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弹性地基上自由边矩形厚板

汪 克 让

（天津大学，1989年3月1日收到）

摘　要

弹性地基上自由边矩形厚板的分析，由于其难度较大，一直没有得到很好的解决．本文采用单三角级数和重三角级数相叠加的方法，求得该问题的精确解．文中所用方法简单明了．所得结果完全满足边界条件并与王克林等[2]的结果完全一致．

关键词　弹性地基　文克勒假定　厚板

一、引　言

在实际工程中经常遭到厚度较大的弹性地基板，如水工建筑物的底板、机场面板等．一般说，这些板应按厚板理论进行计算．但过去由于没有合适的计算方法，故通常按薄板理论导出的近似方法计算，其计算结果的误差可想而知．弹性地基上的矩形厚板过去曾有学者研究过，但由于其控制方程比较复杂，只是对于某些特殊边界条件才得出精确解．近年来，我国学者引入滑支边和广义滑支边的概念，并应用叠加法解决了四边自由弹性地基上矩形厚板的问题[2]，其过程显得较为繁琐．本文把一般问题分解为双轴对称，双轴反对称和对称反对称等四种情况，采用单三角级数和重三角级数叠加的方法得出了该问题的精确解．方法简单明了．

二、基 本 理 论

根据 E. Reissner 提出的厚板理论，在 Winkler 假定下，弹性地基上厚板的平衡微分方程为

$$D\nabla^4 w=q(x,\ y)-p(x,\ y)-\frac{h^2}{10}\frac{2-\mu}{1-\mu}\nabla^2(q-p) \tag{2.1}$$

式中右边最后一项代表横向剪力和法向应力的影响．取反力$p(x,y)=kw$并令 $\frac{h^2}{10}\ \frac{2-\mu}{1-\mu}=c^2$，则式(2.1)变为

$$D\nabla^4 w-c^2k\nabla^2 w+kw=q-c^2\nabla^2 q \tag{2.2}$$

* 张福范推荐．

令$w=w_0+w_1$，w_0是齐次方程

$$D\nabla^4 w_0-c^2k\nabla^2 w_0+kw_0=0 \tag{2.3}$$

的解，w_1是非齐次方程

$$D\nabla^4 w_1-c^2k\nabla^2 w_1+kw_1=q-c^2\nabla^2 q \tag{2.4}$$

的解。

板中的弯矩和扭矩表达式为

$$\left.\begin{aligned}
M_x&=-D\left[\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right]+\frac{h^2}{5}\frac{\partial Q_x}{\partial x}-\frac{\mu h^2}{10(-\mu)}(q-kw)\\
M_y&=-D\left[\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right]+\frac{h^2}{5}\frac{\partial Q_y}{\partial y}-\frac{\mu h^2}{10(-\mu)}(q-kw)\\
M_{xy}&=(1-\mu)D\frac{\partial w}{\partial x\partial y}-\frac{h^2}{10}\left(\frac{\partial Q_x}{\partial y}+\frac{\partial Q_y}{\partial x}\right)
\end{aligned}\right\} \tag{2.5}$$

它们不仅与平均挠度有关，而且也与横向剪力和荷载有关。

板中剪力表达式为

$$\left.\begin{aligned}
Q_x-\frac{h^2}{10}\nabla^2 Q_x&=-D\frac{\partial}{\partial x}\nabla^2 w-\frac{h^3}{10(1-\mu)}\frac{\partial}{\partial x}(q-kw)\\
Q_y-\frac{h^2}{10}\nabla^2 Q_y&=-D\frac{\partial}{\partial y}\nabla^2 w-\frac{h^3}{10(1-\mu)}\frac{\partial}{\partial y}(q-kw)
\end{aligned}\right\} \tag{2.6}$$

依据上式的形式，可以把剪力写成齐次解和特解之和的形式[5]，即

$$\left.\begin{aligned}
Q_x&=Q_x^*-D\frac{\partial}{\partial x}\nabla^2 w_0+c^2k\frac{\partial w_0}{\partial x}+\frac{\partial \psi}{\partial y}\\
Q_y&=Q_y^*-D\frac{\partial}{\partial y}\nabla^2 w_0+c^2k\frac{\partial w_0}{\partial y}-\frac{\partial \psi}{\partial x}
\end{aligned}\right\} \tag{2.7}$$

式中 Q_x^* 和Q_y^* 为剪力特解，可由下式解得

$$\left.\begin{aligned}
Q_x^*-\frac{h^2}{10}\nabla^2 Q_x^*&=-D\frac{\partial}{\partial x}\nabla^2 w_1-\frac{h^2}{10(1-\mu)}\frac{\partial}{\partial x}(q-kw_1)\\
Q_y^*-\frac{h^2}{10}\nabla^2 Q_y^*&=-D\frac{\partial}{\partial y}\nabla^2 w_1-\frac{h^2}{10(1-\mu)}\frac{\partial}{\partial y}(q-kw_1)
\end{aligned}\right\} \tag{2.8}$$

