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反对称角铺设层合板的屈曲和后屈曲*

沈 惠 申

（上海交通大学，1989年3月29日收到）

摘 要

本文以挠度为摄动参数，采用文[1]提供的摄动方法研究了四边简支的完善和非完善反对称角铺设层合板在面内压缩作用下的屈曲和后屈曲性态。

本文讨论了面内边界条件、铺设角、铺层数以及初始几何缺陷对层合板后屈曲性态的影响。

关键词 结构稳定性 屈曲 后屈曲 层合板

一、引 言

随着新颖复合材料在近代结构工程中的应用日趋增长，弄清复合材料层合矩形板在面内压缩作用下的后屈曲性态，对充分认识和利用层合板的屈后强度具有十分重要的意义。

一般认为，各向同性矩形板在面内单向压缩作用下具有稳定的后屈曲平衡路径[1]。正交异性矩形板的屈后强度则与材料刚度比 E_y/E_x 有关。一般说来，刚度比 $E_y/E_x>1$，则其屈后强度高于各向同性板，反之，则低于各向同性板[2]。

对于层合板，由于耦合刚度 B_{ij} 的存在，其后屈曲性态出现复杂的变化情况。弄清这一点对优化设计至关重要。反对称角铺设层合板是复合材料层合板中比较复杂而又常用的一种。因此，这一问题理所当然地引起许多学者的注意和研究[4~11]。但是，目前对于层合板的后屈曲性态，特别是对于具有初始几何缺陷的层合板的后屈曲性态人们还缺乏足够的认识。此种情况说明，进一步的理论分析是必要的。

本文的目的在于研究非完善（具有初始几何缺陷）反对称角铺设层合矩形板在面内压缩作用下的屈曲和后屈曲性态。初挠度的形式取作和矩形板小挠度解的形式一致。

本文同时讨论了面内边界条件、铺设角、铺层数对层合板后屈曲性态的影响。

二、分析方法与渐近解

假定四边简支反对称角铺设层合板沿 x 轴的长为 a，沿 y 轴的宽为 b，板厚为 t，铺设角为 θ，铺层数为 N。受到对边均布压力 p_x。

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对于一般层合板，其合力和弯矩为

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{Bmatrix} \tag{2.1}$$

其中

$$\left.\begin{aligned} &\mathbf{N}=[N_x, N_y, N_{xy}]^T, \ \mathbf{M}=[M_x, M_y, M_{xy}]^T \\ &\boldsymbol{\varepsilon}^0=[\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0]^T, \ \boldsymbol{\kappa}=[\kappa_x, \kappa_y, \kappa_{xy}]^T \end{aligned}\right\} \tag{2.2}$$

$\mathbf{A}=(A_{ij})$，$\mathbf{D}=(D_{ij})$，$\mathbf{B}=(B_{ij})$ （i，$j=1,2,6$）分别为拉伸、弯曲和耦合刚度，如文[3]所定义。

反之，有

$$\begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{B}\mathbf{A}^{-1} & \mathbf{D}-\mathbf{B}\mathbf{A}^{-1}\mathbf{B} \end{bmatrix} \begin{Bmatrix} \mathbf{N} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}^* & \mathbf{B}^* \\ -(\mathbf{B}^*)^T & \mathbf{D}^* \end{bmatrix} \begin{Bmatrix} \mathbf{N} \\ \boldsymbol{\kappa} \end{Bmatrix} \tag{2.3}$$

对于反对称角铺设层合板，我们有

$$A_{16}=A_{26}=B_{11}=B_{12}=B_{22}=B_{66}=D_{16}=D_{26}=0 \tag{2.4a}$$

由此导出

$$A_{16}^*=A_{26}^*=B_{11}^*=B_{12}^*=B_{21}^*=B_{22}^*=B_{66}^*=D_{16}^*=D_{26}^*=0 \tag{2.4b}$$

以 W^* 和 W 分别表示板初始的和附加的挠度，以 ϕ 表示应力函数，即使

$$N_x=\frac{\partial^2\phi}{\partial y^2}, \quad N_y=\frac{\partial^2\phi}{\partial x^2}, \quad N_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \tag{2.5}$$

那么，反对称角铺设层合板非线性大挠度方程可表为

$$L_1W-L_3\phi=\left[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2W}{\partial y^2}+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2W^*}{\partial x^2}\right.$$
$$\left.-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2W^*}{\partial y^2}\right] \tag{2.6}$$

$$L_2\phi+L_3W=\left[\left(\frac{\partial^2W}{\partial x\partial y}\right)^2-\frac{\partial^2W}{\partial x^2}\frac{\partial^2W}{\partial y^2}+2\frac{\partial^2W}{\partial x\partial y}\frac{\partial^2W^*}{\partial x\partial y}\right.$$
$$\left.-\frac{\partial^2W}{\partial x^2}\frac{\partial^2W^*}{\partial y^2}-\frac{\partial^2W}{\partial y^2}\frac{\partial^2W^*}{\partial x^2}\right] \tag{2.7}$$

