应用数学和力学，第12卷第9期(1991年9月)
Applied Mathematics and Mechanics

应用数学和力学编委会编
重庆出版社出版

脉动压力脉动速度变形平均项的表达式*

林多敏　　蔡树棠

(上海市应用数学和力学所；上海工业大学)(中国科技大学；上海市应用数学和力学所)

摘　　要

脉动压力脉动速度变形平均项，最初是Rotta[21]改写压力梯度做功项得来的．但是，这项处理起来都很困难．从 Rotta 开始，以后 Launder 等人都对这项做过一些假定．本文根据脉动速度所满足的方程解出脉动压力，然后进而求出脉动压力乘上脉动速度变形的平均值，得到了脉动压力脉动速度变形平均项的完整表达式．这个表达式说明了 Rotta 和 Launder 等人的有限表达式是有一定道理的．本文所得的完整表达式分为两种情形加以讨论，一种是几种涡旋不分开的情形，另一种是三种涡旋分开考虑的情形．由此，本文为雷诺应力模式[7]和三涡旋模式[18]等湍流模式提供了完整的脉动压力脉动速度变形平均项的表达式．

关键词　压力-应变关联项　湍流模式理论　二阶矩封闭

一、引　　言

湍流运动普遍存在于自然界和工程技术问题中．湍流模式理论是研究湍流问题的一个重要组成部分，它以直接应用为目的，不仅具有很大的科学价值，而且具有广泛的应用前景．由于湍流现象的复杂性，以及工程上的各种需要，迄今为止已发展了许多较为成熟的湍流模式，例如：零方程模式(Boussinesq[8]，1877；Prandtl[17]，1925)；一方程模式(Bradshaw[1]，1967)和二方程模式(Daly 和 Harlow[3]，1970；Hanjalic 和 Launder[7]，1972；Rodi[20]，1972；Lannder 和 Spalding[11]，1972)，可供工程计算使用．但是，对于一些较为复杂的湍流流动，例如强旋流、浮力流和湍流诱发的二次流等，目前大多数模式，尤其以 κ-ε 为代表的一类模式，已不能很好的加以预测．原因在于不仅它们是忽视了湍流粘性的各向异性和流线弯曲所产生的附加湍流生成项，而且更重要的是它们是一种梯度形式模型，包含着历史无关性，用局部物理量替代了流动历史的影响．因此，人们开始不断地改进现有模式的模化方法和选用一些较好的模式理论，如雷诺应力模式(Hanjalic 和 Launder[7] 1972)，同时还发展新的湍流模式，如三涡旋模式(Tsai 和 Ma[22]，1987；Lin 和 Tsai[18]，1989)．在模化方法的改进中，目前争论最多的一项是脉动压力脉动速度变形平均项．由于实验的困难，无测量数据作指导，对这一项的模化最为困难．脉动压力脉动速度变形平均项在相应模式相应的应力输运方程中是一个大项，且 Launder[6](1987)认为此项在

* 1990年9月14日收到．

计算旋转流动时起作用，因此，如何正确模化这一项将有着重要的意义．本文中我们将看到现有的这一项的模化方法将会得到进一步的改进．

脉动压力脉动速度变形平均项最初是 Rotta[21](1951)由脉动压力梯度作功项改变形式时出现的．Rotta(1951)把脉动压力梯度乘上脉动速度项拆成脉动压力乘上脉动速度形变项和相应的扩散项，再把脉动压力乘上脉动速度形变项，参照量纲分析原理、张量变更坐标时的性质以及不可压缩流体的连续方程，写成了与二阶矩、平均速度等物理量线性相关的显示表达式．此后，Launder[7]等人(1972)在周培源[2](1945)和 Rotta (1951)的基础上修改了 Rotta 的表达式．迄今为止，这样的线性模型通过经验确定系数的方法已在各种湍流模式理论里得到广泛的应用(例如：Donaldson[4]，1968；Daly 和 Harlow[3] 1970；Hanjalic[6] 1970；Launder 和 Spalding[11] 1972；Reynolds[19] 1970；Rodi[20] 1972；Hanjalic 和 Launder[7] 1972；Launder、Morse、Rodi 和 Spalding[10] 1973；*Naot*、*shavit* 和 Wolfshtein[16] 1973；Launder、Reece 和 Rodi[12] 1975；Launder[9] 1975；Reece[18] 1977；Fu、Launder 和 Leschziner[5] 1987)．林多敏和蔡树棠[13](1989)在三涡旋模式里直接由 Rotta 和 Launder 等人的结果类推了外来大涡旋的脉动压力脉动速度变形平均项的表达式．另外，Lumley 等人[14,15]的工作(Lnmley 和 Khajeh-Nouri，1974；Lumley 和 Newrnan 1977)使得模化压力脉动速度变形项向着非线性模型方向发展．

