac{\partial\bar{F}}{\partial t}-\frac{\partial A}{\partial t}\frac{\partial z}{\partial t}-\frac{\partial B}{\partial t}z \tag{3.2$'$}$$

$$\frac{\partial z}{\partial t}(x,0,\varepsilon)=0,\quad \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)(x,0,\varepsilon)=\psi_0(x) \tag{3.3$'$}$$

$$\frac{\partial z}{\partial t}(0,t,\varepsilon)=0,\quad \frac{\partial z}{\partial t}(l,t,\varepsilon)=0 \tag{3.4$'$}$$

完全类似第一个能量不等式的推导可得第二个能量不等式。

不难验证$\bar{z}(x,t,\varepsilon)$满足

$$L\bar{z}=-F_1(x,t,\varepsilon) \tag{3.7}$$
$$\bar{z}(x,0,\varepsilon)=\varphi_1(x),\ \partial\bar{z}(x,0,\varepsilon)/\partial t=\psi_1(x)$$
$$\bar{z}(0,t,\varepsilon)=0,\ \bar{z}(l,t,\varepsilon)=0$$

用$\partial^3\bar{z}/\partial t\partial x^2$乘上方程的两边，关于变元$x$，$t$分别在$[0,l]\times[0,t]$上积分，注意到$F_1(0,t,\varepsilon)=F_1(l,t,\varepsilon)=0$，利用Gronwall不等式可得第三个不等式。

方程(3.7)对t求导一次，类似第二和第三个不等式的推导，可得第四个不等式。

引理证毕。

注3.1 如果已知$F(0,t)=F(l,t)=0$或若$B(x,t)$，$F(x,t)$与t无关，则无需引入函数$\bar{z}(x,t,\varepsilon)$，而直接关于$z(x,t,\varepsilon)$可得到引理中的四个不等式。

引理3.2 方程(3.6)的解$z_0(x,t,\varepsilon)$有估计式

$$\left|\frac{\partial^k z_0}{\partial x^j\partial t^{k-j}}\right|\leqslant C\varepsilon\left\{1+\varepsilon^{-j/2}\left[\exp[-\beta_1 x/\sqrt{\varepsilon}]+\exp\left[-\beta_1\frac{l-x}{\sqrt{\varepsilon}}\right]\right]+\varepsilon^{-(k-j)}\exp[-\alpha_1 t/\varepsilon]\right\} \tag{3.8}$$

这里$0\leqslant k\leqslant 4$，$0\leqslant j\leqslant k$。

证明 利用(3.5)，类似于[11]，首先不难证明$|\partial^k z_0/\partial x^k|$有估计式(3.8)。方程(3.6)对$t$求导一次，得关于$\partial z_0/\partial t$的类似方程及$\partial z_0(0,t,\varepsilon)/\partial t=\partial z_0(l,t,\varepsilon)/\partial t=0$，类似可证$(\partial^{k-1}/\partial x^{k-1})(\partial z_0/\partial t)$有估计式(3.8)。依次下去，可证明$(\partial^j/\partial x^j)(\partial^{k-j}z_0/\partial t^{k-j})=\partial^k z_0/\partial x^j\partial t^{k-j}$有估计式(3.8)。 引理证毕。

利用(3.5)、引理3.1和引理3.2，可得

$$E_1\leqslant C\varepsilon^2 \tag{3.9a}$$
$$E_2\leqslant C\varepsilon \tag{3.9b}$$
$$E_3\leqslant C\varepsilon \tag{3.9c}$$
$$E_4\leqslant C\varepsilon^{-1} \tag{3.9d}$$

根据(3.9a)，得

$$z^2(x,t,\varepsilon)-z^2(0,t,\varepsilon)=2\int_0^l z\frac{\partial z}{\partial x}dx\leqslant 2\left(\int_0^l z^2dx\right)^{1/2}\left(\int_0^l\left(\frac{\partial z}{\partial x}\right)^2dx\right)^{1/2}\leqslant C\varepsilon^{3/2}$$