其中算子

$$\left.\begin{aligned} &L_1=D_{11}^*\partial^4/\partial x^4+2(D_{12}^*+2D_{66}^*)\partial^4/\partial x^2\partial y^2+D_{22}^*\partial^4/\partial y^4 \\ &L_2=A_{22}^*\partial^4/\partial x^4+2(A_{12}^*+A_{66}^*/2)\partial^4/\partial x^2\partial y^2+A_{11}^*\partial^4/\partial y^4 \\ &L_3=(B_{61}^*-2B_{16}^*)\partial^4/\partial x^3\partial y+(B_{62}^*-2B_{26}^*)\partial^4/\partial x\partial y^3 \end{aligned}\right\} \tag{2.8}$$

假定边界支承为四边简支的，那么边界条件为

$$x=0, \ a; \quad W=0, \quad M_x=B_{61}^*\frac{\partial^2\phi}{\partial x\partial y}-D_{11}^*\frac{\partial^2W}{\partial x^2}-D_{12}^*\frac{\partial^2W}{\partial y^2}=0, \ N_{xy}=0 \tag{2.9a}$$

$$\int_0^b N_x dy+p_x=0 \tag{2.9b}$$

$$y=0, \ b; \quad W=0, \quad M_y=B_{62}^*\frac{\partial^2\phi}{\partial x\partial y}-D_{12}^*\frac{\partial^2W}{\partial x^2}-D_{22}^*\frac{\partial^2W}{\partial y^2}=0, \ N_{xy}=0 \tag{2.10a}$$

$$\int_0^a N_y dx=0 \qquad \text{（纵向边缘可移简支）} \tag{2.10b}$$

$$V=\text{常数} \qquad \text{（纵向边缘不可移简支）} \tag{2.10c}$$

单位轴向缩短

$$\begin{aligned}\Delta_x/a&=-\frac{1}{ab}\int_0^b\int_0^a\frac{\partial U}{\partial x}dxdy\\&=-\frac{1}{ab}\int_0^b\int_0^a\Big[\Big(A_{11}^*\frac{\partial^2\phi}{\partial y^2}+A_{12}^*\frac{\partial^2\phi}{\partial x^2}-2B_{16}^*\frac{\partial^2 W}{\partial x\partial y}\Big)\\&\quad-\frac{1}{2}\Big(\frac{\partial W}{\partial x}\Big)^2-\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\Big]dxdy\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Delta_y/b&=-\frac{1}{ab}\int_0^a\int_0^b\frac{\partial V}{\partial y}dydx\\&=-\frac{1}{ab}\int_0^a\int_0^b\Big[\Big(A_{22}^*\frac{\partial^2\phi}{\partial x^2}+A_{12}^*\frac{\partial^2\phi}{\partial y^2}-2B_{26}^*\frac{\partial^2 W}{\partial x\partial y}\Big)\\&\quad-\frac{1}{2}\Big(\frac{\partial W}{\partial y}\Big)^2-\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\Big]dydx\end{aligned} \tag{2.12}$$

