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发展方程初边值问题的高阶差分-边界积分方程法及误差分析*

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摘 要

本文结合差分方法与边界积分方程方法，提出并研究了一类新的求解发展型方程初边值问题的高阶差分与边界积分方程耦合数值方法．对于有界区域问题与无界区域问题给出了数值计算格式及其误差的先验估计．

关键词 高阶差分 边界元 发展方程 耦合方法 误差估计

近年来，边界元方法在发展型问题中的应用有了很大的进展．对于发展型问题的边界元方法通常有两种类型，一种是利用发展方程的基本解导出时-空边界积分方程和边界变分方程．第二种是利用积分变换或差分离散化将发展方程转换成椭圆型方程．对于前者，其应用与理论研究已相当完整．而对于后者，其理论研究成果相当贫乏．在[1]中，作者对于一类半线性抛物型初边值问题给出了一类差分-边界有限元耦合方法及其先验误差估计，其误差关于时间步长是一阶精度．本文中，我们对于抛物型初边值问题给出一类Crank-Nicolson型差分-边界有限元耦合方法及其先验误差估计，其误差关于时间步长是二阶精度．

设Ω是空间R^3中的有界区域，Γ是其边界曲面，$\Omega'=R^3\backslash\bar{\Omega}$是$\bar{\Omega}=\Omega\cup\Gamma$的外补区域．考虑抛物型方程初边值问题：

$$(\mathrm{P})\qquad\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u=f & (x\in\Omega\text{或}\Omega',\ t\in(0,T])\\ u=\varphi & (x\in\Gamma,\ t\in[0,T])\\ u(x,0)=u_0(x) & (x\in\Omega\text{或}\Omega')\end{cases}$$

我们首先用Crank-Nicolson差分格式离散时间变量，将(P)转换成差分微分方程，然后利用边界有限元方法求解，得到(P)的近似解．

一、差分-边界积分方程与变分公式

取τ为时间步长，记$t_k=k\tau$，$u^k(x)=u(x,t_k)$，$u^{k+\frac{1}{2}}(x)=u\left(x,t_k+\dfrac{\tau}{2}\right)$和$\bar{u}^{k+1/2}=\dfrac{1}{2}\cdot$ $\cdot(u^k+u^{k+1})$．方程(P)可以改写成

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$$u^k-u^{k-1}-\frac{\tau}{2}\Delta(u^k+u^{k-1})=\tau(f^{k-1/2}+Q^k) \qquad (x\in\Omega\text{或}\Omega') \tag{1.1}$$

其中

$$Q^k=\frac{u^k-u^{k-1}}{\tau}-\frac{\partial u^{k-1/2}}{\partial t}+\Delta\left(u^{k-1/2}-\frac{u^k+u^{k-1}}{2}\right)$$

$$=\frac{1}{2\tau}\left\{\int_{t_{k-1/2}}^{t_k}(t_k-t)^2\frac{\partial^3 u}{\partial t^3}dt+\int_{t_{k-1}}^{t_{k-1/2}}(t-t_{k-1})^2\frac{\partial^3 u}{\partial t^3}dt\right\} \tag{1.2}$$

$$+\frac{1}{2}\left\{\int_{t_{k-1/2}}^{t_k}(t_k-t)\Delta\frac{\partial^2 u}{\partial t^2}dt+\int_{t_{k-1}}^{t_{k-1/2}}(t-t_{k-1})\Delta\frac{\partial^2 u}{\partial t^2}dt\right\}$$

略去Q^k即得一个Crank-Nicolson型的半离散差分-微分方程组：

$$\begin{cases} U^k-\frac{\tau}{2}\Delta U^k=U^{k-1}+\frac{\tau}{2}\Delta U^{k-1}+\tau f^{k-1/2} \qquad (x\in\Omega\text{或}\Omega') & (1.3a)\\ U^k=\varphi^k \qquad (x\in\Gamma, \quad k=1,2,\cdots) & (1.3b)\\ U^0=u_0 \qquad (x\in\Omega\text{或}\Omega') & (1.3c)\end{cases}$$

方程组(1.3)是一系列Helmholtz方程的Dirichlet问题。

本文中，我们采用通用的符号$H^r(G)$表示点集G上的Sobolev空间[2,3]。由微分方程理论知[2]：当$\varphi^k\in H^{\frac{1}{2}}(\Gamma)$，$U^{k-1}\in H^1(R^3)$和$f^{k-1/2}\in H^{-1}(\Omega\cup\Omega')$时，Helmholtz 方程 Dirichlet 问题(1.3)有唯一解$U^k\in H^1(R^3)$。于是边值问题组(1.3)存在唯一解。Helmholtz方程(1.3a)的基本解为：

$$E(x,y)=\frac{1}{4\pi}\exp\left[-\sqrt{\frac{2}{\tau}}|x-y|\right]\frac{1}{|x-y|} \qquad (x,y\in R^3) \tag{1.4}$$

利用递推关系，(1.3a)可以改写成

$$U^k-\frac{\tau}{2}\Delta U^k=\sum_{j=1}^{k-1}(-1)^{j+1}(2U^{k-j}+\tau f^{k+1/2-j})$$

$$+(-1)^{k+1}\left[u_0-\frac{\tau}{2}\Delta u_0+\tau f^{1/2}\right] \qquad (x\in\Omega\text{或}\Omega') \tag{1.5}$$

或

$$U^k+U^{k-1}-\frac{\tau}{2}\Delta(U^k+U^{k-1})=2U^{k-1}+\tau f^{k-1/2} \qquad (x\in\Omega\text{或}\Omega') \tag{1.6}$$

利用基本解 $E(x,y)$，可以写出方程(1.5)或(1.6)的解的积分表达式，从而得到边界积分方程。我们用边界有限元方法求边界积分方程的近似解。设V_h 是 $H^{-\frac{1}{2}}(\Gamma)$ 的一个有限维子空间族。考虑下述逼近格式。记$\partial/\partial n$ 为Γ处外法向导算子。

对于内区域问题，逼近格式A：

(A1) $U_h^0(x)=u_0(x) \quad (x\in\Omega)$， $g_h^0(x)=\frac{\partial u_0}{\partial n}(x) \quad (x\in\Gamma)$；

(A2) $$\tilde{\varphi}_h^k(x)=\frac{\varphi^k}{2}(x)+\int_\Gamma\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)dS_y$$

$$+\frac{2}{\tau}\int_\Omega\left\{\sum_{j=1}^{k}(-1)^j[2U_h^{k-j}(y)+\tau f^{k+1/2-j}(y)]E(x,y)\right\}dy$$

$$+(-1)^{k+1}\left\{\frac{\varphi^0(x)}{2}+\int_\Gamma\left[\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)\right.\right.$$

$$\left.\left.-\frac{\partial u_0}{\partial n}(y)E(x,y)\right]dS_y\right\} \qquad (x\in\Gamma) \tag{1.7a}$$

或

$$\tilde{\varphi}_h^k(x)=\frac{1}{2}(\varphi^k(x)+\varphi^{k-1}(x))+\int_\Gamma\left[(\varphi^k(y)\right.$$

$$\left.+\varphi^{k-1}(y))\frac{\partial}{\partial n_y}E(x,y)-g_h^{k-1}(y)E(x,y)\right]dS_y$$

$$-\frac{2}{\tau}\int_\Omega[2U_h^{k-1}(y)+\tau f^{k-1/2}(y)]E(x,y)dy \qquad (x\in\Gamma) \tag{1.7b}$$