从而得到

$$|z(x,t,\varepsilon)|\leqslant C\varepsilon^{3/4} \tag{3.10a}$$

由(3.9)类似可得

$$\left|\frac{\partial z}{\partial t}(x,t,\varepsilon)\right|\leqslant C\varepsilon^{1/4},\ \left|\frac{\partial^2 z}{\partial t^2}(x,t,\varepsilon)\right|\leqslant C\varepsilon^{-1/2} \tag{3.10b}$$

以及

$$\left|\frac{\partial z}{\partial x}\right|\leqslant C\varepsilon^{1/4},\ \left|\frac{\partial^2 z}{\partial t\partial x}\right|\leqslant C\varepsilon^{-1/2},\ \left|\frac{\partial^2 z}{\partial x^2}\right|\leqslant C\varepsilon^{-3/4}$$

再由(3.1)可得到$R(x,t,\varepsilon)$的同样估计，从而得到$u(x,t,\varepsilon)=\bar{u}_0(x,t,\varepsilon)+R(x,t,\varepsilon)$的估计：

$$|u(x,t,\varepsilon)|\leqslant C,\ \left|\frac{\partial u}{\partial t}\right|\leqslant C(1+\varepsilon^{-1}\exp[-\alpha_1 t/\varepsilon])$$

$$\left|\frac{\partial u}{\partial x}\right|\leqslant C\left(1+\varepsilon^{-1/2}\exp[-\beta_1 x/\sqrt{\varepsilon}]+\varepsilon^{-1/2}\exp\left[-\beta_1\frac{l-x}{\sqrt{\varepsilon}}\right]\right)$$

$$\left|\frac{\partial^2 u}{\partial t^2}\right|\leqslant C(\varepsilon^{-1/4}+\varepsilon^{-2}\exp[-\alpha_1 t/\varepsilon])$$

$$\left|\frac{\partial^2 u}{\partial t\partial x}\right|\leqslant C(\varepsilon^{-1/2}+\varepsilon^{-1}\exp[-\alpha_1 t/\varepsilon])$$

$$\left|\frac{\partial^2 u}{\partial x^2}\right|\leqslant C\left(\varepsilon^{-3/4}+\varepsilon^{-1}\exp\left[-\beta_1\frac{x}{\sqrt{\varepsilon}}\right]+\varepsilon^{-1}\exp\left[-\beta_1\frac{l-x}{\sqrt{\varepsilon}}\right]\right)$$

原方程(1.1)两端对x,t逐步求导，用类似的方法可得$u(x,t,\varepsilon)$的更高阶导数估计。综上所述得

定理3.1 关于渐近解的余项$R(x,t,\varepsilon)$，估计式

$$|R(x,t,\varepsilon)|\leqslant C\varepsilon^{3/4} \tag{3.11a}$$

$$\left|\frac{\partial^2 R}{\partial t^2}(x,t,\varepsilon)\right|\leqslant C\varepsilon^{-1/2} \tag{3.11b}$$

成立，关于问题(1.1)～(1.3)的解$u(x,t,\varepsilon)$，有

$$\left|\frac{\partial^k u}{\partial x^j\partial t^{k-j}}\right|\leqslant C[\varepsilon^{-(k-j)}+\varepsilon^{-j/2}]\qquad 0\leqslant k\leqslant 4,0\leqslant j\leqslant k \tag{3.12}$$

注3.2 如果增加渐近解的项数，则关于z及其导数估计会更好，从而可得到u的更为精致的导数估计.

