

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = - \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{00} \end{bmatrix} \begin{Bmatrix} k_{11} \\ k_{22} \\ 2k_{12} \end{Bmatrix}, \quad \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix} \quad (1.2)$$

板平衡方程是

$$-M_{\alpha\beta,\beta} + Q_{\alpha} + m_{\alpha} = 0, \quad Q_{\alpha,\alpha} + q = 0 \quad (1.3)$$

将方程(1.1), (1.2)代入(1.3), 演化后得

$$\left. \begin{aligned} D \left(\frac{1-\bar{\mu}}{2} \nabla^2 \psi_{\alpha} + \frac{1+\bar{\mu}}{2} \psi_{\beta,\beta\alpha} \right) + \bar{C} (w_{,\alpha} - \psi_{\alpha}) + m_{\alpha} = 0 \\ \bar{C} (\nabla^2 w - \psi_{\alpha,\alpha}) + q = 0 \end{aligned} \right\} \quad (1.4a, b)$$

式中 \bar{D} , \bar{C} 和 $\bar{\mu}$ 分别是假想各向同性板的名义刚度和名义泊松比

$$\bar{D} = D_{12} + 2D_{00}, \quad \bar{C} = C_{00}, \quad \bar{\mu} = D_{12}/\bar{D} \quad (1.5)$$

而

$$m_{\alpha} = m_{\alpha} + \delta m_{\alpha}, \quad q = q + \delta q \quad (1.6)$$

m_{α} 和 q 是实载, δm_{α} 和 δq 是虚载

$$\left. \begin{aligned} \delta m_1 &= (D_{11} - \bar{D})k_{11,1} + C_{22}\gamma_1 \\ \delta m_2 &= (D_{22} - \bar{D})k_{22,2} + C_{11}\gamma_2 \\ \delta q &= C_{22}\gamma_{1,1} + C_{11}\gamma_{2,2} \end{aligned} \right\} \quad (1.7)$$

(1.4)式就是正交各向异性 Reissner 型板的各向同性化控制方程。

假想各向同性板的名义内力定义为

$$M_{\alpha\beta} = -\bar{D}[(1-\bar{\mu})k_{\alpha\beta} + \bar{\mu}k_{\gamma\gamma}\delta_{\alpha\beta}], \quad \bar{Q}_{\alpha} = \bar{C}\gamma_{\alpha} \quad (1.8)$$

而名义内力和实际内力间的转换可按式(1.8)进行

$$\{M_{\alpha\beta}\} = [T_M]\{\bar{M}_{\alpha\beta}\}, \quad \{Q_{\alpha}\} = [T_Q]\{\bar{Q}_{\alpha}\} \quad (1.9)$$

式中

$$\begin{aligned} [T_M] &= \frac{1}{\bar{D}(1-\bar{\mu}^2)} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{00} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\mu} & 0 \\ -\bar{\mu} & 1 & 0 \\ 0 & 0 & 2(1+\bar{\mu}) \end{bmatrix} \\ [T_Q] &= \frac{1}{\bar{C}} \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \end{aligned} \quad (1.10)$$

二、Kirchhoff型各向异型板

只要取 $C_{11} = C_{22} = \infty$, $\gamma_{\alpha} = w_{,\alpha} - \psi_{\alpha} = 0$ 并将 Q_{α} 看作导出量, 就可由前节公式直接得到 Kirchhoff 型板的相应公式

$$\psi_{\alpha} = w_{,\alpha}, \quad k_{\alpha\beta} = w_{,\alpha\beta} \quad (2.1)$$

$$\{M_{\alpha\beta}\} = -[D]\{k_{\alpha\beta}\} \quad (2.2)$$

$$-M_{\alpha\beta,\beta} + Q_{\alpha} + m_{\alpha}^{\pm} = 0, \quad Q_{\alpha,\alpha} + q = 0 \quad (2.3)$$

$$D \nabla^2 \nabla^2 w = q - (m_{\alpha}^{\pm})_{,\alpha} \quad (2.4)$$

$$\bar{D} = D_{12} + 2D_{00}, \quad \bar{\mu} = D_{12}/\bar{D} \quad (2.5)$$

$$m_{\alpha}^{\pm} = m_{\alpha} + \delta m_{\alpha}^{\pm} \quad (2.6)$$

$$\delta m_1^i = (D_{11} - \bar{D})k_{11,11}, \quad \delta m_2^i = (D_{22} - \bar{D})k_{22,22} \quad (2.7)$$

$$\bar{M}_{\alpha\beta} = -\bar{D}[(1-\bar{\mu})k_{\alpha\beta} + \bar{\mu}k_{\gamma\gamma}\delta_{\alpha\beta}] \quad (2.8)$$

$$\{M_{\alpha\beta}\} = [T_M]\{\bar{M}_{\alpha\beta}\} \quad (2.9)$$

$$[T_M] = \frac{1}{\bar{D}(1-\bar{\mu}^2)} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\mu} & 0 \\ -\bar{\mu} & 1 & 0 \\ 0 & 0 & 2(1+\bar{\mu}) \end{bmatrix} \quad (2.10)$$