(A3) $\left\{\begin{array}{l}\text{求 } g_h^k\in V_h\text{，满足}\\ a(g_h^k,q_h)=\langle\tilde{\varphi}_h^k,q_h\rangle \qquad (\forall q_h\in V_h)\end{array}\right.$

(A4) $$U_h^k(x)=\int_\Gamma\left[g_h^k(y)E(x,y)-\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)\right]dS_y$$

$$-\frac{2}{\tau}\int_\Omega\left\{\sum_{j=1}^k(-1)^j[2U_h^{k-j}(y)+\tau f^{k+1/2-j}(y)]E(x,y)\right\}dy$$

$$+(-1)^k\left[u_0(x)+\int_\Gamma\left(\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)\right.\right.$$

$$\left.\left.-\frac{\partial u_0}{\partial n}(y)E(x(y)\right)dS_y\right] \qquad (x\in\Omega) \tag{1.8a}$$

或

$$U_h^k(x)=-U_h^{k-1}(x)+\int_\Gamma\left\{(g_h^k(y)+g_h^{k-1}(y))E(x,y)\right.$$

$$\left.-(\varphi^k(y)+\varphi^{k-1}(y))\frac{\partial}{\partial n_y}E(x,y)\right\}dS_y+\frac{2}{\tau}\int_\Omega[2U_h^{k-1}(y)$$

$$+\tau f^{k-1/2}(y)]E(x,y)dy \qquad (x\in\Omega,\ k=1,2,3,\cdots) \tag{1.8b}$$

易验证，在逼近格式A中，采用(1.7a)和(1.8a)与采用(1.7b)和(1.7b)是等价的。

对于外区域问题，逼近格式B：

(B1) $U_h^0(x)=u_0(x) \qquad (x\in\Omega')$, $\quad g_h^0(x)=\frac{\partial u_0}{\partial n}(x) \qquad (x\in\Gamma)$,

(B2) $$\tilde{\varphi}_h^k(x)=\frac{\varphi^k(x)}{2}-\int_\Gamma\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)dS_y$$

$$+\frac{2}{\tau}\int_{\Omega'}\left\{\sum_{j=1}^k(-1)^j[2U_h^{k-j}(y)+\tau f^{k+1/2-j}(y)]E(x,y)\right\}dy$$

$$+(-1)^{k+1}\left\{\frac{\varphi^0(x)}{2}-\int_\Gamma\left[\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)\right.\right.$$

$$\left.\left.-\frac{\partial u_0}{\partial n}(y)E(x,y)\right]dS_y\right\} \qquad (x\in\Gamma) \tag{1.9a}$$

或

$$\widetilde{\varphi}_h^k(x)=\frac{1}{2}(\varphi^k(x)+\varphi^{k-1}(x))-\int_\Gamma[(\varphi^k(y)+\varphi^{k-1}(y)]\frac{\partial}{\partial n_y}E(x,y)$$

$$-g_h^{k-1}(y)E(x,y)]dS_y-\frac{2}{\tau}\int_{\Omega'}[2U_h^{k-1}(y)$$

$$+\tau f^{k-1/2}(y)]E(x,y)dy \quad (x\in\Gamma); \tag{1.9b}$$

(B3) $\begin{cases} \text{求} g_h^k\in V_h\text{，满足} \\ a(g_h^k,q_h)=-\langle\widetilde{\varphi}_h^k,q_h\rangle \quad (\forall q_h\in V_h) \end{cases}$

(B4) $$U_h^k(x)=-\int_\Gamma\left[g_h^k(y)E(x,y)-\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)\right]dS_y$$

$$-\frac{2}{\tau}\int_{\Omega'}\sum_{j=1}^{k}(-1)^j[2U_h^{k-j}(y)+\tau f^{k+1/2-j}(y)]E(x,y)dy$$

$$+(-1)^k\left[u_0(x)-\int_\Gamma\left(\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)\right.\right.$$

$$\left.\left.-\frac{\partial u_0}{\partial n}(y)E(x,y)\right)dS_y\right] \quad (x\in\Omega'); \tag{1.10a}$$

或

$$U_h^k(x)=-U_h^{k-1}(x)-\int_\Gamma\left\{(g_h^k(y)+g_h^{k-1}(y))E(x,y)\right.$$

$$\left.-(\varphi^k(y)+\varphi^{k-1}(y))\frac{\partial}{\partial n_y}E(x,y)\right\}dS_y+\frac{2}{\tau}\int_{\Omega'}[2U_h^{k-1}(y)$$

$$+\tau f^{k-1/2}(y)]E(x,y)dy \quad (x\in\Omega',\ k=1,2,\cdots) \tag{1.10b}$$

逼近格式A和逼近格式B给出了求解(P)的一类Crank-Nicolson型的差分-边界有界元耦合方法，其中只有(A3)和(B3)是在曲面Γ上隐式求解，其余均是显式计算．又注意到对每一固定点x，当y远离x时，积分核$E(x,y)$迅速趋于零，所以在实际计算中，积分只需在点x的某一个有限邻域内计算．逼近格式A和B是适于计算的．

二、近似解的存在唯一性

在[1]中，证明了下述引理。

引理2.1 存在常数M与$\alpha>0$，使

$$a(g,q)\leqslant M\|g\|_{H^{-1/2}(\Gamma)}\|q\|_{H^{-1/2}(\Gamma)} \quad (\forall g,q\in H^{-\frac{1}{2}}(\Gamma)) \tag{2.1}$$

$$a(g,g)\geqslant\alpha\sqrt{\tau}\|g\|^2_{H^{-1/2}(\Gamma)} \quad (\forall g\in H^{-\frac{1}{2}}(\Gamma)) \tag{2.2}$$

由引理2.1与Lax-Milgram定理即得

定理2.1 变分问题(A3)和(B3)分别有唯一解．

下面两节考虑误差估计．设有限元空间族V_h具有逼近性质：对某个实数$r\geqslant0$，存在常数$C>0$，使对任何$q\in W^{r+1,p}(\Gamma)$ $(1\leqslant p\leqslant\infty)$

$$\inf_{q_h\in V}\|q-q_h\|_{L^p(\Gamma)}\leqslant Ch^s\|q\|_{W^{s,p}(\Gamma)} \quad (0\leqslant S\leqslant r+1) \tag{2.3}$$

引理2.2 设V_h具有逼近性质(2.3)，则

$$\inf_{q_h\in V_h}\|q-q_h\|_{H^{-s}(\Gamma)}\leqslant Ch^{r+1+s}\|q\|_{H^{r+1}(\Gamma)} \quad (0\leqslant s\leqslant r+1) \tag{2.4}$$