Q_x^*和Q_y^*满足平衡方程

$$\frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}+q-kw_1=0 \tag{2.9}$$

式(2.7)中的函数ψ满足微分方程

$$\nabla^2\psi-\frac{10}{h^2}\psi=0 \tag{2.10}$$

边界条件为

$$\begin{aligned}
&Q_x=M_x=M_{xy}=0 \qquad (\text{在}x=\pm a/2\text{边})\\
&Q_y=M_y=M_{xy}=0 \qquad (\text{在}y=\pm b/2\text{边})
\end{aligned} \tag{1.11}$$

荷载$q(x, y)$可展成重 Fourier 级数

$$\begin{aligned}
q(x,\ y)=\sum_{m=1,3,\cdots}\ \sum_{n=1,3,\cdots}&(q_{mn}^{(1)}\cos a_m x\cos b_n y+q_{mn}^{(2)}\sin a_m x\sin b_n y\\
&+q_{mn}^{(3)}\sin a_m x\cos b_n y+q_{mn}^{(4)}\cos a_m x\sin b_n y)
\end{aligned} \tag{2.12}$$

式中

$$\left.\begin{aligned}q_{mn}^{(1)}&=\frac{16}{a_0b_0}\int_0^{\frac{a_0}{2}}\int_0^{\frac{b_0}{2}}q_1(x,\ y)\cos a_m x\cos b_n y dxdy\\q_{mn}^{(2)}&=\frac{16}{a_0b_0}\int_0^{\frac{a_0}{2}}\int_0^{\frac{b_0}{2}}q_2(x,\ y)\sin a_m x\sin b_n y dxdy\\q_{mn}^{(3)}&=\frac{16}{a_0b_0}\int_0^{\frac{a_0}{2}}\int_0^{\frac{b_0}{2}}q_3(x,\ y)\sin a_m x\cos b_n y dxdy\\q_{mn}^{(4)}&=\frac{16}{a_0b_0}\int_0^{\frac{a_0}{2}}\int_0^{\frac{b_0}{2}}q_4(x,\ y)\cos a_m x\sin b_n y dxdy\end{aligned}\right\}\tag{2.13}$$

其中$a_m=m\pi/a_0$，$b_n=n\pi/b_0$，a_0和b_0分别稍大于a和b，本文中取其为$1.1a$和$1.1b$。(2.12)式右端四项分别对应于双轴对称、双轴反对称、关于x轴对称关于y轴反对称和关于x轴反对称关于y轴对称等四种情况。q_1，q_2，q_3和q_4为与上述四种情况对应的荷载，其值均为$q(x,y)/4$。下面按照上述四种情况分别求解。

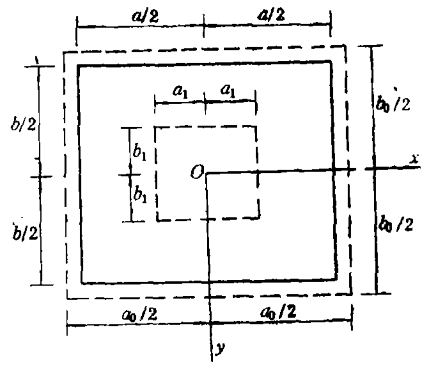

图 1

三、双轴对称问题

在双轴对称情况下，$q_{mn}^{(2)}=q_{mn}^{(3)}=q_{mn}^{(4)}=0$。取非齐次解为

$$w_1=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}a_{mn}\cos a_m x\cos b_n y\tag{3.1}$$

代入(2.4)式解得

$$a_{mn}=\frac{q_{mn}^{(1)}(1+c^2d_{mn})}{Dd^2_{mn}+c^2kd_{mn}+k}\tag{3.2}$$

式中 $d_{mn}=a_m^2+b_n^2$。取齐次解为

$$w_0=\sum_{m=1,3,\cdots}[A_{1m}Y_{1m}(y)+A_{2m}Y_{2m}(y)]\cos a_m x+\sum_{n=1,3,\cdots}[B_{1n}X_{1n}(x)+B_{2n}X_{2n}(x)]\cos b_n y\tag{3.3}$$