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$$\left.\begin{aligned}&\bar{x}=\frac{\pi}{a}x,\quad \bar{y}=\frac{\pi}{b}y,\ \beta=\frac{a}{b},\ \bar{W}=\frac{W}{\sqrt[4]{A_{11}^*A_{22}^*D_{11}^*D_{22}^*}}\\&\bar{W}^*=\frac{W^*}{\sqrt[4]{A_{11}^*A_{22}^*D_{11}^*D_{22}^*}},\ \bar{\phi}=\frac{\phi}{\sqrt{D_{11}^*D_{22}^*}}\\&\gamma_{11}=\frac{D_{12}^*+2D_{66}^*}{D_{11}^*},\quad \gamma_{12}^2=\sqrt{\frac{D_{22}^*}{D_{11}^*}},\ \gamma_{31}=\frac{A_{12}^*+A_{66}^*/2}{A_{22}^*}\\&\gamma_{32}^2=\sqrt{\frac{A_{11}^*}{A_{22}^*}},\ \gamma_5=-\frac{A_{12}^*}{A_{22}^*},\quad \gamma_{21}=\frac{(B_{61}^*-2B_{26}^*)}{D_{11}^*}\sqrt[4]{\frac{D_{11}^*D_{22}^*}{A_{11}^*A_{22}^*}}\\&\gamma_{13}=\frac{(B_{62}^*-2B_{16}^*)}{D_{11}^*}\sqrt[4]{\frac{D_{11}^*D_{22}^*}{A_{11}^*A_{22}^*}},\quad \gamma_{16}=\frac{B_{16}^*}{D_{11}^*}\sqrt[4]{\frac{D_{11}^*D_{22}^*}{A_{11}^*A_{22}^*}}\\&\gamma_{26}=\frac{B_{26}^*}{D_{11}^*}\sqrt[4]{\frac{D_{11}^*D_{22}^*}{A_{11}^*A_{22}^*}},\quad \bar{M}_x=\frac{M_x a^2}{\pi^2 D_{11}^*\sqrt[4]{A_{11}^*A_{22}^*D_{11}^*D_{22}^*}}\\&\bar{M}_y=\frac{M_y a^2}{\pi^2 D_{11}^*\sqrt[4]{A_{11}^*A_{22}^*D_{11}^*D_{22}^*}}\\&\lambda_x=\frac{\sigma_x b^2 t}{4\pi^2\sqrt{D_{11}^*D_{22}^*}},\ \delta_x=\frac{b^2}{4\pi^2\sqrt{A_{11}^*A_{22}^*D_{11}^*D_{22}^*}}\cdot\frac{\Delta_x}{a}\\&\delta_y=\frac{b^2}{4\pi^2\sqrt{A_{11}^*A_{22}^*D_{11}^*D_{22}^*}}\cdot\frac{\Delta_y}{b}\end{aligned}\right\} \tag{2.13}$$

那么方程(2.6)，(2.7)可化为如下无量纲形式（略去字母上的“—”号）

$$L_1^*W-L_3^*\phi=\gamma_{12}^2\beta^2\Big[\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W}{\partial y^2}$$

$$+\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}\Big] \tag{2.14}$$

$$L_2^*\phi+\frac{\gamma_{12}^2}{\gamma_{12}^2}L_3^*W=\gamma_{12}^2\beta^2\Big[\Big(\frac{\partial^2 W}{\partial x\partial y}\Big)^2-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2 W^*}{\partial x\partial y}$$

$$-\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W^*}{\partial y^2}-\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W^*}{\partial x^2}\Big] \tag{2.15}$$

其中

$$\left.\begin{aligned}L_1^*&=\partial^4/\partial x^4+2\gamma_{11}\beta^2\partial^4/\partial x^2\partial y^2+\gamma_{12}^4\beta^4\partial^4/\partial y^4\\L_2^*&=\partial^4/\partial x^4+2\gamma_{21}\beta^2\partial^4/\partial x^2\partial y^2+\gamma_{12}^4\beta^4\partial^4/\partial y^4\\L_3^*&=\gamma_{31}\beta\partial^4/\partial x^3\partial y+\gamma_{13}\beta^3\partial^4/\partial x\partial y^3\end{aligned}\right\} \tag{2.16}$$

边界条件化为

$$x=0,\ \pi;\quad W=0,\ M_x=0,\ \phi_{,xy}=0 \tag{2.17a}$$

$$\frac{1}{\pi}\int_0^\pi\beta^2\frac{\partial^2\phi}{\partial y^2}dy+4\lambda_x\beta^2=0 \tag{2.17b}$$

$$y=0,\ \pi;\quad W=0,\ M_y=0,\ \phi_{,xy}=0 \tag{2.18a}$$

$$\frac{1}{\pi}\int_0^\pi\frac{\partial^2\phi}{\partial x^2}dx=0\qquad\text{（纵向边缘可移简支）} \tag{2.18b}$$

$$\delta_y=0\qquad\text{（纵向边缘不可移简支）} \tag{2.18c}$$

单位轴向缩短化为

$$\delta_x=-\frac{1}{4\pi^2\beta^2\gamma_{12}^2}\int_0^\pi\int_0^\pi\Big[\Big(\gamma_{12}^4\beta^2\frac{\partial^2\phi}{\partial y^2}-\gamma_5\frac{\partial^2\phi}{\partial x^2}-2\gamma_{16}\frac{\gamma_{12}^2}{\gamma_{12}^2}\beta\frac{\partial^2 W}{\partial x\partial y}\Big)$$

$$-\frac{1}{2}\gamma_{12}^2\Big(\frac{\partial W}{\partial x}\Big)^2-\gamma_{12}^2\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x}\Big]dxdy \tag{2.19}$$

$$\delta_y=-\frac{1}{4\pi^2\beta^2\gamma_{12}^2}\int_0^\pi\int_0^\pi\Big[\Big(\frac{\partial^2\phi}{\partial x^2}-\gamma_5\beta^2\frac{\partial^2\phi}{\partial y^2}-2\gamma_{26}\frac{\gamma_{12}^2}{\gamma_{12}^2}\beta\frac{\partial^2 W}{\partial x\partial y}\Big)$$

$$-\frac{1}{2}\gamma_{12}^2\beta^2\Big(\frac{\partial W}{\partial y}\Big)^2-\gamma_{12}^2\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}\Big]dxdy \tag{2.20}$$