三、逼近格式A的误差估计

在本节及其后，我们用C表示不依赖于h的抽象常数，在不同处可取不同的值。

引入辅助函数：

$$u_h^0(x)=u_0(0)\quad (x\in\Omega);\quad u_h^0(x)=0\quad (x\in\Omega')\tag{3.1a}$$

$$\begin{aligned}\tilde{\varphi}^k(x)=&\frac{\varphi^k(x)}{2}+\int_\Gamma\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)dS_y+\frac{2}{\tau}\int_\Omega\sum_{j=1}^k(-1)^j[2u^{k-j}(y)\\&+\tau(f^{k+\frac{1}{2}-j}(y)+Q^{k+1-j}(y))]E(x,y)dy+(-1)^{k+1}\Big\{\frac{\varphi^0(x)}{2}\\&+\int_\Gamma\Big[\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)-\frac{\partial u_0}{\partial n}(y)E(x,y)\Big]dS_y\Big\}\quad (x\in\Gamma)\end{aligned}\tag{3.1b}$$

$$\begin{cases}\tilde{g}_h^k\in V_h,\ \text{满足}\\ a(\tilde{g}_h^k,q_h)=\langle\tilde{\varphi}^k,q_h\rangle\quad(\forall q_h\in V_h)\end{cases}\tag{3.1c}$$

$$\begin{aligned}u_h^k(x)=&\int_\Gamma\Big[\tilde{g}_h^k(y)E(x,y)-\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)\Big]dS_y\\&-\frac{2}{\tau}\int_\Omega\sum_{j=1}^k(-1)^j[2u^{k-j}(y)+\tau(f^{k+1/2-j}(y)+Q^{k+1-j}(y))]E(x,y)dy\\&+(-1)^k\Big[u_0(x)+\int_\Gamma\Big(\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)\\&-\frac{\partial u_0}{\partial n}(y)E(x,y)\Big)dS_y\Big]\quad(x\in\Omega,\ k=1,2,3,\cdots)\end{aligned}\tag{3.1d}$$

引理3.1 令 $g^k=\frac{\partial u^k}{\partial n}$，$\partial_t g^k=g^k-g^{k-1}$，$\partial_t^2 g^k=\partial_t g^{k+1}-\partial_t g^k$，则对任意$1\leqslant k\leqslant T/\tau$，

$$a(g^k-\tilde{g}_h^k,g^k-\tilde{g}_h^k)\leqslant Ch^{2r+3}\|g^k\|^2_{H^{r+1}(\Gamma)}\tag{3.2}$$

$$a(\partial_t(g^k-\tilde{g}_h^k),\partial_t(g^k-\tilde{g}_h^k))\leqslant C\tau h^{2r+3}\int_{t_{k-1}}^{t_k}\Big\|\frac{\partial g}{\partial t}\Big\|^2_{H^{r+1}(\Gamma)}dt\tag{3.3}$$

$$a(\partial_t^2(g^k-\tilde{g}_h^k),\ \partial_t^2(g^k-\tilde{g}_h^k))\leqslant C\tau^3h^{2r+3}\int_{t_{k-1}}^{t_{k+1}}\Big\|\frac{\partial^2 g}{\partial t^2}\Big\|^2_{H^{r+1}(\Gamma)}dt\tag{3.4}$$

证明 注意到u^k可以写成

$$\begin{aligned}u^k(x)=&\int_\Gamma\Big[g^k(y)E(x,y)-\varphi^k(y)\frac{\partial}{\partial n_y}E(x,y)\Big]dS_y\\&-\frac{2}{\tau}\int_\Omega\sum_{j=1}^k(-1)^j[2u^{k-j}(y)+\tau(f^{k+\frac{1}{2}-j}(y)+Q^{k+1-j}(y))]E(x,y)dy\\&+(-1)^k\Big[u_0(x)+\int_\Gamma\Big(\varphi^0(y)\frac{\partial}{\partial n_y}E(x,y)-\frac{\partial u_0}{\partial n}(y)E(x,y)\Big)dS_y\Big]\end{aligned}\tag{3.5}$$

于是，g^k满足

$$a(g^k,q)=\langle\tilde{\varphi}^k,q\rangle,\quad (\forall q\in H^{-\frac{1}{2}}(\Gamma)) \tag{3.6}$$

由(3.1)和(3.6)得

$$a(\partial_t^j(g^k-\tilde{g}_h^k),\ q_h)=0\qquad (\forall q_h\in V_h,\ 0\leqslant j\leqslant 2) \tag{3.7}$$

由(3.7)、(2.1)、(2.2)和(2.4)即得(3.2)、(3.3)和(3.4)。证毕

令$\rho^k=u^k-u_h^k$，$\theta^k=u_h^k-U_h^k$，则

$$\rho^k(x)=\int_\Gamma(g^k(y)-\tilde{g}_h^k(y))E(x,y)dS_y\qquad (x\in\Omega) \tag{3.8a}$$

$$\theta^k(x)=\int_\Gamma(\tilde{g}_h^k(y)-g_h^k(y))E(x,y)dS_y-\frac{2}{\tau}\int_\Omega\sum_{j=1}^k(-1)^j[2(u^{k-j}(y)-U_h^{k-j}(y))+\tau Q^{k-j+1}(y)]E(x,y)dy\qquad (x\in\Omega) \tag{3.8b}$$

将$\rho^k(x)$和$\theta^k(x)$分别按(3.8a)和(3.8b)延拓到Ω'上。

引理3.2 对任意$1\leqslant k\leqslant T/\tau$.

$$\|\rho^k\|^2_{H^0(R^3)}+\tau\|\nabla\rho^k\|^2_{H^0(R^3)}\leqslant C\tau h^{2r+3}\|g^k\|^2_{H^{r+1}(\Gamma)} \tag{3.9}$$

$$\|\partial_t\rho^k\|^2_{H^0(R^3)}+\tau\|\nabla\partial_t\rho^k\|^2_{H^0(R^3)}\leqslant C\tau^2h^{2r+3}\int_{t_{k-1}}^{t_k}\left\|\frac{\partial g}{\partial t}\right\|^2_{H^{r+1}(\Gamma)}dt \tag{3.10}$$

$$\|\partial_t^2\rho^k\|^2_{H^0(R^3)}+\tau\|\nabla\partial_t^2\rho^k\|^2_{H^0(R^3)}\leqslant C\tau^4h^{2r+3}\int_{t_{k-1}}^{t_{k+1}}\left\|\frac{\partial^2 g}{\partial t^2}\right\|^2_{H^{r+1}(\Gamma)}dt \tag{3.11}$$

证明 由(3.8a)得

$$\begin{cases}\partial_t^j\rho^k-\dfrac{\tau}{2}\Delta\partial_t^j\rho^k=0\qquad (x\in\Omega\cup\Omega') & (3.12a)\\[2mm] \partial_t^j\rho^k=\displaystyle\int_\Gamma\partial_t^j(g^k-\tilde{g}_h^k)(y)E(x,y)dS_y\qquad (x\in\Gamma) & (3.12b)\\[2mm] \dfrac{\partial}{\partial n}(\partial_t^j\rho^k)|_{\rm int}-\dfrac{\partial}{\partial n}(\partial_t^j\rho^k)|_{\rm ext}=\partial_t^j(g^k-\tilde{g}_h^k)\qquad (x\in\Gamma) & (3.12c)\end{cases}$$

其中 $v|_{\rm int}$与$v|_{\rm ext}$分别表示函数v在Γ处沿法线的内与外极限值。(3.12)导出

$$\int_{R^3}\left[|\partial_t^j\rho^k|^2+\frac{\tau}{2}|\nabla\partial_t^j\rho^k|^2\right]dx=\frac{\tau}{2}a(\partial_t^j(g^k-\tilde{g}_h^k),\partial_t^j(g^k-\tilde{g}_h^k))$$