其中 A_{1m}，A_{2m}，B_{1n}，B_{2n}为待定常数，函数$X(x)$和$Y(y)$为

$$\left.\begin{aligned}&Y_{1m}(y)=\mathrm{ch}\alpha_m y\cos\beta_m y,\quad Y_{2m}(y)=\mathrm{sh}\alpha_m y\sin\beta_m y\\&Y_{3m}(y)=\mathrm{ch}\alpha_m y\sin\beta_m y,\quad Y_{4m}(y)=\mathrm{sh}\alpha_m y\cos\beta_m y\\&X_{1n}(x)=\mathrm{ch}\lambda_n x\cos\gamma_n x,\quad X_{2n}(x)=\mathrm{sh}\lambda_n x\sin\gamma_n x\\&X_{3n}(x)=\mathrm{ch}\lambda_n x\sin\gamma_n x,\quad X_{4n}(x)=\mathrm{sh}\lambda_n x\cos\gamma_n x\end{aligned}\right\}\tag{3.4}$$

其中

$$\left.\begin{aligned}\alpha_m^2&=\frac{1}{2}\left[\sqrt{a_m^4+a_m^2\frac{c^2k}{D}+\frac{k}{D}}+a_m^2+\frac{c^2k}{2D}\right]\\\beta_m^2&=\frac{1}{2}\left[\sqrt{a_m^4+a_m^2\frac{c^2k}{D}+\frac{k}{D}}-a_m^2-\frac{c^2k}{2D}\right]\end{aligned}\right\}\tag{3.5}$$

在上式中，如用b_n代替a_m，则α_m^2和β_m^2应变为λ_n和γ_n。

由于w_1和q为双轴对称，剪力特解可取为

$$Q_x^* = \sum_{m=1,3,\cdots} \sum_{n=1,3,\cdots} Q_{1mn} \sin a_m x \cos b_n y$$
$$Q_y^* = \sum_{m=1,3,\cdots} \sum_{n=1,3,\cdots} Q_{2mn} \cos a_m x \sin b_n y \tag{3.6}$$

代入(2.8)求得

$$Q_{1mn} = -a_m\left(a_{mn} d_{mn}^{(1)} - d_{mn}^{(2)} q_{mn}^{(1)}\right)$$
$$Q_{2mn} = -b_n\left(a_{mn} d_{mn}^{(1)} - d_{mn}^{(2)} q_{mn}^{(1)}\right) \tag{3.7}$$

式中

$$d_{mn}^{(1)} = \frac{Dd_{mn} + \dfrac{kh^2}{10(1-\mu)}}{1 + \dfrac{h^2}{10} d_{mn}}, \qquad d_{mn}^{(2)} = \frac{\dfrac{h^2}{10(1-\mu)}}{1 + \dfrac{h^2}{10} d_{mn}}$$

根据边界条件的情况，满足方程(2.10)的函数ψ取为

$$\psi = \sum_{m=1,3,\cdots} E_m \sin a_m x \,\mathrm{sh}\, \xi_m y + \sum_{n=1,3,\cdots} H_n \sin b_n y \,\mathrm{sh}\, \eta_n x \tag{3.8}$$

式中　E_m和H_n为待定常数，

$$\xi_m^2 = \frac{10}{h^2} + a_m^2, \qquad \eta_n^2 = \frac{10}{h^2} + b_n^2 \tag{3.9}$$

把式(3.1)，(3.3)，(3.6)和(3.8)代入(2.5)和(2.7)式中，得内力表达式如下

$$\begin{aligned}
Q_x = &- \sum_{m=1,3,\cdots} D[A_{1m}(e^a Y_{1m} - Y''_{1m}) + A_{2m}(e^a Y_{2m} - Y''_{2m})] a_m \sin a_m x \\
&+ \sum_{n=1,3,\cdots} D[B_{1n}(e^b X'_{1n} - X'''_{1n}) + B_{2n}(e^b X'_{2n} - X'''_{2n})] \cos b_n y \\
&+ \sum_{m=1,3,\cdots} E_m \xi_m \sin a_m x \,\mathrm{ch}\, \xi_m y + \sum_{n=1,3,\cdots} H_n b_n \cos b_n y \,\mathrm{sh}\, \eta_n x \\
&- \sum_{m=1,3,\cdots} \sum_{n=1,3,\cdots} a_m E_{mn}^{(1)} \sin a_m x \cos b_n y \\
Q_y = &\sum_{m=1,3,\cdots} D[A_{1m}(e^a Y'_{1m} - Y'''_{1m}) + A_{2m}(e^a Y'_{2m} - Y'''_{2m})] \cos a_m x \\
&- \sum_{n=1,3,\cdots} D[B_{1n}(e^b X_{1n} - X''_{1n}) + B_{2n}(e^b X_{2n} - X''_{2n})] b_n \sin b_n y \\
&- \sum_{m=1,3,\cdots} E_m a_m \cos a_m x \,\mathrm{sh}\, \xi_m y - \sum_{n=1,3,\cdots} H_n \eta_n \sin b_n y \,\mathrm{ch}\, \eta_n x \\
&- \sum_{m=1,3,\cdots} \sum_{n=1,3,\cdots} b_n E_{mn}^{(1)} \cos a_m x \sin b_n y \\
M_x = &\sum_{m=1,3,\cdots} D[A_{1m}(g^a Y_{1m} + s^a Y''_{1m}) + A_{2m}(g^a Y_{2m} + s^a Y''_{2m})] \cos a_m x \\
&+ \sum_{n=1,3,\cdots} D\Big[B_{1n}\Big(f^b X_{1n} + h^b X''_{1n} - \frac{h^2}{5} X''''_{1n}\Big) \\
&\qquad + B_{2n}\Big(f^b X_{2n} + h^b X''_{2n} - \frac{h^2}{5} X''''_{2n}\Big)\Big] \cos b_n y
\end{aligned}$$