方程(2.14)至(2.20)即为四边简支反对称角铺设层合板在面内压缩作用下屈曲问题的控制方程。下面我们将用摄动方法来构造其渐近解。

设方程(2.14)、(2.15)的解为如下渐近展开式

$$W(x,y,\varepsilon)=\sum_{k=1}\varepsilon^k W_k(x,y),\quad \phi(x,y,\varepsilon)=\sum_{k=0}\varepsilon^k\phi_k(x,y) \tag{2.21}$$

并取无量纲初挠度

$$W^*(x,y,\varepsilon)=\varepsilon A_{11}^*\sin mx\sin ny=\varepsilon\mu A_{11}^{(1)}\sin mx\sin ny \tag{2.22}$$

将式(2.21)、(2.22)代入方程(2 14)、(2.15)便可获得各级摄动方程，采用文[1]类似的摄动步骤，逐级摄动，我们可以得到大挠度渐近解。

$$W=\varepsilon[A_{11}^{(1)}\sin mx\sin ny]+\varepsilon^3[A_{13}^{(3)}\sin mx\sin 3ny+A_{31}^{(3)}\sin 3mx\sin ny]$$

$$+\varepsilon^4[A_{22}^{(4)}\sin 2mx\sin 2ny+A_{24}^{(4)}\sin 2mx\sin 4ny+A_{42}^{(4)}\sin 4mx\sin 2ny]$$

$$+O(\varepsilon^5) \tag{2.23}$$

$$\phi=-B_{00}^{(0)}\frac{y^2}{2}-b_{00}^{(0)}\frac{x^2}{2}+\varepsilon[B_{11}^{(1)}\cos mx\cos ny]+\varepsilon^2\Big[-B_{00}^{(2)}\frac{y^2}{2}-b_{00}^{(2)}\frac{x^2}{}$$

$$+B_{20}^{(2)}\cos 2mx+B_{02}^{(2)}\cos 2ny\Big]+\varepsilon^3\Big[B_{13}^{(3)}\cos mx\cos 3ny+B_{31}^{(3)}\cos 3mx\cos ny\Big]$$

$$+\varepsilon^4\Big[-B_{00}^{(4)}\frac{y^2}{2}-b_{00}^{(4)}\frac{x^2}{2}+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny+B_{22}^{(4)}\cos 2mx\cos 2ny$$

$$+B_{40}^{(4)}\cos 4mx+B_{04}^{(4)}\cos 4ny+B_{24}^{(4)}\cos 2mx\cos 4ny+B_{42}^{(4)}\cos 4mx\cos 2ny\Big]+O(\varepsilon^5) \quad (2.24)$$

其中系数 $B_{00}^{(k)}$ 和 $b_{00}^{(k)}(k=0,2,4,\cdots)$ 的关系为

$$\gamma_{12}^2(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}n^2\beta^2)=\frac{1}{1+\mu}\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)$$

$$+\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4}\Big] \quad (2.25a)$$

$$\gamma_{12}^2(\beta^2B_{00}^{(2)}m^2+b_{00}^{(2)}n^2\beta^2)=\frac{1}{16}\frac{\gamma_{12}^2}{\gamma_{12}^2}(m^4+\gamma_{12}^4n^4\beta^4)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)} \quad (2.25b)$$

$$\gamma_{12}^2(\beta^2B_{00}^{(4)}m^2+b_{00}^{(4)}n^2\beta^2)=-\frac{1}{256}(1+2\mu)[2(1+\mu)^2+(1+2\mu)]$$

$$\cdot\frac{\gamma_{12}^4}{\gamma_{12}^4}\Big[\frac{m^8}{g_{13}}+\frac{\gamma_{12}^8n^8\beta^8}{g_{31}}\Big]A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)} \quad (2.25c)$$