由(3.2)~(3.4)即得(3.9)~(3.11)。证毕

引理3.3 对任意$1\leqslant k\leqslant T/\tau$,

$$\|\theta^k\|^2_{H^0(\Omega)}+\sum_{j=1}^k\|\hat{\theta}^{j-1/2}\|^2_{H^0(\Omega')}+\tau\sum_{j=1}^k\|\nabla\hat{\theta}^{j-1/2}\|^2_{H^0(R^3)}$$
$$\leqslant C\left\{h^{2r+3}\int_0^{t_k}\|g\|^2_{H^{r+1}(\Gamma)}dt+\tau^4\int_0^{t_k}\left(\left\|\frac{\partial^2u}{\partial t^2}\right\|^2_{H^2(\Omega)}+\left\|\frac{\partial^3u}{\partial t^3}\right\|^2_{H^0(\Omega)}+\left\|\frac{\partial^2g}{\partial t^2}\right\|^2_{H^{-1/2}(\Gamma)}\right)dt\right\} \tag{3.13}$$

$$\sum_{j=1}^k\|\partial_t\theta^j\|^2_{H^0(\Omega)}+\|\theta^k\|^2_{H^0(\Omega')}+\tau\|\nabla\theta^k\|^2_{H^0(R^3)}$$

$$\leqslant C\tau\Big\{h^{2r+3}\int_0^{t_k}\Big(\|g\|^2_{H^{r+1}(\Gamma)}+\Big\|\frac{\partial g}{\partial t}\Big\|^2_{H^{r+1}(\Gamma)}\Big)dt$$

$$+\tau^4\int_0^{t_k}\Big(\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|^2_{H^2(\Omega)}+\Big\|\frac{\partial^3 u}{\partial t^3}\Big\|^2_{H^0(\Omega)}+\Big\|\frac{\partial^2 g}{\partial t^2}\Big\|^2_{H^{-1/2}(\Gamma)}\Big)dt\Big\} \tag{3.14}$$

$$\sum_{j=2}^{k}\|\partial_t^2\theta^j\|^2_{H^0(\Omega)}+\|\partial_t\theta^k\|^2_{H^0(\Omega')}+\tau\|\nabla\partial_t\theta^k\|^2_{H^0(R^3)}$$

$$\leqslant C\tau^3\Big\{h^{2r+3}\int_0^{T}\Big(\|g\|^2_{H^{r+1}(\Gamma)}+\Big\|\frac{\partial g}{\partial t}\Big\|^2_{H^{r+1}(\Gamma)}+\Big\|\frac{\partial^2 g}{\partial t^2}\Big\|^2_{H^{r+1}(\Gamma)}\Big)dt$$

$$+\tau^4\int_0^{T}\Big(\Big\|\frac{\partial^3 u}{\partial t^3}\Big\|^2_{H^2(\Omega)}+\Big\|\frac{\partial^4 u}{\partial t^2}\Big\|^2_{H^0(\Omega)}+\Big\|\frac{\partial^3 g}{\partial t^3}\Big\|^2_{H^{-1/2}(\Gamma)}\Big)dt\Big\} \tag{3.15}$$

证明　由(A3)与(3.1c)得

$$a(\partial_t^j(\tilde g_h^k-g_h^k),q_h)=\langle\partial_t^j(\tilde\varphi^k-\tilde\varphi_h^k),q_h\rangle\qquad(\forall q_h\in V_h,0\leqslant j\leqslant 2) \tag{3.16}$$

由(3.8b)得

$$\theta^k-\frac{\tau}{2}\Delta(\theta^k+\theta^{k-1})=\theta^{k-1}+\rho^{k-1}+\tau Q^k\qquad(x\in\Omega) \tag{3.17a}$$

$$\theta^k+\theta^{k-1}-\frac{\tau}{2}\Delta(\theta^k+\theta^{k-1})=0\qquad(x\in\Omega') \tag{3.17b}$$

$$\theta^k=\int_\Gamma(\tilde g_h^k(y)-g_h^k(y))E(x,y)dS_y-\tilde\varphi^k+\tilde\varphi_h^k\qquad(x\in\Gamma) \tag{3.17c}$$

$$\frac{\partial}{\partial n}(\theta^k+\theta^{k-1})\Big|_{\rm int}-\frac{\partial}{\partial n}(\theta^k+\theta^{k-1})\Big|_{\rm ext}=\tilde g_h^k+\tilde g_h^{k-1}-(g_h^k+g_h^{k-1})\qquad(x\in\Gamma) \tag{3.17d}$$

(3.17)导致

$$\frac{1}{2}\|\theta^k\|^2_{H^0(\Omega)}+2\|\hat\theta^{k-1/2}\|^2_{H^0(\Omega')}+\tau\|\nabla\hat\theta^{k-1/2}\|^2_{H^0(R^3)}$$

$$=\frac{1}{2}\|\theta^{k-1}\|^2_{H^0(\Omega)}+\int_\Omega(\rho^{k-1}+\tau Q^k)\hat\theta^{k-1/2}dx$$

$$+\frac{\tau}{2}[a((\widehat{\tilde g_h-g_h})^{k-1/2},(\widehat{\tilde g_h-g_h})^{k-1/2})-\langle(\widehat{\tilde\varphi-\tilde\varphi_h})^{k-1/2},(\widehat{\tilde g_h-g_h})^{k-1/2}\rangle]$$

由(3.16)知方括号内项为零．注意到

$$\int_\Omega\rho^{k-1}\hat\theta^{k-1/2}dx=\int_{R^3}\rho^{k-1}\hat\theta^{k-1/2}dx-\int_{\Omega'}\rho^{k-1}\hat\theta^{k-1/2}dx$$

$$=\frac{\tau}{2}\int_{R^3}\Delta\rho^{k-1}\hat\theta^{k-1/2}dx-\int_{\Omega'}\rho^{k-1}\hat\theta^{k-1/2}dx$$

$$=\frac{\tau}{2}[\langle g^{k-1}-\tilde g_h^{k-1},\hat\theta^{k-1/2}\rangle-\int_{R^3}\nabla\rho^{k-1}\cdot\nabla\hat\theta^{k-1/2}dx]-\int_{\Omega'}\rho^{k-1}\hat\theta^{k-1/2}dx$$

$$\leqslant\frac{\tau}{2}\|\nabla\hat\theta^{k-1/2}\|^2_{H^0(R^3)}+\frac{\tau}{4T}(\|\hat\theta^k\|^2_{H^0(\Omega)}+\|\hat\theta^{k-1/2}\|^2_{H^0(\Omega')})$$

$$+\|\hat\theta^{k-1/2}\|^2_{H^0(\Omega')}+\frac{1}{4}\|\rho^{k-1}\|^2_{H^0(\Omega')}+C\tau\{\|\nabla\rho^{k-1}\|^2_{H^0(R^3)}$$

$$+\|\theta^{k-1}\|^2_{H^0(\Omega)}+\inf_{q_h\in V_h}\|g^{k-1}-q_h\|^2_{H^{-1/2}(\Gamma)}\}$$

上式利用了$\langle\hat\theta^{k-1/2},q_h\rangle=a((\widehat{\tilde g_h-g_h})^{k-1/2},q_h)-\langle(\widehat{\tilde\varphi-\tilde\varphi_h})^{k-1/2},q_h\rangle=0\quad(\forall q_h\in V_h)$，和迹定理：$H'(\Omega)\nearrow H^{1/2}(\Gamma)$。

$$\tau\int_\Omega Q^k\hat\theta^{k-1/2}dx\leqslant\frac{\tau}{4T}\|\theta^k\|^2_{H^0(\Omega)}+C\tau\{\|\theta^{k-1}\|^2_{H^0(\Omega)}+\|Q^k\|^2_{H^0(\Omega)}\}$$