定理 任一弹性主向与板边平行的正交各向异性简支矩形板(图2)分别按 Reissner 理论和 Kirchhoff 理论计算的转角、曲率和内力均相同, 只是挠度不等。

证明 在 Reissner 理论中引入函数 F , f 和 ϕ , 使得

$$\psi_1 = F_{,1} + f_{,2}, \quad \psi_2 = F_{,2} - f_{,1}, \quad w = F - \frac{\bar{D}}{C} \nabla^2 F - \phi \quad (a)$$

ϕ 是满足方程(1.4a)的一个特解, 要求

$$A: \phi_{, \alpha} = \frac{1}{C} m_{\alpha}, \quad S: \phi = 0 \quad (b)$$

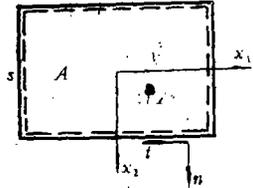


图 2

代入方程(1.4)并简化后得出

$$\frac{1}{2}(1-\bar{\mu})\bar{D}\nabla^2 f - C f = 0 \quad (c)$$

$$\bar{D}\nabla^2 \nabla^2 F = q - m_{\alpha, \alpha} \quad (d)$$

板边简支条件写成

$$S: w = M_{\alpha} = \psi_2 = 0 \quad (e)$$

引入 M_{α} 和式(a), (b), 条件(e)等价于

$$S: F = F_{, \alpha\alpha} = f_{, \alpha\alpha} = 0 \quad (f)$$

由方程(c)和条件 $S: f_{, \alpha\alpha} = 0$ 解得 $f = 0$ 。

如果将 F 理解为相应 Kirchhoff 型板的挠度, 则这两型板的转角、曲率和内力公式完全相同, 而方程(d)也就是方程(2.4), 配合边界条件 $S: F = F_{, \alpha\alpha} = 0$, 正好构成相应 Kirchhoff 型板的定解问题。定理由此得证。

以上重点介绍了正交各向异性板。其实对一般各向异性板也可如法炮制, 只是虚载公式中要增添一些项, 且矩阵 $[D]$, $[C]$, $[T_M]$ 和 $[T_0]$ 是些满阵而已。

三、样条积分方程法

先介绍 Reissner 型各向异性板。积分方程的一般形式写成

$$\begin{aligned} H_i(P) + \int_S (\bar{Q}_i^* w - \bar{M}_i^* \psi_2 - \bar{T}_i^* \psi_1) ds \\ = \int_S (\bar{Q}_i w^* - \bar{M}_i \bar{\psi}_2^* - \bar{T}_i \bar{\psi}_1^*) ds + \int_A (q w^* + m_{\alpha} \bar{\psi}_2^*) dA \end{aligned} \quad (3.1)$$

式中带上扛号的代表假想同性板的名义值, 带 * 号的表示第 i 态基本解(下标 i 略); $H_i(P)$ 因辅态 i 和奇点 P 而异, 比如当 P 位于域内且 i 为 1~5 态时 $H_i(P)$ 分别取 $w(P)$, $\psi_1(P)$, $\psi_2(P)$, $k_{\alpha\beta}(P)$ 和 $\gamma_{\alpha}(P)$ 。变量用样条插值, 解算细节参阅 [3], 域积分项可用散度定理化简,

如

$$(D_{11}-D) \int_A k_{11,1} \bar{\phi}_1^* dA = -(D_{11}-D) \int_A k_{11} \bar{\xi}_{11}^* dA \\ + (D_{11}-D) \int_s k_{11} \bar{\phi}_1^* n_1 ds$$

$$C_{11} \int_A (\gamma_{2,2} w^* + \gamma_2 \bar{\phi}_1^*) dA = -C_{11} \int_A \gamma_2 \bar{\gamma}_1^* dA + \frac{C_{11}}{C} \int_s \bar{Q}_2 w^* n_2 ds$$

等等。

第3、4辅态中的 $k_{\alpha\beta}^*$ 和第5辅态中的 γ_i^* 均呈 $1/r^2$ 阶奇性，但可证明相应域积分存在。边界上的 $k_{\alpha\beta}$ 和 γ_α 应化为名义内力，这样在引入边界条件后在方程组(3.1)中直接解出的将是域内结点上的 $k_{\alpha\beta}$ 、 γ_α 以及边界结点上的名义内力，按式(1.9)就可进一步转换为实际内力。