其它系数皆可表为 $A_{11}^{(1)}$ 的形式，如

$$\left.\begin{aligned}
B_{11}^{(1)}&=\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{mn\beta(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4}A_{11}^{(1)}\\
B_{20}^{(2)}&=\frac{1}{32}\frac{\gamma_{12}^2n^2\beta^2}{m^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\\
B_{02}^{(2)}&=\frac{1}{32}\frac{m^2}{\gamma_{12}^2n^2\beta^2}(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}\\
A_{13}^{(3)}&=\frac{1}{16}\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{m^4}{g_{13}}(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
A_{31}^{(3)}&=\frac{1}{16}\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{\gamma_{12}^4n^4\beta^4}{g_{31}}(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{13}^{(3)}&=\frac{1}{16}\frac{3mn\beta(\gamma_{31}m^2+\gamma_{13}9n^2\beta^2)}{m^4+18\gamma_{21}m^2n^2\beta^2+81\gamma_{12}^4n^4\beta^4}\frac{m^4}{g_{13}}\\
&\quad\cdot(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{31}^{(3)}&=\frac{1}{16}\frac{3mn\beta(\gamma_{31}9m^2+\gamma_{13}n^2\beta^2)}{81m^4+18\gamma_{21}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4}\frac{\gamma_{12}^4n^4\beta^4}{g_{31}}\\
&\quad\cdot(1+\mu)(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{20}^{(4)}&=-\frac{1}{256}\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{\gamma_{12}^2n^2\beta^2}{m^2}\frac{\gamma_{12}^4n^4\beta^4}{g_{31}}\\
&\quad\cdot(1+\mu)^2(1+2\mu)A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}\\
B_{02}^{(4)}&=-\frac{1}{256}\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{m_2}{\gamma_{12}^2n^2\beta^2}\frac{m^4}{g_{13}}(1+\mu)^2(1+2\mu)\\
&\quad\cdot A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}A_{11}^{(1)}
\end{aligned}\right\} \quad (2.26)$$

其中

$$
\left.\begin{aligned}
g_{13}=&\Big[(m^4+18\gamma_{11}m^2n^2\beta^2+81\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{12}^2}{\gamma_{12}^2}\\
&\cdot\frac{9m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}9n^2\beta^2)^2}{m^4+18\gamma_{21}m^2n^2\beta^2+81\gamma_{12}^4n^4\beta^4}\Big]-\gamma_{12}^2(\beta^2B_{00}^{(0)}m^2+b_{00}^{(0)}9n^2\beta^2)\\
g_{31}=&\Big[(81m^4+18\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{12}^2}{\gamma_{12}^2}\\
&\cdot\frac{9m^2n^2\beta^2(\gamma_{31}9m^2+\gamma_{13}n^2\beta^2)^2}{81m^4+18\gamma_{21}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4}\Big]-\gamma_{12}^2(\beta^2B_{00}^{(0)}9m^2+b_{00}^{(0)}n^2\beta^2)
\end{aligned}\right\}\tag{2.27}
$$

比照文[2]我们发现，由于耦合刚度B_{ij}的存在，大挠度渐近解的形式和正交异性复合板的情况有明显的不同。只有当耦合刚度B_{ij}趋于零时，此时γ_{31}和γ_{13}都趋于零，由以得到$B_{11}^{(1)}=B_{13}^{(3)}=B_{31}^{(3)}=0$，式(2.23)、(2.24)回到文[2]所得结果。

三、屈曲和后屈曲性态

将式(2.24)代入边界条件(2.17b)我们有

$$4\lambda_s\beta^2=\beta^2B_{00}^{(0)}+\varepsilon^2\beta^2B_{00}^{(2)}+\varepsilon^4\beta^2B_{00}^{(4)}+\cdots\tag{3.1}$$

当$x=\pi/2m$，$y=\pi/2n$时，由式(2.23)我们有

$$w_m=A_{11}^{(1)}\varepsilon-(A_{13}^{(3)}+A_{31}^{(3)})\varepsilon^3+\cdots\tag{3.2}$$

反之，我们有

$$A_{11}^{(1)}\varepsilon=w_m+\frac{1}{16}(1+\mu)(1+2\mu)\frac{\gamma_{12}^2}{\gamma_{12}^2}\Big[\frac{m^4}{g_{13}}+\frac{\gamma_{12}^4n^4\beta^4}{g_{31}}\Big]w_m^3+\cdots\tag{3.3}$$

其中g_{13}和g_{31}由式(2.27)给出。

下面我们考虑两种不同的面内边界条件

(1) 纵向边缘可移简支

将式(2.24)代入边界条件(2.18b)我们有

$$b_{00}^{(k)}=0\qquad(k=0,2,4,\cdots)\tag{3.4}$$

由此我们导得

$$
\begin{aligned}
\lambda_s=&\frac{1}{4\beta^2\gamma_{12}^2}\Big\{\frac{1}{(1+\mu)m^2}\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)\\
&+\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4}\Big]+\frac{1}{16}\frac{\gamma_{12}^2}{\gamma_{12}^2}\frac{m^4+\gamma_{12}^4n^4\beta^4}{m^2}(1+2\mu)w_m^2\\
&+\frac{1}{256}\frac{\gamma_{12}^4}{\gamma_{12}^4}\Big\langle 2(1+\mu)^2(1+2\mu)^2\Big[\frac{m^4(m^4+\gamma_{12}^4n^4\beta^4)}{g_{13}}+\frac{\gamma_{12}^4n^4\beta^4(m^4+\gamma_{12}^4n^4\beta^4)}{g_{31}}\Big]\\
&-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]\Big[\frac{m^8}{g_{13}}+\frac{\gamma_{12}^8n^8\beta^8}{g_{31}}\Big]\Big\rangle w_m^4+\cdots\Big\}
\end{aligned}\tag{3.5}
$$