于是

$$\|\theta^k\|^2_{H^0(\Omega)}+\|\hat\theta^{k-1/2}\|^2_{H^0(\Omega')}+\tau\|\nabla\hat\theta^{k-1/2}\|^2_{H^0(R^3)}\leqslant\|\theta^{k-1}\|^2_{H^0(\Omega)}$$
$$+\frac{\tau}{2T}\|\theta^k\|^2_{H^0(\Omega)}+C\tau\Big\{\|\theta^{k-1}\|^2_{H^0(\Omega)}+h^{2r+3}\|g^k\|^2_{H^{r+1}(\Gamma)}$$
$$+\tau^3\int_{t_{k-1}}^{t_k}\Big(\Big\|\frac{\partial^2u}{\partial t^2}\Big\|^2_{H^2(\Omega)}+\Big\|\frac{\partial^3u}{\partial t^3}\Big\|^2_{H^0(\Omega)}\Big)dt\Big\}$$

关于k求和并注意到$\theta^0\equiv0$得

$$\|\theta^k\|^2_{H^0(\Omega)}+\sum_{j=1}^k\|\hat\theta^{j-1/2}\|^2_{H^0(\Omega')}+\tau\sum_{j=1}^k\|\nabla\hat\theta^{j-1/2}\|^2_{H^0(R^3)}$$
$$\leqslant C\Big\{\tau\sum_{j=1}^{k-1}\|\theta^j\|^2_{H^0(\Omega)}+h^{2r+3}\int_0^{t_k}\|g\|^2_{H^{r+1}(\Gamma)}dt$$
$$+\tau^4\int_0^{t_k}\Big(\Big\|\frac{\partial^2u}{\partial t^2}\Big\|^2_{H^2(\Omega)}+\Big\|\frac{\partial^3u}{\partial t^3}\Big\|^2_{H^0(\Omega)}\Big)dt\Big\}$$

利用离散型Bellman不等式即得(3.13)。

由(3.17)又得

$$\|\partial_t\theta^k\|^2_{H^0(\Omega)}+\|\theta^k\|^2_{H^0(\Omega')}+\frac{\tau}{2}\|\nabla\theta^k\|^2_{H^0(R^3)}$$
$$=\frac{\tau}{2}\|\nabla\theta^{k-1}\|^2_{H^0(R^3)}+\|\theta^{k-1}\|^2_{H^0(\Omega')}+\int_\Omega(\rho^{k-1}+\tau Q^k)\partial_t\theta^k dx$$

关于k求和得

$$\sum_{j=1}^k\|\partial_t\theta^j\|^2_{H^0(\Omega)}+\|\theta^k\|^2_{H^0(\Omega')}+\frac{\tau}{2}\|\nabla\theta^k\|^2_{H^0(R^3)}$$
$$\leqslant\int_\Omega\rho^{k-1}\theta^k dx-\sum_{j=1}^{k-1}\int_\Omega\partial_t\rho^j\theta^j dx+\frac{1}{2}\sum_{j=1}^k\|\partial_t\theta^j\|^2_{H^0(\Omega)}$$
$$+\tau^2\sum_{j=1}^k\|Q^j\|^2_{H^0(\Omega)}$$

注意到

$$\int_\Omega\rho^{k-1}\theta^k dx=\int_{R^3}\rho^{k-1}\theta^k dx-\int_{\Omega'}\rho^{k-1}\theta^k dx$$
$$=\frac{\tau}{2}\Big[\langle g^{k-1}-\tilde g_h^{k-1},\theta^k\rangle-\int_{R^3}\nabla\rho^{k-1}\cdot\nabla\theta^k dx\Big]-\int_{\Omega'}\rho^{k-1}\theta^k dx$$

$$\leqslant\frac{1}{2}\|\theta^k\|^2_{H^0(\Omega')}+\frac{\tau}{4}\|\nabla\theta^k\|^2_{H^0(R^3)}+\|\rho^{k-1}\|^2_{H^0(\Omega')}$$

$$+C\tau\left\{\inf_{q_h\in V_h}\|g^{k-1}-q_h\|^2_{H^{-1/2}(\Gamma)}+\|\nabla\rho^{k-1}\|^2_{H^0(R^3)}+\|\theta^k\|^2_{H^0(\Omega)}\right\}$$

$$\int_\Omega\partial_t\rho^j\theta^j dx\leqslant\tau^2\|\theta^j\|^2_{H^1(R^3)}+\tau\|\theta^j\|^2_{H^0(\Omega')}+\frac{1}{\tau}\|\partial_t\rho^j\|^2_{H^0(\Omega')}$$

$$+C\left\{\|\nabla\partial_t\rho^j\|^2_{H^0(R^3)}+\inf_{q_h\in V_h}\|\partial_t g^j-q_h\|^2_{H^{-1/2}(\Gamma)}\right\}$$

于是

$$\sum_{j=1}^{k}\|\partial_t\theta^j\|^2_{H^0(\Omega)}+\|\theta^k\|^2_{H^0(\Omega')}+\tau\|\nabla\theta^k\|^2_{H^0(R^3)}\leqslant\tau\sum_{j=1}^{k-1}\Big(\|\theta^j\|^2_{H^0(\Omega')}$$

$$+\tau\|\nabla\theta^j\|^2_{H^0(R^3)}\Big)+C\tau\left\{h^{2r+3}\int_0^{t_k}\left(\|g\|^2_{H^{r+1}(\Gamma)}+\left\|\frac{\partial g}{\partial t}\right\|^2_{H^{r+1}(\Gamma)}\right)dt\right.$$

$$\left.+\tau^4\int_0^{t_k}\left(\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{H^2(\Omega)}+\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{H^0(\Omega)}\right)dt\right\}$$

利用离散型Bellman不等式即得(3.14)。

由(3.17)得

$$\begin{cases}\partial_t\theta^k-\partial_t\theta^{k-1}-\dfrac{\tau}{2}\Delta(\partial_t\theta^k+\partial_t\theta^{k-1})=\partial_t\rho^{k-1}+\tau\partial_t Q^k \qquad (x\in\Omega) & (3.18a)\\ \partial_t\theta^k+\partial_t\theta^{k-1}-\dfrac{\tau}{2}\Delta(\partial_t\theta^k+\partial_t\theta^{k-1})=0 \qquad (x\in\Omega') & (3.18b)\end{cases}$$

(3.18)与(3.16)导出

$$\|\partial_t^2\theta^{k-1}\|^2_{H^0(\Omega)}+\|\partial_t\theta^k\|^2_{H^0(\Omega')}+\frac{\tau}{2}\|\nabla\partial_t\theta^k\|^2_{H^0(R^3)}$$

$$=\|\partial_t\theta^{k-1}\|^2_{H^0(\Omega')}+\frac{\tau}{2}\|\nabla\partial_t\theta^{k-1}\|^2_{H^0(R^3)}+\int_\Omega(\partial_t\rho^k+\tau\partial_t Q^k)\partial_t^2\theta^{k-1}dx$$