四、差分格式的建立

在x方向采用非均匀网格，在t方向采用均匀网格，得到整个区域上$\bar{G}$的网格$\bar{G}_{h,k}=\bar{G}_h\times\bar{G}_k$，这里$\bar{G}_h=\{x_i,0=x_0<x_1<\cdots<x_{N-1}<x_N=l,\ i=0,1,\cdots,N\}$，$\bar{G}_k=\{t_j=jk,j=0,1,\cdots,[T/k]\}$，$h_i=x_i-x_{i-1}$，$h=\max\limits_i h_i$，$\hbar_i=(h_i+h_{i+1})/2$

用$u^d(x,t)$表示$u(x,t)$的近似值，定义差分算子((x,t)表示(x_i,t_j))：

$$L_\varepsilon^{(h,k)}u^d(x,t)\equiv r(x,t,k)u^d_{t\bar{t}}(x,t)-\varepsilon[u^d_{\bar{x}\hat{x}}(x,t+k)+u^d_{\bar{x}\hat{x}}(x,t-k)]/2$$

$$+A(x,t)u^d_{\mathring{t}}(x,t)+B(x,t)u^d(x,t)=F(x,t),\qquad (x,t)\in G_{h,k} \tag{4.1}$$

其中

$$r(x,t,k)=\frac{1}{2}A(x,t)k\coth\left(\frac{1}{2}A(x,t)k/\varepsilon\right)$$

$$u^d_{\bar{x}\hat{x}}(x,t)=[(u^d(x+h_{i+1},t)-u^d(x,t))/h_{i+1}-(u^d(x,t)-u^d(x-h_i,t))/h_i]/\hbar_i$$

$$u^d_{\mathring{t}}(x,t)=(u^d(x,t+k)-u^d(x,t-k))/(2k)$$

以及 $u^d_{\hat{x}}(x,t)=(u^d(x+h_{i+1},t)-u^d(x,t))/\hbar_i$

下面考虑初始及边界条件的近似. 第一初始条件和边界条件的差分近似为

$$u^d(x,0)=\varphi(x),\ u^d(0,t)=u^d(l,t)=0 \tag{4.2}$$

设第二初始条件的差分近似为

$$\sigma u^d_t(x,0)=\psi(x)/\varepsilon \tag{4.3}$$

其中 $u^d_t(x,0)=(u^d(x,k)-u^d(x,0))/k$，$\sigma$为拟合因子. 我们要求$v_0^{(0)}(x,t/\varepsilon)$精确满足(4.3)，

则得到

$$\sigma=\rho[1-\exp[-A(x,0)\rho]]^{-1}A(x,0),\quad \rho=k/\varepsilon \tag{4.4}$$

下面我们研究(4.3)对第二初始条件的逼近误差．

从σ的选取过程，我们有

$$\sigma(v_0^{(0)})_t(x,0)=\psi(x)/\varepsilon=\partial u(x,0)/\partial t$$

设 $u_0=\bar{u}_0-v_0^{(0)}$，并由 $u=\bar{u}_0+R=u_0+R+v_0^{(0)}$，得

$$\sigma u_t(x,0)-\sigma u_t^d(x,0)=\sigma u_t(x,0)-\partial u(x,0)/\partial t$$

$$=\sigma(u(x,k)-u(x,0))/k-\sigma(v_0^{(0)}(x,k)-v_0^{(0)}(x,0))/k=\mathrm{I}+\mathrm{II}$$

其中 $\mathrm{I}=\sigma(R(x,k,\varepsilon)-R(x,0,\varepsilon))/k$, $\mathrm{II}=\sigma(u_0(x,k,\varepsilon)-u_0(x,0,\varepsilon))/k$．易证 $\mathrm{I}=\sigma k\partial^2R(x,\xi,\varepsilon)/\partial t^2$，由定理 3.1 知$|\mathrm{I}|\leqslant C\sigma k/\varepsilon^{1/2}$；$\mathrm{II}=\sigma\partial u_0(x,\xi,\varepsilon)/\partial t$，又由 $\partial u_0(x,0,\varepsilon)/\partial t=0$，我们还有 $\mathrm{II}=\sigma k\partial^2u_0(x,\xi,\varepsilon)/\partial t^2$，所以 $|\mathrm{II}|\leqslant C\min(\max|\partial u_0/\partial t|,\ \max|\partial^2u_0/\partial t^2|k)\sigma$．据(2.11)，得$|\partial u_0/\partial t|\leqslant C$，$|\partial^2u_0/\partial t^2|\leqslant C\varepsilon^{-1}$，所以$|\mathrm{II}|\leqslant C\min(1,k/\varepsilon)\sigma$．因此$|\sigma u_t(x,0)-\sigma u_t^d(x,0)|\leqslant C[k/\varepsilon^{1/2}+\min(1,k/\varepsilon)]\sigma$．即

$$|(u-u^d)_t(x,0)|\leqslant C[\min(1,k/\varepsilon)+k/\varepsilon^{1/2}] \tag{4.5}$$