$$
\begin{aligned}
\delta_s=&\lambda_s\gamma_{12}^2+\frac{1}{32}\frac{m^2}{\beta^2}(1+2\mu)w_m^2+\frac{1}{256}\frac{m^4}{\beta^2}(1+\mu)^2(1+2\mu)^2\frac{\gamma_{12}^2}{\gamma_{12}^2}\Big[\frac{m^4}{g_{13}}\\
&+\frac{\gamma_{12}^4n^4\beta^4}{g_{31}}\Big]w_m^4+\cdots
\end{aligned}\tag{3.6}
$$

其中

$$\left.\begin{aligned}g_{13}=m^2\Big\{&\Big[(m^4+18\gamma_{11}m^2n^2\beta^2+81\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{9m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}9n^2\beta^2)^2}{m^4+18\gamma_{21}m^2n^2\beta^2+81\gamma_{22}^4n^4\beta^4}\Big]\\&\cdot(1+\mu)-\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big]\Big\}\\g_{31}=m^2\Big\{&\Big[(81m^4+18\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{9m^2n^2\beta^2(\gamma_{31}9m^2+\gamma_{13}n^2\beta^2)^2}{81m^4+18\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big]\\&\cdot(1+\mu)-9\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big]\Big\}\end{aligned}\right\}\tag{3.7}$$

（2） 纵向边缘不可移简支

将式(2.23)、(2.24)代入边界条件(2.18c)，令摄动小参数$\varepsilon\to 0$，我们有

$$b_{00}^{(0)}=\gamma_5\beta^2B_{00}^{(0)}\tag{3.8}$$

由因我们导得

$$\begin{aligned}\lambda_s=\frac{1}{4\beta^2\gamma_{12}^2}\Big\{&\frac{1}{(1+\mu)(m^2+\gamma_5n^2\beta^2)}\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)\\&+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big]+\frac{1}{16}\,\frac{\gamma_{12}^2}{\gamma_{22}^2}\,\frac{m^4+3\gamma_{22}^4n^4\beta^4}{m^2+\gamma_5n^2\beta^2}(1+2\mu)w_m^2\\&+\frac{1}{256}\,\frac{\gamma_{12}^4}{\gamma_{22}^4}\Big\langle 2(1+\mu)^2(1+2\mu)^2\Big[\frac{m^4(m^4+3\gamma_{22}^4n^4\beta^4)}{g_{13}}+\frac{\gamma_{22}^4n^4\beta^4(m^4+3\gamma_{22}^4n^4\beta^4)}{g_{31}}\Big]\\&-(1+\mu)(1+2\mu)[2(1+\mu)^2+(1+2\mu)]\Big[\frac{m^8}{g_{13}}+\frac{\gamma_{22}^8n^8\beta^8}{g_{31}}\Big]\Big\rangle w_m^4+\cdots\Big\}\end{aligned}\tag{3.9}$$

$$\begin{aligned}\delta_s=&\lambda_s\Big(\gamma_{22}^2-\frac{\gamma_5^2}{\gamma_{22}^2}\Big)+\frac{1}{32}\,\frac{m^2+\gamma_5n^2\beta^2}{\beta^2}(1+2\mu)w_m^2\\&+\frac{1}{256}\,\frac{(m^2+\gamma_5n^2\beta^2)^2}{\beta^2}(1+\mu)^2(1+2\mu)^2\frac{\gamma_{12}^2}{\gamma_{22}^2}\Big[\frac{m^4}{g_{13}}+\frac{\gamma_{22}^4n^4\beta^4}{g_{31}}\Big]w_m^4+\cdots\end{aligned}\tag{3.10}$$