目前，脉动压力脉动速度变形平均项的线性和非线性模型表面上看都试图由泊松方程求解压力脉动来模化而得到，实际上，这一项的模化考虑都是从张量的数学性质和量纲分析的角度进行的，都是用局部物理量来描述的，因此，这些表达式在模式理论用于复杂湍流流动或者考虑大小尺度涡影响时，无疑是有一定的局限性．本文根据脉动速度所满足的方程解出脉动压力，然后进而求出脉动压力乘上脉动速度变形的平均值，得到了脉动压力脉动速度变形平均项的完整表达式．这个表达式说明了 Rotta 和 Launder 等人的有限表达式是有一定道理的．本文所得的完整表达式分为两种情形加以讨论，一种是几种涡旋不分开的情形，另一种是三种涡旋分开考虑的情形．由此，本文为雷诺应力模式和三涡旋模式等湍流模式提供了完整的脉动压力脉动速度变形平均项的表达式．

二、大小涡旋分不开时的脉动压力

对于不可压缩流体，我们有N-S方程和连续方程

$$\begin{cases} \rho\left(\dfrac{\partial U_i}{\partial t}+U_j\dfrac{\partial U_i}{\partial x_j}\right)=-\dfrac{\partial P}{\partial x_i}+\mu\nabla^2 U_i & (2.1a)\\ \dfrac{\partial U_j}{\partial x_j}=0 & (2.1b)\end{cases}$$

其中 ρ 为流体密度，U_i 为流速，P 为压力，t 为时间，x_i 为坐标，μ 为粘性系数．我们令 $\bar{A}$ 表示物理量 A 的平均量，a 表示物理量 A 的脉动量．对(2.1a)和(2.1b)进行雷诺平均，我们得到雷诺方程和平均连续方程，即

$$\begin{cases} \rho\left(\dfrac{\partial \bar{U}_i}{\partial t}+\bar{U}_j\dfrac{\partial \bar{U}_i}{\partial x_j}\right)=-\dfrac{\partial \bar{P}}{\partial x_i}-\dfrac{\partial}{\partial x_j}(\rho\overline{u_i u_j})+\mu\nabla^2\bar{U}_i & (2.2a)\\ \dfrac{\partial \bar{U}_j}{\partial x_j}=0 & (2.2b)\end{cases}$$

从(2.1a)减去(2.2a)，从(2.1b)减去(2.2b)，我们得到脉动量的方程式，

$$\begin{cases}\rho\left(\frac{\partial u_i}{\partial t}+\bar{U}_j\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial u_i}{\partial x_i}+u_j\frac{\partial \bar{U}_i}{\partial x_j}\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}(\rho\overline{u_i u_j})+\mu\nabla^2 u_i & (2.3a)\\ \frac{\partial u_j}{\partial x_j}=0 & (2.3b)\end{cases}$$

以下我们用a_i表示运流脉动加速度，也就是定义

$$a_i\equiv\frac{\partial u_i}{\partial t}+\bar{U}_j\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial u_i}{\partial x_i} \tag{2.4}$$

那么，(2.3a)就变成

$$\frac{\partial P}{\partial x_i}=\frac{\partial}{\partial x_j}(\rho\overline{u_i u_j})+\mu\nabla^2 u_i-\rho a_i-\rho u_j\frac{\partial \bar{U}_i}{\partial x_j} \tag{2.5}$$

设M_0和M为空间两点，相应坐标分别为$\mathbf{x}_0$和$\mathbf{x}$，其中M点远离壁面，M_0点在壁面间歇区的分界线上，M_0点和M点脉动速度关联为零。我们对(2.5)由M_0点到M点进行积分，则得脉动压力为

$$P=\int_{M_0}^{M}\left[\frac{\partial}{\partial x_j}(\rho\overline{u_i u_j})+\mu\nabla^2 u_i-\rho a_i-\rho u_j\frac{\partial \bar{U}_i}{\partial x_j}\right]' dx_i' \tag{2.6}$$

其中上标$'$表示方括号内x_i用x_i'来替代，x_i'为哑标。

三、大小涡旋不分开时脉动压力脉动速度变形平均项

利用(2.6)，我们用脉动速度变形项$\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)$乘上脉动压力，然后再算平均，则得

$$\overline{\rho\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}=\int_{M_0}^{M}\left[\mu\overline{\nabla'^2 u_\alpha'\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}-\rho\overline{a_\alpha'\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}\right.$$
$$\left.-\rho\overline{u_\beta'\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}\frac{\partial \bar{U}_\alpha'}{\partial x_\beta'}\right]dx_\alpha' \tag{3.1}$$