再由(1.2)和(4.2)得

$$|(u-u^d)_t(x,k)|\leqslant Ck[\min(1,k/\varepsilon)+k/\varepsilon^{1/2}] \tag{4.6}$$

五、离散能量不等式

离散问题(4.1)～(4.3)的能量不等式的推导与[10]相同．但由于篇幅所限，[10]只给出证明思路．本文给出详细证明作为[10]的补充．首先叙述结果．

定理5.1 设$u^d(x,t)$是问题(4.1)～(4.3)的解，则

$$\|u^d\|_s^2+\varepsilon\|u_{\bar{x}}^d\|_{*s}^2+\max(\varepsilon,k)\|u_{\bar{t}}^d\|_{s-1}^2$$

$$\leqslant C\left[k\sum_{j=1}^{s-1}\sum_{i=0}^{N-1}\hbar_iF^2+\max(\varepsilon,k)\|u_t^d\|_0^2+\|u^d\|_1^2+\|u^d\|_0^2\right] \tag{5.1}$$

其中 $s=1,\cdots,[T/k]$，$\|u^d\|_s^2=\sum_{i=0}^{N-1}\hbar_i[u^d(x_i,sk)]^2$，$\|u^d\|_{*s}^2=\sum_{i=0}^{N-1}h_{i+1}[u^d(x_i,sk)]^2$

证明 为叙述方便起见，我们记$y=u^d$．由[10]和[12]列出的差分关系式；可以验证(设$H_i=h_{i+1}/\hbar_i$)：

$$ry_{t\bar{t}}(y_t+y_{\bar{t}})=r(y_t^2)_{\bar{t}}=(ry_t^2)_{\bar{t}}-r_{\bar{t}}y_{\bar{t}}^2 \tag{5.2a}$$

$$\left.\begin{aligned}y_ty_{\bar{x}\hat{x}t}&=(y_ty_{t\bar{x}})_{\hat{x}}-H_iy_{t\bar{x}}^2\\ y_{\bar{t}}y_{\bar{x}\hat{x}\bar{t}}&=(y_{\bar{t}}y_{\bar{t}\bar{x}})_{\hat{x}}-H_iy_{\bar{t}\bar{x}}^2\end{aligned}\right\} \tag{5.2b}$$

$$\left.\begin{aligned}y_ty_{t\bar{x}\hat{x}\bar{t}}&=(y_ty_{t\bar{t}\bar{x}})_{\hat{x}}-H_i(y_{t\bar{x}}^2)_{\bar{t}}/2-kH_iy_{t\bar{t}\bar{x}}^2/2\\ y_{\bar{t}}y_{t\bar{x}\hat{x}t}&=(y_{\bar{t}}y_{t\bar{t}\bar{x}})_{\hat{x}}-H_i(y_{\bar{t}\bar{x}}^2)_t/2+kH_iy_{t\bar{t}\bar{x}}^2/2\end{aligned}\right\} \tag{5.2c}$$

$$\left.\begin{aligned}2y_{\bar t}y_{\bar x\hat x}&=2(y_{\bar t}y_{\bar x})_{\hat x}-H_i(y_x^2)_{\bar t}-kH_iy_{\bar t x}^2\\2y_ty_{\bar x\hat x}&=2(y_ty_{\bar x})_{\hat x}-H_i(y_x^2)_t+kH_iy_{tx}^2\end{aligned}\right\}\tag{5.2d}$$

$$\left.\begin{aligned}y_ty_{\bar x\hat x\bar t}&=y_ty_{\bar x\hat x t}-ky_ty_{t\bar x\hat x\bar t}\\y_{\bar t}y_{\bar x\hat x t}&=y_{\bar t}y_{\bar x\hat x\bar t}+ky_{\bar t}y_{t\bar x\hat x\bar t}\end{aligned}\right\}\tag{5.2e}$$