其中

$$\left.\begin{aligned}g_{13}=&\Big[(m^4+18\gamma_{11}m^2n^2\beta^2+81\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{9m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}9n^2\beta^2)^2}{m^4+18\gamma_{21}m^2n^2\beta^2+81\gamma_{22}^4n^4\beta^4}\Big]\\&\cdot(m^2+\gamma_5n^2\beta^2)(1+\mu)-\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)\\&+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big](m^2+9\gamma_5n^2\beta^2)\\g_{31}=&\Big[(81m^4+18\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)+\frac{\gamma_{22}^2}{\gamma_{12}^2}\,\frac{9m^2n^2\beta^2(\gamma_{31}9m^2+\gamma_{13}n^2\beta^2)^2}{81m^4+18\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big]\\&\cdot(m^2+\gamma_5n^2\beta^2)(1+\mu)-\Big[(m^4+2\gamma_{11}m^2n^2\beta^2+\gamma_{12}^4n^4\beta^4)\\&+\frac{\gamma_{22}^2}{\gamma_{22}^2}\,\frac{m^2n^2\beta^2(\gamma_{31}m^2+\gamma_{13}n^2\beta^2)^2}{m^4+2\gamma_{21}m^2n^2\beta^2+\gamma_{22}^4n^4\beta^4}\Big](9m^2+\gamma_5n^2\beta^2)\end{aligned}\right\}\tag{3.11}$$

对于完善层合板($W^*=0$，即$\mu=0$)，令$w_m=0$，由式(3.5)或(3.9)我们容易得到线性临界值。

四、结果和讨论

作为数值分析例子，表1、表2和表3分别给出完善反对称角铺设层合板屈曲载荷比较。材料为高模量石墨一环氧，其弹性常数为（$1\text{psi}=6.89\times10^3\text{Pa}$）

$E_{11}=3.0\times10^6\text{psi}$，$E_{22}=0.75\times10^6\text{psi}$，$G_{12}=0.375\times10^6\text{psi}$，$\nu_{12}=0.25$

表中本文结果按纵边可移简支给出。比较发现，除铺设角$\theta=60^\circ$和$\theta=75^\circ$，铺层数$N=4$和$N=6$的情况外（参见表1），本文结果和文[4，6]的结果相当一致。表中(m,n)为屈曲模态。可以看出，随着铺设角θ的增加将会出现屈曲模态的变化。因此，层合板的优化设计必须计及屈曲模态的变化。

表1 反对称角铺设层合方板屈曲载荷比较

$\beta=1.0$		$\sigma_x(b/t)^2/E_{22}$					
		$N=4$		$N=6$		$N=\infty$	
θ	(m,n)	本文	Jones[6]	本文	Jones[6]	本文	Jones[6]
0°	(1,1)	35.8307	35.831	35.8307	35.831	35.8307	35.831
15°	(1,1)	38.2534	38.253	41.3126	41.313	43.7600	43.760
30°	(1,1)	49.8240	49.824	55.2654	55.265	59.6186	59.619
45°	(1,1)	56.0881	56.088	62.4546	62.455	67.5478	67.548
60°	(2,1)	47.1985	45.434	51.0411	50.257	54.1151	54.115
75°	(2,1)	18.1206	22.075	22.0139	23.772	25.1286	25.129
90°	(3,1)	13.1317	13.132	13.1317	13.132	13.1317	13.132

表2 反对称角铺设层合方板屈曲载荷比较

$\beta=1.0$ $\theta=45^\circ$	$\sigma_x(b/t)^2/E_{22}$		
N	本文	Jones[6]	Chia[4]
2	21.7089	21.709	21.7101
4	56.0881	56.088	56.0894
∞	67.5478	67.548	67.5458

表3 反对称角铺设层合矩形板屈曲载荷比较

$\theta=45^\circ$ $N=4$		$\sigma_x(b/t)^2/E_{22}$	
β	(m,n)	本文	Chia[4]
1.0	(1,1)	56.0881	56.0894
1.5	(2,1)	58.6420	58.64
2.0	(2,1)	56.0881	56.09

图1为石墨一环氧(GR)和硼一环氧(BO)层合方板（$\theta=45^\circ$，$N=4$）后屈曲载荷一挠度曲线比较。可以看出，对于石墨一环氧层合板，当挠度较大时（$W/t>1.0$），本文结果较之文[4]结果为低；对于硼一环氧层合板，当挠度较大时（$W/t>2.0$）。本文结果较之文[4]结果为高。而当挠度较小时（$W/t<1.0$），本文结果和文[4]结果几乎完全一致。

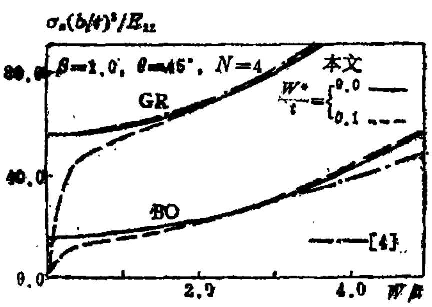


图1 反对称角铺设层合方板后屈曲载荷一挠度曲线比较

进而，我们计算了($\beta=3.0$)反对称角铺设层合板对应两种面内边界条件的后屈曲平衡路径。计算中铺设角取$\theta=45^\circ$，铺层数取$N=6$，材料弹性常数取