把原来的坐标变量x_i和x_i'换成x_i和$\xi_k=x_k'-x_k$，那么，(3.1)就可写成

$$\overline{P\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}=\int_{M_0}^{M}\left[\left(\frac{\partial}{\partial x_k}-\frac{\partial}{\partial \xi_k}\right)\frac{\partial^2}{\partial \xi_l\partial \xi_l}\overline{u_i u_\alpha'}\right.$$
$$\left.+\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial \xi_i}\right)\frac{\partial^2}{\partial \xi_l\partial \xi_l}\overline{u_k u_\alpha'}\right]d\xi_\alpha$$
$$-\rho\int_{M_0}^{M}\left[\left(\frac{\partial}{\partial x_k}-\frac{\partial}{\partial \xi_k}\right)\overline{u_i a_\alpha'}+\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial \xi_i}\right)\overline{u_k a_\alpha'}\right]d\xi_\alpha$$
$$-\rho\int_{M_0}^{M}\left[\left(\frac{\partial}{\partial x_k}-\frac{\partial}{\partial \xi_k}\right)\overline{u_i u_\beta'}\frac{\partial \bar{U}_\alpha'}{\partial \xi_\beta}\right.$$
$$\left.+\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial \xi_i}\right)\overline{u_k u_\beta'}\frac{\partial \bar{U}_\alpha'}{\partial \xi_\beta}\right]d\xi_\alpha \tag{3.2}$$

四、脉动运流加速度的表达式

根据 Reynolds 假定[8]，我们可以认为在空间一个固定点附近，流场平均速度变化不

大，而且在时间的过程里流场平均速度的变化也不快。我们考虑空间中一个湍流团在流场中的运动。我们把湍流团看成是在平均流场中以附加速度u_i运动的球体，在运动过程中，它将受到普通的粘性阻力和湍流涡粘性阻力的作用。我们用R代表球体半径，则球体上所受粘性阻力和涡粘性阻力大致可以表示为

$$-C_2'\mu\frac{u_i}{R}4\pi R^2-C_1'\mu_T\frac{u_i}{R}4\pi R^2=-4\pi(\mu C_2'+\mu_T C_1')Ru_i \tag{4.1}$$