关于k求和得

$$\sum_{j=1}^{k-1}\|\partial_t^2\theta^j\|^2_{H^0(\Omega)}+\|\partial_t\theta^k\|^2_{H^0(\Omega')}+\frac{\tau}{2}\|\nabla\partial_t\theta^k\|^2_{H^0(R^3)}$$

$$=\|\partial_t\theta^1\|^2_{H^0(\Omega')}+\frac{\tau}{2}\|\nabla\partial_t\theta^1\|^2_{H^0(R^3)}+\int_\Omega\partial_t\rho^{k-1}\partial_t\theta^k dx$$

$$-\sum_{j=1}^{k-1}\int_\Omega\partial_t^2\rho^{j-1}\theta^j dx-\int_\Omega\partial_t\rho^1\theta^1 dx+\tau\sum_{j=2}^{k}\int_\Omega\partial_t Q^j\partial_t^2\theta^{j-1}dx$$

注意到

$$\|\partial_t\theta^1\|^2_{H^0(R^3)}+\frac{\tau}{2}\|\nabla\partial_t\theta^1\|^2_{H^0(R^3)}=\|\theta^1\|^2_{H^0(R^3)}+\frac{\tau}{2}\|\nabla\theta^1\|^2_{H^0(R^3)}$$

$$=\tau\int_\Omega Q^1\theta^1 dx$$

于是

$$\|\partial_t\theta^1\|^2_{H^0(R^3)}+\tau\|\nabla\partial_t\theta^1\|^2_{H^0(R^3)}\leqslant C\tau^2\|Q^1\|^2_{H^0(\Omega)}$$

$$\leqslant C\tau^6\int_0^{t_1}\left(\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{H^2(\Omega)}+\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{H^0(\Omega)}\right)dt$$

$$\leqslant C\tau^{7}\int_{0}^{T}\left(\left\|\frac{\partial^{3}u}{\partial t^{3}}\right\|_{H^{2}(\Omega)}^{2}+\left\|\frac{\partial^{4}u}{\partial t^{4}}\right\|_{H^{0}(\Omega)}^{2}\right)dt \tag{3.19a}$$

$$\int_{\Omega}\partial_{t}\rho^{1}\theta^{1}dx\leqslant C\tau^{3}\left\{h^{2r+3}\int_{0}^{T}\left(\left\|\frac{\partial g}{\partial t}\right\|_{H^{r+1}(\Gamma)}^{2}+\left\|\frac{\partial^{2}g}{\partial t^{2}}\right\|_{H^{r+1}(\Gamma)}^{2}\right)dt\right.$$
$$\left.+\tau^{4}\int_{0}^{T}\left(\left\|\frac{\partial^{2}u}{\partial t^{2}}\right\|_{H^{2}(\Omega)}^{2}+\left\|\frac{\partial^{3}u}{\partial t^{3}}\right\|_{H^{2}(\Omega)}^{2}+\left\|\frac{\partial^{4}u}{\partial t^{4}}\right\|_{H^{0}(\Omega)}^{2}\right)dt\right\} \tag{3.19b}$$

于是，类似地推导可得

$$\sum_{j=1}^{k-1}\|\partial_{t}^{2}\theta^{j}\|_{H^{0}(\Omega)}^{2}+\|\partial_{t}\theta^{k}\|_{H^{0}(\Omega')}^{2}+\tau\|\nabla\partial_{t}\theta^{k}\|_{H^{0}(R^{3})}^{2}$$
$$\leqslant\tau\sum_{j=1}^{k-1}\left(\|\partial_{t}\theta^{j}\|_{H^{0}(\Omega')}^{2}+\tau\|\nabla\partial_{t}\theta^{j}\|_{H^{0}(R^{3})}^{2}\right)$$
$$+C\tau^{3}\left\{h^{2r+3}\int_{0}^{T}\left(\|g\|_{H^{r+1}(\Gamma)}^{2}+\left\|\frac{\partial g}{\partial t}\right\|_{H^{r+1}(\Omega)}^{2}+\left\|\frac{\partial^{2}g}{\partial t^{2}}\right\|_{H^{r+1}(\Gamma)}\right)dt\right.$$
$$\left.+\tau^{4}\int_{0}^{T}\left(\left\|\frac{\partial^{3}u}{\partial t^{3}}\right\|_{H^{2}(\Omega)}^{2}+\left\|\frac{\partial^{4}u}{\partial t^{4}}\right\|_{H^{0}(\Omega)}^{2}\right)dt\right\} \tag{3.19c}$$

利用离散型Beuman不等式即得(3.15). 证毕.

引理3.4 对任意$1\leqslant k\leqslant T/\tau$,

$$a(\tilde{g}_{h}^{k}-g_{h}^{k},\ \tilde{g}_{h}^{k}-g_{h}^{k})\leqslant C\left\{h^{2r+3}\int_{0}^{T}\left(\|g\|_{H^{r+1}(\Gamma)}^{2}\right.\right.$$
$$\left.+\left\|\frac{\partial g}{\partial t}\right\|_{H^{r+1}(\Gamma)}^{2}+\left\|\frac{\partial^{2}g}{\partial t^{2}}\right\|_{H^{r+1}(\Gamma)}^{2}\right)dt+\tau^{4}\int_{0}^{T}\left(\left\|\frac{\partial^{3}u}{\partial t^{3}}\right\|_{H^{2}(\Omega)}^{2}\right.$$
$$\left.\left.+\left\|\frac{\partial^{4}u}{\partial t^{4}}\right\|_{H^{0}(\Omega)}^{2}\right)dt\right\} \tag{3.20}$$

$$a(\hat{g}_{h}^{k-1/2}-g_{h}^{k-1/2},\ \hat{g}^{k-1/2}-g_{h}^{k-1/2})\leqslant C\left\{h^{2r+3}\int_{0}^{t_{k}}\left(\|g\|_{H^{r+1}(\Gamma)}^{2}\right.\right.$$
$$\left.\left.+\left\|\frac{\partial g}{\partial t}\right\|_{H^{r+1}(\Gamma)}^{2}\right)dt+\tau^{4}\int_{0}^{t_{k}}\left(\left\|\frac{\partial^{2}u}{\partial t^{2}}\right\|_{H^{2}(\Omega)}^{2}+\left\|\frac{\partial^{3}u}{\partial t^{3}}\right\|_{H^{0}(\Omega)}^{2}\right)dt\right\} \tag{3.21}$$

证明 注意到

$$a(\tilde{g}_{h}^{k}-g_{h}^{k},\tilde{g}_{h}^{k}-g_{h}^{k})=\langle\tilde{\varphi}^{k}-\tilde{\varphi}_{h}^{k},\tilde{g}_{h}^{k}-g_{h}^{k}\rangle$$
$$=\frac{2}{\tau}\left\langle\int_{\Omega}\sum_{j=1}^{k}(-1)^{j}[2(u^{k-j}(y)-U_{h}^{k-j}(y))+\tau Q^{k+1-j}(y)]E(x,y)dy,\right.$$
$$\left.\tilde{g}_{h}^{k}-g_{h}^{k}\right\rangle$$
$$=\frac{2}{\tau}\int_{\Omega}\sum_{j=1}^{k}(-1)^{j}[2(\rho^{k-j}(y)+\theta^{k-j}(y))+\tau Q^{k+1-j}(y)]\int_{\Gamma}(\tilde{g}_{h}^{k}$$
$$-g_{h}^{k})(y)E(x,y)dS_{x}dy$$
$$\leqslant\frac{4}{\tau}\left\{\sum_{j=1}^{k}\|\partial_{t}\rho^{j}\|_{H^{0}(\Omega)}+\tau\sum_{j=1}^{k}\|\partial_{t}Q^{j}\|_{H^{0}(\Omega)}\right\}\|w^{k}\|_{H^{0}(\Omega)}$$
$$+\frac{4}{\tau}\sum_{j=1}^{k}\int_{\Omega}(-1)^{j}\theta^{k-j}w^{k}dy$$