$$
\left.\begin{aligned}
&+\sum_{m=1,3,\cdots}E_m\frac{h^2}{5}a_m\xi_m\cos a_mx\,\mathrm{ch}\xi_my+\sum_{n=1,3,\cdots}H_n\frac{h^2}{5}b_n\eta_n\cos b_ny\,\mathrm{ch}\eta_nx\\
&+\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}\left[Da_{mn}u_{mn}-\frac{h^2}{5}a_n^2E_{mn}^{(1)}-\frac{\mu h^2}{10(1-\mu)}q_{mn}^{(1)}\right]\\
&\qquad\cdot\cos a_mx\cos b_ny\\
M_y=&\sum_{m=1,3,\cdots}D\left[A_{1m}\left(f^aY_{1m}+h^aY''_{1m}-\frac{h^2}{5}Y''''_{1m}\right)+\right.\\
&\left.A_{2m}\left(f^aY_{2m}+h^aY''_{2m}-\frac{h^2}{5}Y''''_{2m}\right)\right]\cos a_mx\\
&+\sum_{n=1,3,\cdots}D[B_{1n}(g^bX_{1n}+s^bX''_{1n})+B_{2n}(g^bX_{2n}+s^bX''_{2n})]\cos b_ny\\
&-\sum_{m=1,3,\cdots}E_m\frac{h^2}{5}a_m\xi_m\cos a_mx\,\mathrm{ch}\xi_my-\sum_{n=1,3,\cdots}H_n\frac{h^2}{5}b_n\eta_n\cos b_ny\,\mathrm{ch}\eta_nx\\
&+\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}\left[Da_{mn}v_{mn}-\frac{h^2}{5}b_n^2E_{mn}^{(1)}-\frac{\mu h^2}{10(1-\mu)}q_{mn}^{(1)}\right]\cos a_mx\cos b_ny\\
M_{xy}=&-\sum_{m=1,3,\cdots}D\left[A_{1m}\left(r^aY'_{1m}+\frac{h^2}{5}Y'''_{1m}\right)+A_{2m}\left(r^aY'_{2m}+\frac{h^2}{5}Y'''_{2m}\right)\right]a_m\sin a_mx\\
&-\sum_{n=1,3,\cdots}D\left[B_{1n}\left(r^bX'_{1n}+\frac{h^2}{5}X'''_{1n}\right)+B_{2n}\left(r^bX'_{2n}+\frac{h^2}{5}X'''_{2n}\right)\right]b_n\sin b_ny\\
&-\sum_{m=1,3,\cdots}E_m\frac{h^2}{10}(a_n^2+\xi_n^2)\sin a_mx\,\mathrm{sh}\xi_my+\sum_{n=1,3,\cdots}H_n\frac{h^2}{10}(b_n^2+\eta_n^2)\sin b_ny\,\mathrm{sh}\eta_nx\\
&-\sum_{m=1,3\cdots}\sum_{n=1,3\cdots}a_mb_n\left[\frac{h^2}{5}E_{mn}^{(1)}-a_{mn}D(1-\mu)\right]\sin a_mx\sin b_ny
\end{aligned}\right\}\quad(3.10)
$$

由于w_0和w_1已满足双轴对称条件，Q_x^*和Q_y^*以及ψ的一阶导数满足剪力的反对称条件，故由边界条件在$x=a/2$处$Q_x=M_x=M_{xy}=0$和在$y=b/2$处 $Q_y=M_y=M_{xy}=0$，即可求得

$$
\begin{aligned}
&-\sum_{m=1,3,\cdots}D[A_{1m}(e^a+b_n^2)c_{1n}+A_{2m}(e^a+b_n^2)c_{2n}]a_m\sin\frac{a_ma}{2}\\
&+D\left\{B_{1n}\left[e^bX'_1\left(\frac{a}{2}\right)-X'''_{1n}\left(\frac{a}{2}\right)\right]+B_{2n}\left[e^bX'_{2n}\left(\frac{a}{2}\right)\right.\right.\\
&\left.\left.-X'''_{2n}\left(\frac{a}{2}\right)\right]\right\}+\sum_{m=1,3,\cdots}E_m\xi_mc_{6n}\sin\frac{a_ma}{2}\\
&+H_nb_n\,\mathrm{sh}\frac{\eta_na}{2}-\sum_{m=1,3,\cdots}a_mE_{mn}^{(1)}\sin\frac{a_ma}{2}=0\quad(n=1,\ 3,\ 5,\ \cdots)\\
&D\left\{A_{1m}\left[e^aY'_{1m}\left(\frac{b}{2}\right)-Y'''_{1m}\left(\frac{b}{2}\right)\right]+A_{2m}\left[e^aY'_{2m}\left(\frac{b}{2}\right)-Y'''_{2m}\left(\frac{b}{2}\right)\right]\right\}
\end{aligned}
$$