其中C_1'和C_2'为无量纲系数，μ为粘性系数，μ_T为涡粘性系数。对湍流团我们可以写出运动方程式，即

$$\frac{4}{3}\pi R^3\rho a_i=-4\pi(\mu C_2'+\mu_T C_1')Ru_i \tag{4.2}$$

于是得到湍流团加速度a_i

$$a_i=-\frac{3(C_1'\mu_T+C_2'\mu)}{\rho R^2}u_i \tag{4.3}$$

因为$\mu_T\gg\mu$，所以我们可以把与μ有关的项略去，则(4.3)有

$$a_i=-\frac{3C_1'\mu_T}{\rho R^2}u_i \tag{4.4}$$

假设湍流团的半径和某一宏观尺度l成正比，并且假设μ_T与这一长度l和脉动速度的均方根值q以及密度ρ等成正比，我们就有

$$R\propto l,\quad \mu_T\propto\rho lq \tag{4.5}$$

于是，(4.4)有

$$a_i=-C_1\frac{q}{l}u_i \tag{4.6}$$

其中 C_1为无量纲系数。

五、脉动压力脉动速度变形平均项表达式

若把(4.6)代入(3.2)，我们就得到大小涡旋不分开时的脉动压力脉动速度变形平均项的表达式，即

$$\begin{aligned}
\overline{P\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)}=&\mu\int_{M_0}^{M}\frac{\partial^3\overline{u_iu'}}{\partial x_k\partial\xi_l\partial\xi_l}d\xi_a+\mu\int_{M_0}^{M}\frac{\partial^3\overline{u_ku_a'}}{\partial x_i\partial\xi_l\partial\xi_l}d\xi_a\\
&-\mu\int_{M_0}^{M}\frac{\partial^2}{\partial\xi\partial_l\xi_l}\left[\frac{\partial}{\partial\xi_k}\overline{u_iu_a'}+\frac{\partial}{\partial\xi_i}\overline{u_ku_a'}\right.\\
&\left.+\frac{\partial}{\partial\xi_a}(\overline{u_iu_k'}+\overline{u_ku_i'})\right]d\xi_a-\mu\left[\frac{\partial^2}{\partial\xi_l\partial\xi_l}(\overline{u_iu_k'}+\overline{u_ku_i'})\right]\Bigg|_{\xi=0}\\
&+C_1\rho\int_{M_0}^{M}\frac{\partial}{\partial x_k}\left(\frac{q'}{l'}\overline{u_iu_a'}\right)d\xi_a+C_1\rho\int_{M_0}^{M}\frac{\partial}{\partial x_i}\left(\frac{q'}{l'}\overline{u_ku_a'}\right)d\xi_a\\
&-C_1\rho\int_{M_0}^{M}\left[\frac{\partial}{\partial\xi_k}\left(\frac{q'}{l'}\overline{u_iu_a'}\right)+\frac{\partial}{\partial\xi_i}\left(\frac{q'}{l'}\overline{u_ku_a'}\right)\right.\\
&\left.+\frac{\partial}{\partial\xi_a}\left(\frac{q'}{l'}\overline{u_iu_k'}+\frac{q'}{l'}\overline{u_ku_i'}\right)\right]d\xi_a-2C_1\rho\frac{q}{l}\overline{u_iu_k}\\
&-\rho\int_{M_0}^{M}\frac{\partial}{\partial x_k}\overline{u_iu_\beta'}\frac{\partial U_a'}{\partial\xi_\beta}d\xi_a-\rho\int_{M_0}^{M}\frac{\partial}{\partial x_i}\overline{u_ku_\beta'}\frac{\partial U_a'}{\partial\xi_\beta}d\xi_a
\end{aligned}$$

$$+\rho\int_{M_0}^{M}\left[\frac{\partial}{\partial\xi_k}\overline{u_i u'_\beta}\frac{\partial U'_a}{\partial\xi_\beta}+\frac{\partial}{\partial\xi_i}\overline{u_k u'_\beta}\frac{\partial U'_a}{\partial\xi_\beta}-\frac{\partial}{\partial\xi_a}\left(\overline{u_i u'_\beta}\frac{\partial U'_k}{\partial\xi_\beta}\right.\right.$$

$$\left.\left.+\overline{u_k u'_\beta}\frac{\partial U'_i}{\partial\xi_\beta}\right)\right]d\xi_a+\rho\left(\overline{u_i u_\beta}\frac{\partial U'_k}{\partial\xi_\beta}+\overline{u_k u_\beta}\frac{\partial U'_i}{\partial\xi_\beta}\right) \tag{5.1}$$

由(5.1)可知：Rotta 和 Launder 等人所假定的脉动压力脉动速度变形平均项的表达式和我们所推导的表达式中的某些项是共同的。但是，他们采用的形式是有限的，不够完整。由于(5.1)中的路径积分并不能简单地由δ_{ij}项来处理，因此，他们的表达式中为了满足连续方程所补上去的项，实际上并没有什么根据，正确的项应该是某些偏导数项的积分形式，由具体流动来决定。

六、三涡旋模式中的脉动压力脉动速度变形平均项

在三涡旋模式里（林多敏和蔡树棠[13]，1989），外来大涡旋脉动速度u_i^N所满足的方程式为

$$\begin{cases}\left(\frac{\partial u_i^N}{\partial t}+U_j\frac{\partial u_i^N}{\partial x_j}+u_j^N\frac{\partial u_i^N}{\partial x_i}+u_j^N\frac{\partial U_i}{\partial x_j}\right)=-\frac{1}{\rho}\frac{\partial P^N}{\partial x_i}+\frac{\partial}{\partial x_j}(\overline{u_i^N u_j^N})+v\nabla^2 u_i^N \\ \qquad -\frac{\partial}{\partial x_j}[\langle u_i^P u_j^P\rangle-\overline{u_i^P u_j^P}]-\frac{\partial}{\partial x_j}[\langle\widetilde{u_i^S u_j^S}\rangle-\overline{u_i^S u_j^S}] & (6.1a)\\ \frac{\partial u_j^N}{\partial x_j}=0 & (6.1b)\end{cases}$$

式中符号说明请见文献[13]。我们仍采用文献[13]中的表达式：

$$\widetilde{u_i^S u_j^S}\approx\frac{1}{3}q_s^2\delta_{ij}-v_T\left[\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right]-v_T\left[\frac{\partial u_i^N}{\partial x_j}+\frac{\partial u_j^N}{\partial x_i}\right]-v_T\left[\frac{\partial u_i^P}{\partial x_j}+\frac{\partial u_N^P}{\partial x_i}\right] \tag{6.2}$$

$$\langle\widetilde{u_i^S u_j^S}\rangle\approx\frac{1}{3}q_s^2\delta_{ij}-v_T\left[\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right]-v_T\left[\frac{\partial u_i^N}{\partial x_j}+\frac{\partial u_j^N}{\partial x_i}\right] \tag{6.3}$$

$$\overline{u_i^S u_j^S}\approx\frac{1}{3}q_s^2\delta_{ij}-v_T\left[\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right] \tag{6.4}$$