对Kirchhoff型各向异性板积分方程通式表为

$$H_i(P) + \int_s (\bar{V}_i^* w - \bar{M}_i^* \bar{\phi}_i^*) ds + \sum [T_i^*] w \\ = \int_s (\bar{V}_i w^* - \bar{M}_i \bar{\phi}_i^*) ds + \int_A (q \bar{\phi}^* + m_i \bar{\phi}_i^*) dA + \sum [T_i] w^* \quad (3.2)$$

字母意义同上述，解算细节阅[2]，[4]。域积分项仍可用散度定理化简。解出域内结点上的 $k_{\alpha\beta}$ 和边界名义内力后再转换为实际内力。 Σ 表示对角点值求和。

用本文方法解算双参数弹性地基上的板时式(3.2)中的 q 应改为 $(q - Kw + Tk_{\alpha\alpha})$ ， K ， T 是地基参数⁽⁵⁾。可见除 w 应列为待解值外并不增添其他未知量。

四、算 例

例1 一周边固支正交各向异性圆薄板承受均布荷载 q 。设弹性主向与坐标轴平行，已知

$$D_{11} = 1.4250 \times 10^6 h^3 / 12$$

$$D_{22} = 0.1190 \times 10^6 h^3 / 12$$

$$D_{12} = 0.0546 \times 10^6 h^3 / 12$$

$$D_{66} = 0.1200 \times 10^6 h^3 / 12$$

剖分如图3所示，各变量沿径向和环向分别按2次和3次样条函数插值，根据Kirchhoff型板理论计算，结果见表1。

表 1

| 点 | $w \frac{10^6 h^3}{12 q a^4}$ | $M_{11} / q a^2$ | $M_{22} / q a^2$ | 点 | $Q_r / q a$ | $T_\theta / q a^2$ | $M_\theta / q a^2$ | | |
|----|-------------------------------|------------------|------------------|---------|-------------|--------------------|--------------------|---------|----------|
| 0 | * | 0.0239 | 0.142 | 0.0166 | 16 | * | -0.875 | 0 | -0.273 |
| | Δ | 0.023941 | 0.14169 | 0.01662 | | Δ | -0.8752 | 0 | -0.27293 |
| 6 | * | 0.0135 | 0.038 | 0.0099 | 18 | * | -0.500 | 0.0625 | -0.102 |
| | Δ | 0.013467 | 0.03804 | 0.00985 | | Δ | -0.5000 | 0.06253 | -0.10214 |
| 10 | * | 0.0134 | 0.104 | 0.0065 | 20 | * | -0.126 | 0 | -0.023 |
| | Δ | 0.013467 | 0.10365 | 0.00677 | | Δ | -0.1248 | 0 | -0.02279 |

* 本文解。 $M_n = \frac{D'_{11}}{D} \bar{M}_n$ ， $Q_n = \bar{V}_n$ ， $T_n = \frac{D'_{16}}{D} \bar{M}_n$ ； D'_{11} ， D'_{16} 为 n ， t 系中的弹性常数⁽⁶⁾

Δ 精确解⁽⁷⁾

例2 一正交各向异性简支方板承受正弦形荷载 $q=q_0 \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}$ 。弹性主向与板边平行, 已知 $D_{11}=5.0625 \times 10^5 h^3/12$, $D_{22}=1 \times 10^5 h^3/12$, $D_{12}=0.10125 \times 10^5 h^3/12$, $D_{66}=1.074375 \times 10^5 h^3/12$ 。剖分如图4所示, 各变量双向均按2次样条函数插值, 并根据 Reissner 型板理论计算, 结果见表2。不论 C_{11} , C_{22} 怎样取值, 表中数值均不变。