$$-\sum_{n=1,3,\cdots} D[B_{1n}(e^b+a_n^2)c_{1m}+B_{2n}(e^b+a_n^2)c_{2m}]b_n\sin\frac{b_nb}{2}-E_m a_m \mathrm{sh}\frac{\xi_m b}{2}$$

$$-\sum_{n=1,3,\cdots} H_n\eta_n c_{8m}\sin\frac{b_nb}{2}-\sum_{n=1,3,\cdots} b_n E_{mn}^{(1)}\sin\frac{b_nb}{2}=0 \quad (m=1,3,5,\cdots)$$

$$\sum_{m=1,3,\cdots} D[A_m(g^a-s^ab_n^2)c_{1n}+A_{2m}(g^a-s^ab_n^2)c_{2n}]\cos\frac{a_ma}{2}$$

$$+D\Big\{B_{1n}\Big[f^bX_{1n}\Big(\frac{a}{2}\Big)+h^bX''_{1n}\Big(\frac{a}{2}\Big)-\frac{h^2}{5}X^{IV}_{1n}\Big(\frac{a}{2}\Big)\Big]$$

$$+B_{2n}\Big[f^bX_{2n}\Big(\frac{a}{2}\Big)+h^bX''_{2n}\Big(\frac{a}{2}\Big)-\frac{h^2}{5}X^{IV}_{2n}\Big(\frac{a}{2}\Big)\Big]\Big\}$$

$$-\sum_{m=1,3,\cdots} E_m\frac{h^2}{5}a_m\xi_m c_{8n}\cos\frac{a_ma}{2}+H_n\frac{h^2}{5}b_n\eta_n\mathrm{ch}\frac{\eta_na}{2}$$

$$+\sum_{m=1,3,\cdots}\Big[Da_{mn}u_{mn}-\frac{h^2}{5}a_m^2E_{mn}^{(1)}-\frac{\mu h^2}{10(1-\mu)}q_{mn}^{(1)}\Big]\cos\frac{a_ma}{2}=0 \quad (n=1,3,5\cdots)$$

$$D\Big\{A_{1m}\Big[f^aY_{1m}\Big(\frac{b}{2}\Big)+h^aY''_{1m}\Big(\frac{b}{2}\Big)-\frac{h^2}{5}Y^{IV}_{1m}\Big(\frac{b}{2}\Big)\Big]$$

$$+A_{2m}\Big[f^aY_{2m}\Big(\frac{b}{2}\Big)+h^aY_{2m}\Big(\frac{b}{2}\Big)-\frac{h^2}{5}Y^{IV}_{2m}\Big(\frac{b}{2}\Big)\Big]\Big\}$$

$$+\sum_{n=1,3,\cdots} D[B_{1n}(g^b-s^ba_m^2)c_{1m}+B_{2n}(g^b-s^ba_m^2)c_{2m}]\cos\frac{b_nb}{2}$$

$$-E_m\frac{h^2}{5}a_m\xi_m\mathrm{ch}\frac{\xi_mb}{2}-\sum_{n=1,3,\cdots} H_n\frac{h^2}{5}b_n\eta_n c_{8m}\cos\frac{b_nb}{2}$$

$$+\sum_{n=1,3,\cdots}\Big[Da_{mn}v_{mn}-\frac{h^2}{5}b_n^2E_{mn}^{(1)}-\frac{\mu h^2}{10(1-\mu)}q_{mn}^{(1)}\Big]\cos\frac{b_nb}{2}=0 \quad (m=1,3,5,\cdots)$$

$$-\sum_{m=1,3,\cdots} D\Big[A_{1m}\Big(\frac{h^2}{5}b_n^2-r^a\Big)c_{1n}+A_{2m}\Big(\frac{h^2}{5}b_n^2-r^a\Big)c_{2n}\Big]a_mb_n\sin\frac{a_ma}{2}$$

$$+D\Big\{B_{1n}\Big[r^bX'_{1n}\Big(\frac{a}{2}\Big)+\frac{h^2}{5}X'''_{1n}\Big(\frac{a}{2}\Big)\Big]+B_{2n}\Big[r^bX'_{2n}\Big(\frac{a}{2}\Big)+\frac{h^2}{5}X'''_{2n}\Big(\frac{a}{2}\Big)\Big]\Big\}b_n$$