由(5.2b)～(5.2e)，可以有

$$\begin{aligned}\mathrm{III}&=(y_t+y_{\bar t})(y_{\bar x\hat x}(x,t+k)+y_{\bar x\hat x}(x,t-k))\\&=(y_t+y_{\bar t})(ky_{\bar x\hat x t}-ky_{\bar x\hat x\bar t}+2y_{\bar x\hat x})\\&=k^2(y_{\bar t}y_{\bar x\hat x t\bar t}+y_ty_{\bar x\hat x t\bar t})+2y_ty_{\bar x\hat x}+2y_{\bar t}y_{\bar x\hat x}\\&=2(y_{\bar t}y_{\bar x})_{\hat x}+2(y_ty_{\bar x})_{\hat x}-H_i[(y_x^2)_{\bar t}+ky_{\bar t x}^2+(y_x^2)_t-ky_{xt}^2]\\&\quad+k^2\{(y_{\bar t}-y_{t\bar t\bar x})_{\hat x}+(y_ty_{t\bar t\bar x})_{\hat x}-H_i[(y_{tx}^2)\ +(y_{\bar t x}^2)_t]/2\}\end{aligned}$$

注意到 $y_{tx}^2-y_{\bar t x}^2-k(y_{tx}^2)_{\bar t}/2-k(y_{\bar t x}^2)_t/2=0$，所以我们有

$$\begin{aligned}\mathrm{III}&=2(y_{\bar t}y_{\bar x})_{\hat x}+2(y_ty_{\bar x})_{\hat x}-H_i[(y_x^2)_{\bar t}+(y_x^2)_t]\\&\quad+k^2[(y_{\bar t}y_{t\bar t\bar x})_{\hat x}+(y_ty_{t\bar t\bar x})_{\hat x}]\end{aligned}\tag{5.2f}$$

另外还有

$$\begin{aligned}B(x,t)y(y_t+y_{\bar t})&=\frac{1}{2}[B(x,t-k)y^2]_t-\frac{1}{2}B_{\bar t}(x,t)y^2\\&\quad-\frac{1}{2}kB(x,t)y_t^2+\frac{1}{2}[B(x,t)y^2]_{\bar t}-\frac{1}{2}B_{\bar t}(x,t)y^2(x,t-k)\\&\quad+\frac{1}{2}kB(x,t)y_{\bar t}^2\end{aligned}\tag{5.2g}$$

所以用$2y_{\hat t}=(y_t+y_{\bar t})$乘方程(4.1)两边，利用(5.2a)，(5.2f)，(5.2g)，得

$$\begin{aligned}&(ry_t^2)_{\bar t}-\frac{\varepsilon}{2}\{2(y_{\bar t}y_{\bar x})_{\hat x}+2(y_ty_{\bar x})_{\hat x}-H_i[(y_x^2)_{\bar t}+(y_x^2)_t]\\&\quad+k^2[(y_{\bar t}y_{t\bar t\bar x})_{\hat x}+(y_ty_{t\bar t\bar x})_{\hat x}]\}+\frac{1}{2}A(x,t)(y_t+y_{\bar t})^2\\&\quad+\frac{1}{2}(B(x,t-k)y^2)_t+\frac{1}{2}(B(x,t)y^2)_{\bar t}+\frac{1}{2}kB(x,t)y_{\bar t}^2\\&=F(x,t)(y_t+y_{\bar t})+r_{\bar t}y_{\bar t}^2+\frac{1}{2}B_{\bar t}(x,t)y^2+\frac{1}{2}kB(x,t)y_t^2\\&\quad+\frac{1}{2}B_t(x,t)y^2(x,t-k)\end{aligned}\tag{5.3}$$