其中 $w^k=\int_\Gamma(\tilde{g}_h^k(y)-g(y)),E(x,y)dS_y$。注意到

$$\|\rho^{k-1}\|^2_{H^0(\Omega)}\leqslant C\tau h^{2r+3}\|g^k\|^2_{H^{r+1}(\Gamma)}$$

$$\sum_{j=1}^k\|\partial_t\rho^j\|_{H^0(\Omega)}\leqslant C\tau^{1/2}h^{r+3/2}\left[\int_0^{t_k}\left\|\frac{\partial g}{\partial t}\right\|^2_{H^{r+1}(\Gamma)}dt\right]^{1/2}$$

$$\|w^k\|^2_{H^0(R^3)}+\frac{\tau}{2}\|\nabla w^k\|^2_{H^0(R^3)}=\frac{\tau}{2}a(\tilde{g}_h^k-g_h^k,\tilde{g}_h^k-g_h^k)$$

$$\sum_{j=1}^k\int_\Omega(-1)^j\theta^{k-j}w^kdy=\sum_{j=1}^k(-1)^j\left[\int_{R^3}\theta^{k-j}w^kdy-\int_{\Omega'}\theta^{k-j}w^kdy\right]$$

$$=\sum_{j=1}^k(-1)^j\left[\frac{\tau}{2}\int_{R^3}\theta^{k-j}\Delta w^kdy-\int_{\Omega'}\theta^{k-j}w^kdy\right]$$

$$\leqslant C[\|w^k\|^2_{H^0(\Omega')}+\tau\|\nabla w^k\|^2_{H^0(R^3)}]^{1/2}\left[\frac{1}{\tau}\sum_{j=1}^k\|\partial_t\theta^j\|^2_{H^0(\Omega')}\right.$$

$$\left.+\sum_{j=1}^k\|\partial_t\nabla\theta^j\|^2_{H^0(R^3)}\right]^{1/2}$$

$$\leqslant C\tau a(\tilde{g}_h^k-g_h^k,\tilde{g}_h^k-g_h^k)^{1/2}\left[h^{2r+3}\int_0^T\left(\|g\|^2_{H^{r+1}(\Gamma)}\right.\right.$$

$$\left.+\left\|\frac{\partial g}{\partial t}\right\|^2_{H^{r+1}(\Gamma)}+\left\|\frac{\partial^2 g}{\partial t^2}\right\|^2_{H^{r+1}(\Gamma)}\right)dt+\tau^4\int_0^T\left(\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{H^2(\Omega)}\right.$$

$$\left.\left.+\left\|\frac{\partial^4 u}{\partial t^4}\right\|^2_{H^0(\Omega)}\right)dt\right]^{1/2}$$

综合上述估计即得(3.20)。又注意到

$$a((\widehat{\tilde{g}_h-g_h})^{k-1/2},(\widehat{\tilde{g}_h-g_h})^{k-1/2})=\langle(\widehat{\tilde{\varphi}-\tilde{\varphi}_h})^{k-1/2},(\widehat{\tilde{g}_h-g_h})^{k-1/2}\rangle$$

$$=\frac{2}{\tau}\left\langle\int_\Omega[2(u^{k-1}(y)-U_h^{k-1}(y))+\tau Q^k(y)]E(x,y)dy,(\widehat{\tilde{g}_h-g_h})^{k-1/2}\right\rangle$$

$$=\frac{2}{\tau}\int_\Omega[2(\rho^{k-1}(y)+\theta^{k-1}(y))+\tau Q^k(y)]$$

$$\cdot\int_\Gamma(\widehat{\tilde{g}_h-g_h})^{k-1/2}(y)E(x,y)dS_xdy$$

类似地论证即得(3.21)。证毕。

综合上述引理，即得到下述先验误差估计。

定理3.1 设u是精确解，$g=\partial u/\partial n$，U_h^k和g_h^k是由逼近格式A确定的近似解，则

$$\max_{1\leqslant k\leqslant T/\tau}\left\{\frac{1}{\tau}\sum_{j=1}^k\|\partial_t(u^j-U_h^j)\|^2_{H^0(\Omega)}+\|u^k-U_h^k\|^2_{H^1(\Omega)}\right.$$

$$+a(\dot{g}^{k-1/2}-\dot{g}_h^{k-1/2},\ \dot{g}^{k-1/2}-\dot{g}_h^{k-1/2})\Big\}\leqslant C\{h^{2r+3}\|g\|^2_{H^1(0,T;H^{r+1}(\Gamma))}$$

$$+\tau^4(\|u\|^2_{H^2(0,T;H^2(\Omega))}+\|u\|^2_{H^3(0,T;H^0(\Omega))})\}\qquad(3.22)$$

$$\max_{1\leqslant k\leqslant T/\tau}\Big\{\frac{1}{\tau^3}\sum_{j=1}^{k}\|\partial_t^2(u^j-U_h^j)\|^2_{H^0(\Omega)}+\frac{1}{\tau^2}\|\partial_t(u^k-U_h^k)\|^2_{H^1(\Omega)}$$

$$+a(g^k-g_h^k,g^k-g_h^k)\Big\}\leqslant C\{h^{2r+3}\|g\|^2_{H^2(0,T;H^{r+1}(\Gamma))}$$

$$+\tau^4(\|u\|^2_{H^3(0,T;H^2(\Omega))}+\|u\|^2_{H^4(0,T;H^0(\Omega))})\}\qquad(3.23)$$

考虑最大模误差估计.

引理3.5 令

$$v(x)=\int_\Gamma(g^k(y)-g_h^k(y))E(x,y)dS_y\qquad(x\in\bar{\Omega}),$$

则

$$\|v\|_{L^\infty(\bar{\Omega})}\leqslant C\tau^{-1/4}h^{-1/2}(1+|\ln h|^{1/2})\{h^{r+3/2}\|g\|_{H^2(0,T;H^{r+1}(\Gamma))}$$
$$+\tau^2(\|u\|_{H^2(0,T;H^2(\Omega))}+\|u\|_{H^3(0,T;H^0(\Omega))})\}\qquad(3.24)$$

引理3.5的证明类似于[1]中引理4.7的证明.