$$+\sum_{m=1,3,\cdots} E_m\frac{h^2}{10}\Big((a_m^2+\xi_m^2)c_{5n}\sin\frac{a_ma}{2}-H_n\frac{h^2}{10}(b_n^2+\eta_n^2)\mathrm{sh}\frac{\eta_na}{2}$$

$$+\sum_{m=1,3,\cdots} a_mb_n\Big[\frac{h^2}{5}E_{mn}^{(1)}-a_{mn}D(1-\mu)\Big]\sin\frac{a_ma}{2}=0 \quad (n=1,3,5,\cdots)$$

$$D\Big\{A_{1m}\Big[r^aY'_{1m}\Big(\frac{b}{2}\Big)+\frac{h^2}{5}Y'''_{1m}\Big(\frac{b}{2}\Big)\Big]+A_{2m}\Big[r^aY'_{2m}\Big(\frac{b}{2}\Big)+\frac{h^2}{5}Y'''_{2m}\Big(\frac{b}{2}\Big)\Big]\Big\}a_m$$

$$-\sum_{n=1,3,\cdots} D\Big[B_{1n}\Big(\frac{h^2}{5}a_m^2-r^b\Big)c_{1m}+B_{2n}\Big(\frac{h^2}{5}a_m^2-r^b\Big)c_{2m}\Big]a_mb_n\sin\frac{b_nb}{2}$$

$$+E_m\frac{h^2}{10}(a_m^2+\xi_m^2)\text{sh}\frac{\xi_m b}{2}-\sum_{n=1,3,\cdots}H_n\frac{h^2}{10}(b_n^2+\eta_n^2)c_{6m}\sin\frac{b_n b}{2}$$
$$+\sum_{n=1,3,\cdots}a_m b_n\left[\frac{h^2}{5}E_{mn}^{(1)}-a_{mn}D(1-\mu)\right]\sin\frac{b_n b}{2}=0 \qquad (m=1,3,5,\cdots) \tag{3.11}$$

在(3.10)和(3.11)中

$$e^a=a_m^2+\frac{c^2h}{D},\quad s^a=\frac{h^2}{5}a_m^2-\mu,\quad f^a=\frac{\mu h^2k}{10(1-\mu)D}+\mu a_m^2,$$
$$h^a=\frac{h^2}{5}\frac{c^2k}{D}+\frac{h^2}{5}a_m^2-1,\quad g^a=a_m^2+\frac{\mu h^2k}{10(1-\mu)D}-\frac{h^2}{5}a_m^4-\frac{h^2}{5}\frac{c^2k}{D}a_m^2,$$
$$r^a=(1-\mu)-\frac{h^2}{5}a_m^2-\frac{h^2}{5}\frac{c^2k}{D},\quad u_{mn}=a_m^2+\mu b_n^2+\frac{\mu h^2k}{10(1-\mu)D},$$
$$v_{mn}=b_n^2+\mu a_n^2+\frac{\mu h^2k}{10(1-\mu)D},\quad E_{mn}^{(1)}=a_{mn}d_{mn}^{(1)}-d_{mn}^{(2)}q_{mn}^{(1)}.$$

如果把e^a，s^a，f^a，h^a，g^a和r^a中的a_m用b_n替换，即得e^b，s^b，f^b，h^b，g^b和r^b。$c_{1n}\sim c_{4n}$和$c_{1m}\sim c_{4m}$分别为$Y_{1m}\sim Y_{4m}$和$X_{1n}\sim X_{4n}$的 Fourier 展开系数。c_{5n}，c_{6n}，c_{5m}和c_{6m}分别为$\text{sh}\xi_m y$，$\text{ch}\xi_m y$，$\text{sh}\eta_n x$和$\text{ch}\eta_n x$的 Fourier 展开系数。

如果A_{1m}，A_{2m}，B_{1n}，B_{2n}，E_m和H_n各取K个，则方程组(3.11)中共有$6K$个方程，由此可解得$6K$个待定常数。

四、双轴反对称问题

在双轴反对称情况下，$q_{mn}^{(1)}=q_{mn}^{(3)}=q_{mn}^{(4)}=0$。取非齐次解为

$$w_1=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}a_{mn}\sin a_m x\sin b_n y \tag{4.1}$$

代入(2.4)解得

$$a_{mn}=\frac{q_{mn}^{(2)}(1+c^2d_{mn})}{Dd_{mn}^2+c^2kd_{mn}+k}$$

取齐次解为

$$w_0=\sum_{m=1,3,\cdots}[A_{3m}Y_{3m}(y)+A_{4m}Y_{4m}(y)]\sin a_m x$$
$$+\sum_{n=1,3,\cdots}[B_{3n}X_{3n}(x)+B_{4n}X_{4n}(x)]\sin b_n y \tag{3.2}$$