所以

$$-\frac{\partial}{\partial x_j}[\langle\widetilde{u_i^S u_j^S}\rangle-\overline{u_i^S u_j^S}]=\frac{\partial}{\partial x_j}\left[v_T\left(\frac{\partial u_i^N}{\partial x_j}+\frac{\partial u_j^N}{\partial x_i}\right)\right] \tag{6.5}$$

同样我们也令

$$a_i^N\equiv\frac{\partial u_i^N}{\partial t}+U_j\frac{\partial u_i^N}{\partial x_j}+u_j^N\frac{\partial u_i^N}{\partial x_i} \tag{6.6}$$

同样我们也可以由流体团速度的减慢来得到外来大涡旋湍流团的加速度a_i^N

$$a_i^N=-\frac{3(C_1^N\mu_T^*+C_2^N\mu)}{\rho R_N^2}u_i^N \tag{6.7}$$

其中 C_1^N和C_2^N为无量纲系数，μ_T^*为涡粘性系数，R_N为外来大涡旋湍流团的半径。因为μ远较湍流粘性系数要小，相比之下，我们可略去$C_2^N\mu$项。我们假定R_N正比于外来大涡旋特征长度l_N，即

$$R_N\propto l_N \tag{6.8}$$

而涡粘性系数μ_T^*有

$$\mu_T^*=C_1^*\rho q_N l_N+C_2^*\rho q_P l_P+C_3^*\rho v_T \tag{6.9}$$

其中 C_1^*，C_2^*，C_3^*为无量纲系数．于是，(6.7)有

$$a_i^N=-\left[C_2\frac{q_N}{l_N}+C_3\frac{q_P l_P}{l_N^2}+C_4\frac{v_T}{l_N^2}\right]u_i^N \tag{6.10}$$

其中 C_2，C_3，C_4为无量纲系数．最后，由(6.1a)可得外来大涡旋脉动压力脉动速度变形平均项的表达式，即

$$\begin{aligned}\overline{p^N\left[\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right]}=&-\rho\int_{M_0}^{M}\overline{a_\alpha'^N\left(\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right)}\,dx_\alpha'\\&-\rho\int_{M_0}^{M}\overline{u_\beta'^N\left(\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right]}\frac{\partial \overline{U}_\alpha'}{\partial x_\beta'}dx_\alpha'\\&+\mu\int_{M_0}^{M}\overline{(\nabla'^2u_\alpha'^N)\left(\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right)}dx_\alpha'\\&+\int_{M_0}^{M}\frac{\partial}{\partial x_\beta'}\overline{\left[\mu_T'\left(\frac{\partial u_\alpha'^N}{\partial x_\beta'}+\frac{\partial u_\beta'^N}{\partial x_\alpha'}\right)\right]\left(\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right)}\,dx_\alpha'\\&-\int_{M_0}^{M}\frac{\partial}{\partial x_\beta'}\overline{\langle u_\alpha'^P u_\beta'^P\rangle\left(\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right)}\,dx_\alpha'\end{aligned} \tag{6.11}$$