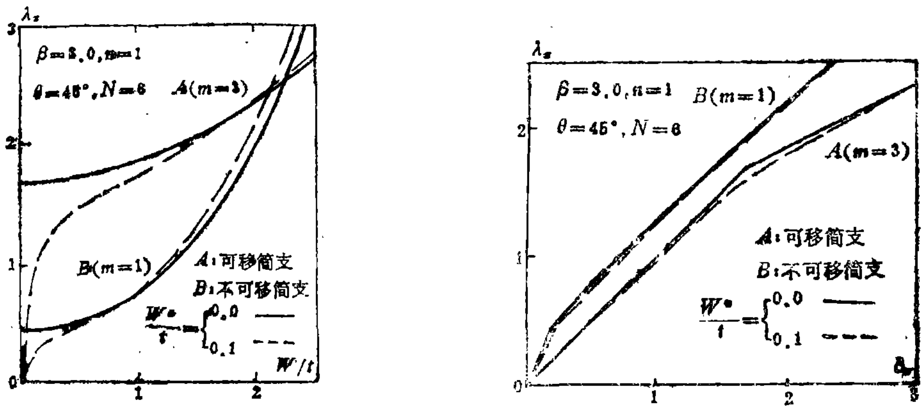
$$E_{11}=130.0\text{GPa},\ E_{22}=9.0\text{GPa},\ G_{12}=4.8\text{GPa},\ \nu_{12}=0.28$$

计算结果如图 2 所示。结果表明，纵边可移简支层合板屈曲模态取$(m,\ n)=(3,1)$，纵边不可移简支层合板屈曲模态取$(m,\ n)=(1,1)$，其所对应的屈曲载荷要比纵边可移简支情况低得多，但却具有较高的屈后强度。

图 3 给出铺设角$\theta=22.5^\circ$，45°和67.5°时的计算结果。图示表明，当铺设角取$\theta=45^\circ$时，较之$\theta=22.5^\circ$和$\theta=67.5^\circ$层合板具有较高的屈曲载荷，但其后屈曲平衡路径相对要平缓得多。显见，对应不同铺设角时，层合板后屈曲载荷—挠度曲线具有显著差别。

图 4 给出铺层数$N=24$和$N=4$时的计算结果（此时铺设角取$\theta=45^\circ$）。可以看出，铺层数增加初始屈曲载荷增加，而后屈曲平衡路径逐趋平缓。需要说明，由于采用无量纲参数，随着铺层数N的增加，初始屈曲载荷的增加不如表1那么明显。

图1至图4中分别给出完善$(W^*/t=0.0)$和非完善$(W^*/t=0.1)$层合板的计算结果。可以看出，初始几何缺陷对层合板的影响类似对正交异性矩形板的影响[2]。一般说来，层合板后屈曲平衡路径对初始几何缺陷表现不敏感。



(a)载荷—挠度曲线　　(b)载荷—端部缩短曲线

图2　对应两种面内边界条件，后屈曲平衡路径

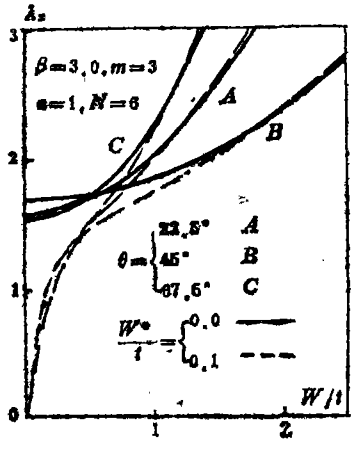


图3　对应不同铺设角，载荷—挠度曲线

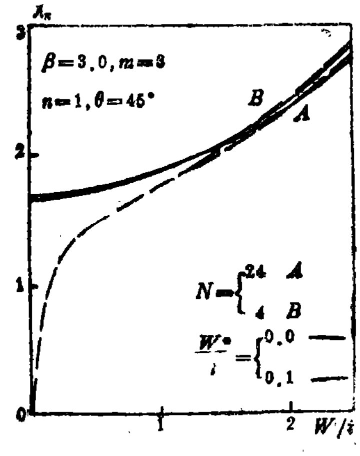


图4　对应不同铺层数，载荷—挠度曲线

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Buckling and Postbuckling Behavior of Antisymmetrically Angle-Ply Laminated Composite Plates

Shen Hui-shen

(*Shanghai Jiaotong University, Shanghai*)

Abstract

The buckling and postbuckling behaviors of perfect and imperfect antisymmetrically angle-ply laminated composite plates under uniaxial compression have been studied by perturbation technique which takes deflection as its perturbation parameter.

In this paper, the effects of in-plane boundary conditions angles, total number of layers and initial geometric imperfection on the postbuckling behavior of laminated plates have been discussed.

Key words structural stability, buckling, postbuckling, laminated plate