剪力特解为

$$\left.\begin{aligned}Q_x^*&=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}Q_{1mn}\cos a_m x\sin b_n y\\ Q_y^*&=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}Q_{2mn}\sin a_m x\cos b_n y\end{aligned}\right\} \tag{4.3}$$

式中

$$Q_{1mn}=a_m E_{mn}^{(2)}, \qquad Q_{2mn}=b_n E_{mn}^{(2)}$$

$$E_{mn}^{(2)}=a_{mn}d_{mn}^{(1)}-d_{mn}^{(2)}q_{mn}^{(2)}$$

由于边界条件的反对称性，满足式(2.10)的函数ψ可取为

$$\psi=-\sum_{m=1,3,\cdots}E_m\cos a_m x\operatorname{ch}\xi_m y-\sum_{n=1,3,\cdots}H_n\cos b_n y\operatorname{ch}\eta_n x \tag{4.4}$$

把(4.1)～(4.4)代入(2.5)和(2.7)中可得与式(3.10)类似的内力表达式。如果在(3.10)式中分别用A_{3m}，A_{4m}，B_{3n}，B_{4n}，Y_{3m}，Y_{4m}，X_{3n}，X_{4n}，$-\cos a_m x$，$\cos b_n y$，$\sin a_m x$，$\sin b_n y$，$\operatorname{ch}\eta_n x$，$\operatorname{sh}\eta_n x$，$\operatorname{ch}\xi_m y$，$\operatorname{sh}\xi_m y$和$E_{mn}^{(2)}$代替$A_{1m}$，$A_{2m}$，$B_{1n}$，$B_{2n}$，$Y_{1m}$，$Y_{2m}$，$X_{1n}$，$X_{2n}$，$\sin a_m x$，$\sin b_n y$，$\cos a_m x$，$\cos b_n y$，$\operatorname{sh}\eta_n x$，$\operatorname{ch}\eta_n x$，$\operatorname{sh}\xi_m y$，$\operatorname{ch}\xi_m y$和$E_{mn}^{(1)}$，即可得本情况的内力表达式。

同样进行上面的变换，并注意分别用c_{3n}，c_{4n}，c_{3m}和c_{4m}代替c_{1n}，c_{2n}，c_{1m}和c_{3m}，可以把(3.11)变换成本情况的方程组。如果A_{3m}，A_{4m}，B_{3n}，B_{4n}，E_m和H_n各取K个，可得由$6K$个方程构成的方程组，从而可解出$6K$个待定常数。

五、对称反对称问题

在荷载关于x轴对称、关于y轴反对称的情况下，$q_{mn}^{(1)}=q_{mn}^{(2)}=q_{mn}^{(4)}=0$。采用与前述类似的方法得

$$w_1=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}a_{mn}\sin a_m x\cos b_n y \tag{5.1}$$

$$w_0=\sum_{m=1,3,\cdots}[A_{1m}Y_{1m}+A_{2m}Y_{2m}]\sin a_m x+\sum_{n=1,3,\cdots}[B_{3n}X_{3n}+B_{4n}X_{4n}]\cos b_n y \tag{5.2}$$

$$\left.\begin{aligned}Q_x^*&=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}Q_{1mn}\cos a_m x\cos b_n y\\Q_y^*&=\sum_{m=1,3,\cdots}\sum_{n=1,3,\cdots}Q_{2mn}\sin a_m x\sin b_n y\end{aligned}\right\} \tag{4.3}$$

$$\psi=-\sum_{m=1,3,\cdots}E_n\cos a_n x\operatorname{sh}\xi_n y+\sum_{n=1,3,\cdots}H_n\operatorname{ch}\eta_n x\sin b_n y \tag{5.4}$$

式中

$$a_{mn}=\frac{q^{(3)}{}_{mn}(1+c^2d_{mn})}{Dd_m^2+c^2kd_{mn}+k}$$

$$Q_{1mn}=a_m E_{mn}^{(3)}, \qquad Q_{2mn}=-b_n E_{mn}^{(3)}$$

$$E_{mn}^{(3)}=a_{mn}d_{mn}^{(1)}-d_{mn}^{(2)}q_{mn}^{(3)}$$

把双轴对称内力表达式中 B_{1n}，B_{2n}，X_{1n}，X_{2n}，$\sin\alpha_m x$，$\cos\alpha_m x$，$\mathrm{sh}\eta_n x$，$\mathrm{ch}\eta_n x$ 和 $E_{mn}^{(1)}$分别用B_{3n}，B_{4n}，X_{3n}，X_{4n}，$-\cos\alpha_m x$，$\sin\alpha_m x$，$\mathrm{ch}\eta_n x$，$\mathrm{sh}\eta_n x$和$E_{mn}^{(3)}$代替，即得本情况的内力表达式。代入边界条件，并注意到用c_{3m}和c_{4m}代替c_{1m}和c_{2m}，可以把(3.11)变换成本情况的方程组，如果A_{1m}，A_{2m}，B_{3n}，B_{4n}，E_m和H_n各取k个，可得$6k$个方程构成的方程组，从而可解得$6k$个待定常数。