把x_i和x_i'换成x_i和$\xi_k=x_k'-x_k$，则(6.11)为

$$\begin{aligned}\overline{p^N\left(\frac{\partial u_i^N}{\partial x_k}+\frac{\partial u_k^N}{\partial x_i}\right)}=&\mu\int_{M_0}^{M}\frac{\partial^3\overline{u_i^N u_\alpha'^N}}{\partial x_k\partial\xi_l\partial\xi_l}d\xi_\alpha+\mu\int_{M_0}^{M}\frac{\partial^3\overline{u_k^N u_\alpha'^N}}{\partial x_i\partial\xi_l\partial\xi_l}d\xi_\alpha\\&-\mu\int_{M_0}^{M}\frac{\partial^2}{\partial\xi_l\partial\xi_l}\left[\frac{\partial}{\partial\xi_k}\overline{u_i^N u_N'^N}+\frac{\partial}{\partial\xi_i}\overline{u_k^N u_\alpha'^N}+\frac{\partial}{\partial\xi_\alpha}(\overline{u_i^N u_k'^N}\right.\\&\left.+\overline{u_k^N u_i'^N})\right]d\xi_\alpha-\mu\left[\frac{\partial^2}{\partial\xi_l\partial\xi_l}(\overline{u_i^N u_k'^N}+\overline{u_k^N u_i'^N})\right]\bigg|_{\xi=0}\\&+\rho\int_{M_0}^{M}\left\{\frac{\partial}{\partial x_k}\left[\left(C_2\frac{q_N'}{l_N'}+C_3\frac{q_P'l_P'}{l_N'^2}+C_4\frac{v_T'}{l_N'^2}\right)\overline{u_i^N u_\alpha'^N}\right]\right.\\&+\frac{\partial}{\partial x_i}\left[\left(C_2\frac{q_N'}{l_N'}+C_3\frac{q_P'l_P'}{l_N'^2}+C_4\frac{v_T'}{l_N'^2}\right)\overline{u_k^N u_\alpha'^N}\right]\\&-\frac{\partial}{\partial\xi_k}\left[\left(C_2\frac{q_N'}{l_N'}+C_3\frac{q_P'l_P'}{l_N'^2}+C_4\frac{v_T'}{l_N'^2}\right]\overline{u_i^N u_\alpha'^N}\right]\\&-\frac{\partial}{\partial\xi_i}\left[\left(C_2\frac{q_N'}{l_N'}+C_3\frac{q_P'l_P'}{l_N'^2}+C_4\frac{v_T'}{l_N'^2}\right]\overline{u_i^N u_\alpha'^N}\right]\\&\left.-\frac{\partial}{\partial\xi_\alpha}\left[\left(C_2\frac{q_N'}{l_N'}+C_3\frac{q_P'l_P'}{l_N'^2}+C_4\frac{v_T'}{l_N'^2}\right)\left(\overline{u_i^N u_k'^N}+\overline{u_k^N u_i'^N}\right)\right]\right\}d\xi_\alpha\\&-2\rho\left(C_2\frac{q_N}{l_N}+C_3\frac{q_P l_P}{l_N^2}+C_4\frac{v_T}{l_N^2}\right)\overline{u_i^N u_k^N}\\&-\rho\int_{M_0}^{M}\frac{\partial}{\partial x_k}\left(\overline{u_i^N u_\beta'^N}\right)\frac{\partial\overline{U}_\alpha'}{\partial\xi_\beta}d\xi_\alpha-\rho\int_{M_0}^{M}\frac{\partial}{\partial x_i}\left(\overline{u_k^N u_\beta'^N}\right)\frac{\partial\overline{U}_\alpha'}{\partial\xi_\beta}d\xi_\alpha\\&+\rho\int_{M_0}^{M}\left[\frac{\partial}{\partial\xi_k}\left(\overline{u_i^N u_\beta'^N}\right)\frac{\partial\overline{U}_\alpha'}{\partial\xi_\beta}+\frac{\partial}{\partial\xi_i}\left(\overline{u_k^N u_\beta'^N}\right)\frac{\partial\overline{U}_\alpha'}{\partial\xi_\beta}\right.\\&\left.-\frac{\partial}{\partial\xi_\alpha}\left(\overline{u_i^N u_\beta'^N}\frac{\partial\overline{U}_k'}{\partial\xi_\beta}+\overline{u_k^N u_\beta'^N}\frac{\partial\overline{U}_i'}{\partial\xi_\beta}\right)\right]d\xi_\alpha-\rho\left(\overline{u_i^N u_\beta^N}\frac{\partial\overline{U}_k}{\partial\xi_\beta}+\overline{u_k^N u_\beta^N}\frac{\partial\overline{U}_i}{\partial\xi_\beta}\right)\end{aligned}$$