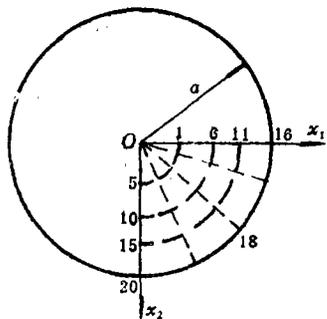


图 3

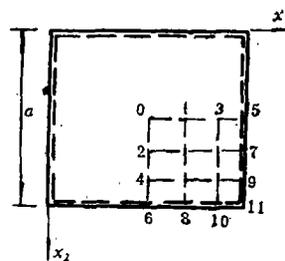


图 4

表 2

| 点 | | $M_{11}/q_0 a^2$ | $M_{22}/q_0 a^2$ | $Q_1/q_0 a$ | 点 | | $\psi_n \frac{10^5 h^3}{12 q_0 a^3}$ | $Q_n/q_0 a$ | $T_n/q_0 a^2$ |
|----|----------|--|------------------|-------------|----|----------|--------------------------------------|-------------|---------------|
| 0 | * | 0.0496 | 0.0105 | 0 | 5 | * | -0.00306 | -0.220 | 0 |
| | Δ | 0.049533 | 0.010584 | 0 | | Δ | -0.0030534 | -0.22037 | 0 |
| 1 | * | 0.0429 | 0.0091 | -0.110 | 7 | * | -0.00263 | -0.190 | -0.0102 |
| | Δ | 0.042897 | 0.009149 | -0.11018 | | Δ | -0.0026443 | -0.19084 | -0.010306 |
| 3 | * | 0.0248 | 0.0052 | -0.190 | 9 | * | -0.00152 | -0.110 | -0.0177 |
| | Δ | 0.024767 | 0.005282 | -0.19084 | | Δ | -0.0015267 | -0.11018 | -0.017850 |
| 2 | * | 0.0428 | 0.0092 | 0 | 6 | * | -0.00306 | -0.096 | 0 |
| | Δ | 0.042897 | 0.009149 | 0 | | Δ | -0.0030534 | -0.09794 | 0 |
| 4 | * | 0.0247 | 0.0053 | 0 | 8 | * | -0.00265 | -0.084 | +0.0103 |
| | Δ | 0.024767 | 0.005282 | 0 | | Δ | -0.0026443 | -0.08482 | +0.010306 |
| 11 | * | $T_n/q_0 a^2 = \begin{cases} 0.0205 \\ 0.020612 \end{cases}$ | | | 10 | * | -0.00153 | -0.048 | +0.0178 |
| | Δ | | | | | Δ | -0.0015267 | -0.04897 | +0.017850 |

Δ 精确解, 按 Kirchhoff 型板公式计算, 取 $w = \frac{q_0 a^4}{\pi^4 (D_{11} + 2D + D_{22})} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}$

• 本文解

五、讨 论

(1) 本文在对各向异性板控制方程作出各向同性化比拟后可以避免采用繁复的各向异性板基本解, 又能一次给出全部物理量, 且板的形状、受载和支承都不受什么限制, 在稀疏剖分下也能有良好的计算精度。

(2) 本文方法用来计算双参数弹性地基上的各向异性板并不增添多大工作量, 积分方程中采用的仍是熟知的各向同性板基本解。

(3) 在正交各向异性简支矩形板中两类板理论的相通性可使计算模型的选取更为灵活方便。

(4) 本文方法可望推广到各向异性的扁壳和二三维问题上,

参 考 文 献

- [1] Wu, B. C. and N. J. Altiero, A new numerical method for the analysis of anisotropic thin plate bending problem, *Comp. Meth. Appl. Mech. Engng.*, 25 (1981), 343-353.
- [2] Wang, Y. C. et al., SBEM for plate bending problems, *Boundary Elements* (Ed. Du Q. H.), Pergamon Press, Beijing (1986), 427-436.
- [3] Wang, Y. C. et al., SBEM for Reissner's plates and its application to foundation plates, *Boundary Elements*, IX (Ed. C. A. Brebbia), 2, Springer-Verlag (1987), 111-125.
- [4] 王有成, Kirchhoff型板样条边界元, 计算结构力学及其应用, 3, 1 (1986), 41-50.
- [5] Selvadurai, A. P. S., *Elastic Analysis of Soil-Foundation Interaction*, Elsevier Scient. Publ. Comp., Amsterdam (1979).
- [6] 列赫尼斯基, 《各向异性板》, 科学出版社 (1963).
- [7] Timoshenko, S. et al., *Theory of Plates and Shells*, McGraw-Hill (1959).

Isotropicalized Spline Integral Equation Method for the Analysis of Anisotropic Plates

Wang You-cheng Wang Zuo-hui

(Hefei University of Technology, Hefei)

Abstract

In the view of Reissner's and Kirchhoff's theories, respectively, we formulate the isotropicalized governing equations for the anisotropic plates, and give the proof of the equivalence relation between these two plate-models for the simply-supported rectangular orthotropic plates. The well-known fundamental solutions of the isotropic plates are utilized for the spline integral equation analysis of anisotropic plates. Even with sparse meshes the satisfactory results can be obtained. The analysis of plates on two-parameter elastic foundation is so simple as the common case that only a few terms should be added to the formulas of fictitious loads.

Key words anisotropic plates, spline integral equation method, isotropicalized process