六、计 算 实 例

我们计算了一矩形板在双轴对称、双轴反对称和对称反对称等四种情况在集中力作用下的挠度和内力。具体数据为$a=2\mathrm{m}$，$b=1.75\mathrm{m}$，$P=10\mathrm{tf}$，$E=3.5\times10^6\mathrm{tf/m^2}$，$\mu=0.15$，$k=1.4\times10^4\mathrm{tf/m^3}$，板厚$h=0.34\mathrm{m}$，荷载作用位置$a_1=1\mathrm{m}$，$b_1=0.875\mathrm{m}$。为了便于比较，下面列出文献[2]中所有的双轴对称情况的结果。计算时取$K=34$，所得反力为9.98tf（1tf=9.8kN）

表 1　　挠度值（单位mm）

y(m) \ x(m)	0	0.4	0.8	1.2	1.6	2.0
0	0.2208 0.2208	0.2218 0.2218	0.2212 0.2214	0.2126 0.2128	0.1950 0.1954	0.1740 0.1745
0.35	0.2208 0.2208	0.2223 0.2224	0.2232 0.2235	0.2146 0.2150	0.1956 0.1960	0.1737 0.1743
0.7	0.2191 0.2192	0.2218 0.2220	0.2273 0.2278	0.2187 0.2193	0.1952 0.1957	0.1717 0.1732
1.05	0.2132 0.2133	0.2160 0.2162	0.2215 0.2220	0.2129 0.2153	0.1894 0.1900	0.1660 0.1667
1.4	0.2028 0.2030	0.2043 0.2048	0.2053 0.2057	0.1967 0.1973	0.1779 0.1788	0.1564 0.1571
1.75	0.1904 0.1907	0.1909 0.1913	0.1897 0.1902	0.1813 0.1820	0.1649 0.1656	0.1446 0.1445

表 2　　弯矩M_x（单位tf·m/m）

y(m) \ x(m)	0	0.40	0.80	1.20	1.6	2.0
0	−0.128 −0.136	0.097 0.091	0.550 0.548	0.606 0.607	0.238 0.237	0.001 0.000
0.35	−0.179 −0.186	0.065 0.058	0.671 0.674	0.730 0.735	0.221 0.219	0.001 0.000
0.70	−0.272 −0.282	−0.029 −0.042	0.949 0.940	1.011 1.003	0.151 0.143	0.001 0.000
1.05	−0.269 −0.279	−0.026 −0.038	0.953 0.944	1.015 1.008	0.152 0.145	0.002 0.000
1.40	−0.165 −0.171	0.075 0.071	0.673 0.679	0.732 0.738	0.226 0.224	0.003 0.000
1.75	−0.122 −0.124	0.100 0.100	0.559 0.567	0.617 0.625	0.235 0.236	0.038 −0.002

表3　　　　**弯矩M_y**（单位tf·m/m）

y(m) \ x(m)	0	0.4	0.8	1.2	1.6	2.0
0	−0.001 −0.006	−0.064 −0.071	−0.205 −0.216	−0.198 −0.208	−0.042 −0.047	0.016 0.015
0.35	0.123 0.120	0.108 0.103	0.013 0.000	0.020 0.008	0.128 0.126	0.143 0.144
0.70	0.349 0.350	0.485 0.488	0.887 0.888	0.895 0.896	0.502 0.507	0.376 0.383
1.05	0.372 0.374	0.514 0.518	0.925 0.927	0.932 0.934	0.528 0.534	0.396 0.403
1.40	0.165 0.165	0.178 0.178	0.134 0.127	0.137 0.130	0.187 0.186	0.169 0.172
1.75	0.003 0.001	0.003 0.000	0.003 0.000	0.002 0.000	0.001 0.000	0.074 −0.002

表中每一格中上边一行为本文结果，下边一行为文[2]的结果。

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Thick Rectangular Plates with Free Edges on Elastic Foundations

Wang Ke-rang

(*Tianjin University, Tianjin*)

Abstract

In this paper an accurate solution for the thick rectangular plate with free edges laid on elastic foundation is presented. The superposition method of trigonometric series is used. The method can solve this kind of plates directly and simply. Its results completely satisfy the boundary conditions of the four free edges and nicely agree with the solutions by Wang Ke-lin and Huang Yi.

Key words elastic foundation, Winkler's assumption, thick plate.