$$
\begin{aligned}
&+\int_{M_0}^{M}\frac{\partial}{\partial x_k}\left(\mu_T'\frac{\partial^2}{\partial\xi_l\partial\xi_l}\overline{u_i^N u_\alpha'^N}\right)d\xi_\alpha+\int_{M_0}^{M}\frac{\partial}{\partial x_i}\left(\mu_T'\frac{\partial^2}{\partial\xi_l\partial\xi_l}\overline{u_k^N u_\alpha'^N}d\xi_\alpha\right.\\
&-\int_{M_0}^{M}\left[\frac{\partial}{\partial\xi_k}\left(\mu_T'\frac{\partial^2}{\partial\xi_l\partial\xi_l}\overline{u_i^N u_\alpha'^N}\right)-\frac{\partial}{\partial\xi_i}\left(\mu_T'\frac{\partial^2}{\partial\xi_l\partial\xi_l}\overline{u_k^N u_\alpha'^N}\right)\right.\\
&-\frac{\partial}{\partial\xi_\alpha}\left(\mu_T'\frac{\partial^2}{\partial\xi_l\partial\xi_l}\overline{u_i^N u_k'^N}+\mu_T'\frac{\partial^2}{\partial\xi_l\partial\xi_l}\overline{u_k^N u_i'^N}\right)\\
&-\mu_T\left[\frac{\partial^2}{\partial\xi_l\partial\xi_l}\left(\overline{u_i^N u_k'^N}+\overline{u_k^N u_i'^N}\right)\right]\Bigg|_{\xi=0}\\
&+\int_{M_0}^{M}\frac{\partial}{\partial x_k}\left(\frac{\partial\mu_T'}{\partial\xi_\beta}\frac{\partial}{\partial\xi_\beta}\overline{u_i^N u_\alpha'^N}\right)d\xi_\alpha+\int_{M_0}^{M}\frac{\partial}{\partial x_i}\left(\frac{\partial\mu_T'}{\partial\xi_\beta}\frac{\partial}{\partial\xi_\beta}\overline{u_k^N u_\alpha'^N}\right)d\xi_\alpha\\
&-\int_{M_0}^{M}\frac{\partial}{\partial\xi_k}\left(\frac{\partial\mu_T'}{\partial\xi_\beta}\frac{\partial}{\partial\xi_\beta}\overline{u_i^N u_\alpha'^N}\right)d\xi_\alpha-\int_{M_0}^{M}\frac{\partial}{\partial\xi_i}\left(\frac{\partial\mu_T'}{\partial\xi_\beta}\frac{\partial}{\partial\xi_\beta}\overline{u_k^N u_\alpha'^N}\right)d\xi_\alpha\\
&+\int_{M_0}^{M}\frac{\partial}{\partial x_k}\left(\frac{\partial\mu_T'}{\partial\xi_\beta}\frac{\partial}{\partial\xi_\alpha}\overline{u_i^N u_\beta'^N}\right)d\xi_\alpha+\int_{M_0}^{M}\frac{\partial}{\partial x_i}\left(\frac{\partial\mu_T'}{\partial\xi_\beta}\frac{\partial}{\partial\xi_\alpha}\overline{u_k^N u_\beta'^N}\right)d\xi_\alpha\\
&-\int_{M_0}^{M}\frac{\partial}{\partial\xi_k}\left(\frac{\partial\mu_T'\partial}{\partial\xi_\beta\partial\xi_\alpha}\overline{u_i^N u_\beta'^N}\right)d\xi_\alpha-\int_{M_0}^{M}\frac{\partial}{\partial\xi_i}\left(\frac{\partial\mu_T'\partial}{\partial\xi_\beta\partial\xi_\alpha}\overline{u_k^N u_\beta'^N}\right)d\xi_\alpha\\
&-\int_{M_0}^{M}\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial\xi_\beta}\overline{\langle u_\alpha'^P u_\beta'^P\rangle u_i^N}\right)d\xi_\alpha-\int_{M_0}^{M}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial\xi_\beta}\overline{\langle u_\alpha'^P u_\beta'^P\rangle u_k^N}\right)d\xi_\alpha\\
&+\int_{M_0}^{M}\frac{\partial}{\partial\xi_k}\left(\frac{\partial}{\partial\xi_\beta}\overline{\langle u_\alpha'^P u_\beta'^P\rangle u_i^N}\right)d\xi_\alpha+\int_{M_0}^{M}\frac{\partial}{\partial\xi_i}\left(\frac{\partial}{\partial\xi_\beta}\overline{\langle u_\alpha'^P u_\beta'^P\rangle u_k^N}\right)d\xi_\alpha \quad (6.12)
\end{aligned}
$$

七、结 束 语

在本文之前，人们试图通过求解泊松方程来模化脉动压力脉动速度变形平均项，但是未能获得圆满的成功，仍然停留在张量分析和量纲分析的基础上。本文没有模化任何项，运用湍流团的阻力公式和流体力学基本理论方程得出了脉动压力脉动速度变形平均项的表达式，可以说这是一个完整的表达式。这个完整表达式含有现有的有限表达式中某些合理项，并且说明有限表达式中仅仅含有$\overline{u_i u_j}$这样的项是不够的，还应该含有雷诺应力的微分项。由于本文表达式中含有路径积分，因此，如何简化(5.1)和(6.12)需要针对具体的流动情形。如果我们继续利用周培源教授在文献[2]中的一些展开式，我们将可以进一步简化本文中的表达式，并由此得到优于其它模化方法所得的有限表达式。如果我们对于湍流团加速度a_i的表达式取阻力高次展开项，那么我们也可使本文中的完整表达式达到某种非线性化程度。在本文之后，我们将在数值模拟具体湍流流动时应用本文的表达式，同时进一步探讨有关在壁面附近情形下的结果。

参 考 文 献

[1] Bradshaw, P., D. H. Ferriss and N. P. Atwell, Calculation of boundary-layer -deneloopment using the turbulent energy equation *J. Fluid Mech.*, **28** (1967), 593—616

[2] Chou, P. Y., On the velocity correlations and the solutions of equations of

turbulent fluctuation, *Quart. Appli. Math.*, 3 (1945), 38—54.