容易验证：

$$\left.\begin{aligned}&|r_{\bar t}|=\left|\frac{\partial r}{\partial t}(x,\xi,k)\right|\leqslant c_0\max(k,\varepsilon)\\&c_1\max(k,\varepsilon)\leqslant|r|\leqslant c_2\max(k,\varepsilon)\\&F(y_t+y_{\bar t})\leqslant c_3F^2+c_4(y_t+y_{\bar t})^2,\quad c_3c_4=1/4\end{aligned}\right\}\tag{5.4}$$

其中$c_i, i=0,1,\cdots,4$，是与ε无关的正常数。对(5.3)求和$k\sum_{j=1}^{s-1}\sum_{i=0}^{N-1}\hbar_i$，得(取$c_4<a/2$)

$$c_0\max(k,\varepsilon)\|y_t\|_{s-1}^2+\frac{1}{2}\varepsilon\|y_x\|_{*s}^2+\frac{1}{2}\beta\|y\|_s^2$$

$$\leqslant\|\sqrt{r}\,y_t\|_{s-1}^2+\frac{1}{2}\varepsilon\sum_{i=0}^{N-1}h_{i+1}[(y_x^2)_{s-1}+(y_x^2)_s]$$

$$+k\sum_{j=1}^{s-1}\sum_{i=0}^{N-1}\hbar_i\left(\frac{1}{2}A(x,t)-c_4\right)(y_t+y_{\bar t})^2+\frac{1}{2}\|\sqrt{B(x,t-k)}\,y\|_s^2$$

$$+\frac{1}{2}\|\sqrt{B(x,t)}\,y\|_{s-1}^2+\frac{1}{2}k\sum_{j=1}^{s-1}\sum_{i=0}^{N-1}\hbar_i\cdot kB(x,t)y_{\bar t}^2$$

$$\leqslant Ck\sum_{j=1}^{s-1}\sum_{i=0}^{N-1}\hbar_iF^2+\|\sqrt{r}\,y_t\|_0^2+Ck\max(k,\varepsilon)\|y_{\bar t}\|_1^2$$

$$+C\|y\|_1^2+C\|y\|_0^2+Ck\sum_{j=1}^{s-1}[\max(k,\varepsilon)\|y_t\|_j^2+\|y\|_j^2]$$

由离散Gronwall不等式(见[12])即得(5.1)。 定理证毕。

六、在能量范数意义下的一致收敛性

我们首先讨论古典估计。

引理6.1 设$u^d(x,t)$是问题(4.1)～(4.3)的解，$u(x,t)$是问题(1.1)～(1.3)的解，则

$$\|u(x,t)-u^d(x,t)\|_s\leqslant C\{\max(\varepsilon,k)k^2/\varepsilon^4+h/\varepsilon^{1/2}+\sqrt{\max(\varepsilon,k)}\,[\min(1,k/\varepsilon)+k/\varepsilon^{1/2}]\} \tag{6.1}$$

证明 $L_\varepsilon^{(h,k)}(u^d-u)=(L_\varepsilon-L_\varepsilon^{(h,k)})u=I_1+\varepsilon I_2+A(x,t)I_3$

这里

$$I_1=(\varepsilon-r)\frac{\partial^2 u}{\partial t^2}+r\left(\frac{\partial^2 u}{\partial t^2}-u_{t\bar t}\right)$$

$$I_2=\frac{1}{2}[u_{\bar x\hat x}(x,t+k)+u_{\bar x\hat x}(x,t-k)]-\frac{\partial^2 u}{\partial x^2}$$

$$I_3=\frac{\partial u}{\partial t}-u_{\mathring t}$$

利用Taylor展式，定理3.1及

$$|r-\varepsilon|=\varepsilon|k\coth x-1|\leqslant C\varepsilon\min\left(\frac{k}{\varepsilon},\left(\frac{k}{\varepsilon}\right)^2\right),\quad x=\frac{1}{2}Ak/\varepsilon,$$

不难证明

$$|I_1|\leqslant C\max(\varepsilon,k)k^2/\varepsilon^4,\quad \varepsilon|I_2|\leqslant C(k^2/\varepsilon+h/\varepsilon^{1/2}),\quad |I_3|\leqslant Ck^2/\varepsilon^3$$