引理3.6 对任何$x\in\Omega$,

$$\|E(x,\cdot)\|_{L^2(\Omega)}\leqslant C\tau^{1/4},\qquad(3.25)$$

$$\|E(x,\cdot)\|_{L^{6/5}(\Omega)}\leqslant C\tau^{3/4}.\qquad(3.26)$$

定理3.2 设u是精确解，U_h^k是由逼近格式A确定的近似解，则

$$\max_{1\leqslant k\leqslant T/\tau}\|u^k-U_h^k\|_{L^\infty(\bar{\Omega})}\leqslant C\tau^{-1/4}(1+h^{-1/2}(1+|\ln h|^{1/2})\{h^{r+3/2}+\tau^2\}\qquad(3.27)$$

证明 注意到

$$|u^k(x)-U_h^k(x)|\leqslant\Big|\int_\Gamma(g^k(y)-g_h^k(y))E(x,y)dS_y\Big|$$

$$+\frac{2}{\tau}\Big\{\Big|\int_\Omega(u^{k-1}(y)-U_h^{k-1}(y))E(x,y)dy\Big|$$

$$+\Big|\int_\Omega(u^1(y)-U_h^1(y)E(x,y)dy\Big|$$

$$+\sum_{j=1}^{k-1}\Big|\int_\Omega\partial_t^2(u^j(y)-U_h^j(y))E(x,y)dy\Big|$$

$$+\tau\sum_{j=1}^{k-1}\Big|\int_\Omega\partial_tQ^j(y)E(x,y)dy\Big|\Big\}$$

$$\leqslant\Big\|\int_\Gamma(g^k(y)-g_h^k(y))E(\cdot,y)dS_y\Big\|_{L^\infty(\Omega)}$$

$$+C\tau^{-1}\Big\{(\|u^{k-1}-U_h^{k-1}\|_{H^1(\Omega)}+\|u^1-U_h^1\|_{H^1(\Omega)})\|E(x,\cdot)\|_{L^{6/5}(\Omega)}$$

$$+\tau^{-1/2}\Big[\sum_{j=1}^{k-1}\|\partial_t^2(u^j-U_h^j)\|^2_{H^0(\Omega)}\Big]^{1/2}\|E(x,\cdot)\|_{H^0(\Omega)}$$

$$+\tau^{1/2}\left[\sum_{j=1}^{k}\|\partial_t Q^j\|^2_{H^0(\Omega)}\right]^{1/2}\|E(x,\cdot)\|_{H^0(\Omega)}\Big\}$$

上式利用了嵌入性质：$H^1(\Omega)\hookrightarrow L^6(\Omega)$. 利用(3.22)、(3.23)、(3.24)、(3.25)和(3.26)即得(3.27). 证毕.

四、逼近格式B的误差估计

假定函数f和u_0在无穷远处具有下述渐近性态：对$0<m<\infty$，当$|x|\to\infty$时，

$$f(x)=O\left(\frac{1}{|x|^{m+3}}\right),\quad |u_0(x)|=O\left(\frac{1}{|x|^{m+3}}\right),$$

$$|\nabla u_0(x)|=O\left(\frac{1}{|x|^{m+3}}\right) \tag{4.1}$$

引理4.1 若f与u_0满足(4.1)，则对任意$1\leqslant k\leqslant T/\tau$，当$|x|\to\infty$时

$$|U_h^k(x)|=O\left(\frac{1}{|x|^{m+3}}\right),\quad |\nabla U_h^k(x)|=O\left(\frac{1}{|x|^{m+3}}\right) \tag{4.2}$$

证明 由(1.8b)得

$$|U_h^k(x)\leqslant|U_h^{k-1}(x)|+2\left|\int_\Gamma\left[\vartheta_h^{k-1/2}(y)E(x,y)-\hat\varphi^{k-1/2}(y)\frac{\partial}{\partial n_y}E(x,y)\right]dS_y\right|$$

$$+\frac{2}{\tau}\left|\int_{\Omega'}[2U_h^{k-1}(y)+\tau f^{k-1/2}(y)]E(x,y)dy\right| \tag{4.3}$$

当$|x|\geqslant\max\limits_{y\in\Gamma}|y|/2$时，$|x-y|\geqslant|x|/2$，从而

$$\left|\int_\Gamma\vartheta_h^{k-1/2}(y)E(x,y)dS_y\right|\leqslant C\|\vartheta_h^{k-1/2}\|_{L^\infty(\Gamma)}\exp\left[-\frac{1}{2}\sqrt{\frac{2}{\tau}}|x|\right]$$

$$\left|\int_\Gamma\hat\varphi^{k-1/2}(y)\frac{\partial}{\partial n_y}E(x,y)dS_y\right|\leqslant C\tau^{-1/2}\|\hat\varphi^{k-1/2}\|_{L^\infty(\Gamma)}\exp\left[-\frac{1}{2}\sqrt{\frac{2}{\tau}}|x|\right]$$

$$\left|\int_{\Omega'}f^{k-1/2}(y)E(x,y)dy\right|\leqslant\int_{\Omega'\cap\{|y|\leqslant|x|/2\}}|f^{k-1/2}(y)|E(x,y)dy$$

$$+\int_{\Omega'\cap\{|y|>|x|/2\}}|f^{k-1/2}(y)|E(x,y)dy$$

$$\leqslant\Big\{\|f^{k-1/2}\|_{H^0(\Omega)}\exp\left[-\frac{1}{2}\sqrt{\frac{2}{\tau}}|x|\right]+\sup_{|y|\geqslant|x|/2}|f^{k-1/2}(y)|$$

$$\cdot\int_{\Omega'}E(x,y)dy\Big\}\leqslant C\frac{1}{|x|^{m+3}},$$

利用递推关系(4.3)和上述估计即得(4.2). 证毕.

利用近似解在无穷远处的渐近性态(4.2)，类似于第三节的论证，即得到先验误差估计.

定理4.1 设u是精确解，U_h^k是由逼近格式B确定的近似解，则

$$\max_{1\leqslant k\leqslant T/\tau}\Big\{\frac{1}{\tau}\sum_{j=1}^{k}\|\partial_t(u^j-U_h^j)\|^2_{H^0(\Omega')}+\|u^k-U_h^k\|^2_{H^1(\Omega')}$$

$$+a(g^{k-1/2}-g_h^{k-1/2},\ g^{k-1/2}-g_h^{k-1/2})\Big\}\leqslant C\{h^{2r+3}\|g\|^2_{H^1(0,T;H^{r+1}(\Gamma))}$$

$$+\tau^4(\|u\|^2_{H^2(0,T;H^2(\Omega'))}+\|u\|^2_{H^3(0,T;H^0(\Omega'))})\} \tag{4.4}$$

$$\max_{1\leqslant k\leqslant T/\tau}\Big\{\frac{1}{\tau^3}\sum_{j=1}^{k-1}\|\partial_t^2(u^j-U_h^j)\|^2_{H^0(\Omega')}+\frac{1}{\tau^2}\|\partial_t(u^k-U_h^k)\|^2_{H^1(\Omega')}$$

$$+a(g^k-g_h^k,g^k-g_h^k)\Big\}\leqslant C\{h^{2r+3}\|g\|^2_{H^2(0,T;H^{r+1}(\Gamma))}$$

$$+\tau^4(\|u\|^2_{H^3(0,T;H^2(\Omega'))}+\|u\|^2_{H^4(0,T;H^0(\Omega'))})\} \tag{4.5}$$

$$\max_{1\leqslant k\leqslant T/\tau}\|u^k-U_h^k\|_{L^\infty(\Omega')}\leqslant C\tau^{-1/4}(1+h^{-1/2}(1+|\ln h|^{1/2}))\{h^{r+3/2}+\tau^2\} \tag{4.6}$$

对于二维问题，我们已另文研究，并得到类似的逼近格式与误差估计。

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A Coupling Method of Difference with High Order Accuracy and Boundary Integral Equation for Evolutionary Equation and Its Error Estimates

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Abstract

In the present paper, a new numerical method for solving initial-boundary value problems of evolutionary equations is proposed and studied, combining difference method with high accuracy with boundary integral equation method. The numerical approximate schemes for both problems on a bounded or unbounded domain in R^3 are proposed and their prior error estimates are obtained.

Key words difference with high order accuracy, boundary finite element, evolutionary equation, error estimates