[3] Daly, B. J. and F. H. Harlow, Transport equations of turbulence, *Phys.Fluid*, 13 (1970), 2634—2649.

[4] Donaldson, C. Dup, A computer study of an analytical model of boundary layer transition, *AIAA Paper*, No. 68—38 (1968).

[5] Fu, S., B. E. Launder and M. A. Leschziner, Modelling strongly swirling recirculating jet flow with Reynolds-stress transport closures, *Proc. Sixth Symposium on Turbulent Shear Flows*, Toulouse, France, September (1987), 17—6—1.

[6] Hanjalic, K, Two-dimensional flow in an axisymmetric channel, Ph. D. Thesis, Univ. of London (1970).

[7] Hanjalic, K. and B. E. Launder, A Reynolds stress model of turbulence and its application to thin shear flows, *J. Fluid Mech.*, 52 (1972), 609—638.

[8] Hinze, J. O., *Turbulence*, McGraw-Hill Book Company (1975).

[9] Launder, B. E,, On the effects of gravitational field on the turbulent transport of heat and momentum, *J. Fluid Mech.*, 67 (1975), 569—581.

[10] Launder, B. E., A. Morse, W. Rodi and D. B. Spalding, The prediction of free shear flows——a comparison of the performance of six turbulence models, NASA-SP-321 (1973).

[11] Launder, B. E. and D. B. Spalding, *Lectures in Mathematical Models of Turbulence*, Academic Press, New York (1972).

[12] Launder, B. E., G. J. Reece and W. Rodi, Progress in the development of a Reynolds stress turbulence closure, *J. Fluid Mech.*, 68 (1975), 537—566.

[13] Lin, D. M. and S. T. Tsai, The separation of large vortexes and the closed equation of turbulence model, *Appli. Math. and Mech.*, 10 (10) (1989), 853—864.

[14] Lumley, J. L. and B. Khajeh-Nouri, Computational modelling of turbulent transport, *Advances in Geophysics*, 18A, Academic Press, New York (1974), 169—192.

[15] Lumley, J. L. and G. R. Newman, The return to isotropy of homogeneous turbulence, *J. Fluid Mech.*, 82 (1977), 161—178.

[16] Naot, D., A., Shavit and M. Wolfshtein, Two-point correlation model and the redistribution of Reynolds stresses, *Phys. Fluids*, 16 (1973), 738—743.

[17] Prandtl, L.,Bericht über Untersuchungen zur ausgebildeten Turbulenz, *Z.Annew. Math. Mech.*, 5 (1925), 136—139.

[18] Reece, G. J., A generalized Reynolds stress model of turbulence, Ph. D.Thesis, Faculty of Engineering, Univ. of London (1977).

[19] Reynolds, W. C., Computation of turbulent flows-state-of-the-art, Engng. Dept. Stanford Univ., Rep. MD-27 (1970).

[20] Rodi, W., The prediction of free turbulent boundary layers by use of a two-equation model of turbulence, Ph. D. Dissertation, Mechanical Eng. Dept., Imperial College, London, December (1972).

[21] Rotta, J.C., Statistische Theorie nichthomogener Tubulenz,*Z. Phys.*,129 (1951), 547—572.

[22] Tsai, S. T. and B. K. Ma, A new turbulence model with the separate consideration of large and small vortexes, *Appli Math. and Mech.*, 8 (10) (1987), 849—858.

Expressions for Pressure-Velocity-Gradient Correlations

Lin Duo-min

(Shanghai Institute of Applied Math. and Mech., Shanghai; Shanghai University of Technology, Shanghai)

Tsai Shu-tang

(Dept. of Mod. Mech., University of Science and Technology of China, Hefei; Shanghai Institute of Appied Math. and Mech., Shanghai)

Abstract

The term for pressure-velocity-gradient correlation was initiated by Rotta's[12] rewriting the correlation between the pressure fluctuation gradient and velocity fluctuation. However, it is very difficult to consider the effect of this term. Since Rotta's work, Launder et al.[7] has made some estimates of this term. In this paper, according to the equations for velocity fluctuation, the pressure fluctuation is solved so that the average value of the product of the pressure fluctuation and the velocity fluctuation gradient is obtained. Thus, the whole expressions for the pressure-velocity-gradient correlation are derived. The result explains that the limited expressions by Rotta and Launder are reasonable to a certain degree. The whole expressions in this paper are discussed respectively in two situations: one is without a separate consideration of large and small vortexes; the other is with a separate consideration of three kinds of vortexes. Therefore, the paper gives the whole expressions for pressure-velocity-gradient correlation to the Reynolds stress turbulence model[7] and the three-vortex turbulence model.[13]

Key words pressure-velocity-gradient correlation, turbelence model theory, second-order closure