从而得到

$$|L_\varepsilon^{(h,k)}(u^d-u)|\leqslant C[\max(\varepsilon,k)k^2/\varepsilon^4+h/\varepsilon^{1/2}]$$

由(1.2),(1.3),(4.2),(4.5)，我们知在边界上有

$$u^d(x,0)-u(x,0)=0,\ |u_t^d(x,0)-u_t(x,0)|\leqslant C[\min(1,k/\varepsilon)+k/\varepsilon^{1/2}]$$

$$u^d(0,t)-u(0,t)=0,\ u^d(l,t)-u(l,t)=0$$

由定理5.1，即得引理的结论.

为得到一致收敛性，我们在x方向采用特殊的非均匀网格. 设s_i是区间[0,1]上的网格结点，$s_i=ih_0$，$i=0,1,\cdots,N_0$，$h_0N_0=1$.

引入函数$\lambda(s)=M\varepsilon^{1/2}\ln(1-s)^{-1}$，$s\in[0,1)$，其中$M$是某一正常数. 定义$I=\{i|\lambda(s_i)-\lambda(s_{i-1})\leqslant h_0,\ \lambda(s_i)\leqslant l/2\}$及$m_0=\max\limits_{i\in I} i$. 我们构造区间$[0,l]$上的网格点为

$$x_i=\begin{cases}\lambda(s_i), & \text{当}0\leqslant i\leqslant m_0\text{时}\\ x_{m_0}+(i-m_0)h_0, & \text{当}m_0<i\leqslant m\text{时}\\ l-x_{N-i}, & \text{当}m<i\leqslant N\text{时}\end{cases}$$

其中整数$m\leqslant m_0+(l/2-x_{m_0})/h_0$. 当$m=m_0+(l/2-x_{m_0})/h_0$时，取$N=2m$; 当 $m<m_0+(l/2-x_{m_0})/h_0$时，取$N=2m+1$. 容易验证网格点关于$x=l/2$对称，并且$h\leqslant 2h_0$. 类似于[13]，我们有

引理6.2 设$E_i=\min\{h_{i+1}\varepsilon^{-\frac{1}{2}}e_i+h_i\varepsilon^{-\frac{1}{2}}e_{i-1},\ e_i+e_{i-1}\}$，其中 $e_i=\exp[-\beta_1x_i/\sqrt{\varepsilon}]+\exp[-\beta_1(l-x_i)/\sqrt{\varepsilon}]$. 那么当$M\beta_1\geqslant 1$时有$\max\limits_{1\leqslant i\leqslant N-1}E_i\leqslant Ch_0$.

现在我们来讨论非古典估计.

引理6.3 设$u(x,t)$，$u^d(x,t)$的意义同引理6.1，则

$$\|u(x,t)-u^d(x,t)\|_\varepsilon\leqslant C(\varepsilon^{3/4}+k+h_0+\sqrt{\max(\varepsilon,k)}\min(1,k/\varepsilon)) \tag{6.2}$$

证明 $|L_\varepsilon^{(h,k)}(u^d-\bar u_0)|\leqslant|L_\varepsilon(u-\bar u_0)|+|(L_\varepsilon^{(h,k)}-L_\varepsilon)\bar u_0|$，由(3.2)，(3.5)可知$|L_\varepsilon(u-\bar u_0)|\leqslant C\varepsilon$，下面估计$|(L_\varepsilon^{(h,k)}-L_\varepsilon)\bar u_0|$. 记 $\bar u_0=w+v_0^{(0)}+\varepsilon v_1^{(0)}+v$，这里 $v=v_0^{(1)}+\sqrt{\varepsilon}v_1^{(1)}+v_0^{(2)}+\sqrt{\varepsilon}v_1^{(2)}$. 类似于[10]，有 $|(L_\varepsilon^{(h,k)}-L_\varepsilon)(w+v)|\leqslant C(\varepsilon+k+h+\max\limits_i E_i)\leqslant C(\varepsilon+k+h_0)$. 剩下来考虑 $(L_\varepsilon^{(h,k)}-L_\varepsilon)(v_0^{(0)}+\varepsilon v_1^{(0)})$. 类似于[11]中的方法，可以验证

$$|(L_\varepsilon^{(h,k)}-L_\varepsilon)(v_0^{(0)}+\varepsilon v_1^{(0)})|\leqslant C(\varepsilon h+k+\exp[-\alpha_1t/\varepsilon])$$

所以 $|L_\varepsilon^{(h,k)}(u^d-\bar u_0)|\leqslant C(\varepsilon+k+h_0+\exp[-\alpha_1t/\varepsilon])$

在边界上有 $(u^d-\bar u_0)(x,0)=0$，$(u^d-\bar u_0)_t(x,0)=O(\min(1,k/\varepsilon))$，$(u^d-\bar u_0)(0,t)=R(0,t,\varepsilon)$，$(u^d-\bar u_0)(l,t)=R(l,t,\varepsilon)$. 作变换

$$z^d=(u^d-\bar u_0)(x,t)-R(0,t,\varepsilon)-(x/l)(R(0,t,\varepsilon)-R(l,t,\varepsilon))$$

则 $|L_\varepsilon^{(h,k)}z^d|\leqslant C(\varepsilon+k+h_0+\exp[-\alpha_1t/\varepsilon])$

$$z^d(x,0)=0,\ z_t^d(x,0)=O(\min(1,k/\varepsilon))$$

$$z^d(0,t)=0,\ z^d(l,t)=0$$

由定理5.1即得

$$\|z^d\|_s \leqslant C(\varepsilon+k+h_0+\sqrt{\max(\varepsilon,k)}\ \min(1,k/\varepsilon))$$

从而$\|u^d-\bar{u}_0\|_s$有同上的估计。结合(3.11a)即得引理的结论。

综合引理6.1和引理6.3，可得一致收敛定理。

定理6.1 离散问题(4.1)～(4.3)的解$u^d(x,t)$在能量范数意义下一致收敛于问题(1.1)～(1.3)的解$u(x,t)$，即有

$$\|u-u^d\|_s \leqslant \begin{cases} C\max(h_0^{1/2},k^{1/2}), & \text{当}\varepsilon\leqslant\max(h_0,k)\text{时} \\ C\max(h_0^{3/5},k^{3/5}), & \text{当}\varepsilon\geqslant\max(h_0,k)\text{时} \end{cases} \qquad s=0,1,\cdots,[T/k]$$

证明 仅考虑$h_0\leqslant k$的情形，$h_0>k$情形类似可证。这时(6.1)，(6.2)成为

$$\|u^d-u\|_s \leqslant C[\max(\varepsilon,k)k^2/\varepsilon^4+\sqrt{\max(\varepsilon,k)}\ (\min(1,k/\varepsilon)+k/\varepsilon^{1/2})+k/\varepsilon^{1/2}] \tag{6.3}$$

$$\|u^d-u\|_s \leqslant C[\varepsilon^{3/4}+k+\sqrt{\max(\varepsilon,k)}\ \min(1,k/\varepsilon)] \tag{6.4}$$

当$0<\varepsilon\leqslant k$时，由(6.4)得$\|u^d-u\|_s\leqslant Ck^{1/2}$;

当$k<\varepsilon\leqslant k^{4/5}$时，由(6.4)得$\|u^d-u\|_s\leqslant Ck^{3/5}$;

当$k^{4/5}<\varepsilon$时，由(6.3)得$\|u^d-u\|_s\leqslant Ck^{3/5}$。所以定理的结论成立。

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Numerical Solution of the Singularly Perturbed Problem for the Hyperbolic Equation with Initial Jump

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Abstract

In this paper we consider the initial-boundary value problem for a second order hyperbolic equation with initial jump. The bounds on the derivatives of the exact solution are given. Then a difference scheme is constructed on a non-uniform grid. Finally, uniform convergence of the difference solution is proved in the sense of the discrete energy norm.