应用数学和力学，第13卷第1期（1992年1月）
Applied Mathematics and Mechanics
应用数学和力学编委会编
重庆出版社出版

考虑横向剪切效应的悬臂矩形板的弯曲*

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摘　要

本文以Reissner板理论为基础，利用厚板的广义简支边概念及迭加法，求得了考虑横向剪切效应的悬臂矩形板弯曲的精确解．从所得结果来看，这种方法是有效的．

关键词　Reissner理论　悬臂矩形板　对称和非对称弯曲　广义简支边　精确解

一、引　言

随着各种新型材料的出现和广泛应用，有必要就计及横向剪切效应的板的弯曲进行分析．悬臂矩形板的弯曲长期被认为是一个难度较大的问题．本文以Reissner板理论为基础，分析了悬臂矩形板的对称和非对称弯曲，得到了该问题的精确解．利用厚板的广义简支边概念[3]和迭加法，将问题归结为求解一组无穷维线性代数方程组．由于采用三角级数展开方法，使得方程组中必须含有两个补充方程，这两个方程不同于薄板理论中的角点条件，是由于数学方法而引入的．通过求解方程组，得到了各种情形下的挠度和应力分布，并数值地研究了厚度对弯曲的影响．

二、问题的描述及求解

设各向同性悬臂矩形板受均布载荷q的作用，其长、宽分别为a、b，厚为h．材料常数为E、ν．边$x=0$固定，其它三边自由，如图1所示．

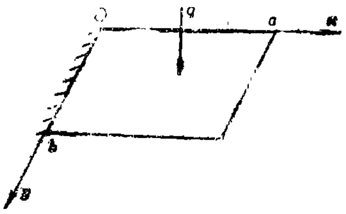

图 1

则Reissner理论的控制方程[2]为：

$$\left.\begin{aligned}&\Delta^2 w=\frac{q}{D}-\frac{(2-\nu)h^2}{10(1-\nu)D}\Delta q\\&\Delta\Phi-\frac{10}{h^2}\Phi=0\end{aligned}\right\}\tag{2.1}$$

$$\left.\begin{aligned}&w=\beta_x=\beta_y=0 &&（当x=0时）\\&M_x=M_{xy}=V_x=0 &&（当x=a时）\end{aligned}\right\}\tag{2.2}$$

* 1990年3月22日收到．兰州大学科学基金资助课题．

$$M_y=M_{xy}=V_y=0 \qquad (\text{当}y=0,\ b\text{时}) \tag{2.3}$$

其中 $D=\dfrac{Eh^3}{12(1-\nu^2)}$，$w$，$\Phi$分别为挠度和应力函数，$\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$，而剪力为

$$\left.\begin{aligned} V_x&=-D\frac{\partial}{\partial x}\Delta w-\frac{(2-\nu)h^2}{10(1-\nu)}\frac{\partial q}{\partial x}+\frac{\partial \Phi}{\partial y}\\ V_y&=-D\frac{\partial}{\partial y}\Delta w-\frac{(2-\nu)h^2}{10(1-\nu)}\frac{\partial q}{\partial y}-\frac{\partial \Phi}{\partial y}\end{aligned}\right\}$$

弯矩及扭矩为

$$\left.\begin{aligned} M_x&=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{h^2}{5}\frac{\partial V_x}{\partial x}-\frac{\nu h^2}{10(1-\nu)}q\\ M_y&=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)+\frac{h^2}{5}\frac{\partial V_y}{\partial y}-\frac{\nu h^2}{10(1-\nu)}q\\ M_{xy}&=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}+\frac{h^2}{10}\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_x}{\partial x}\right)\end{aligned}\right\}$$

转角为

$$\left.\begin{aligned} \beta_x&=-\frac{\partial w}{\partial x}+\frac{h^2}{5(1-\nu)D}V_x\\ \beta_y&=-\frac{\partial w}{\partial y}+\frac{h^2}{5(1-\nu)D}V_y\end{aligned}\right\}$$

一般来说，边值问题(2.1)～(2.3)的求解是十分困难的．下面我们利用厚板的广义简支[8]概念和迭加原理来讨论边值问题(2.1)～(2.3)的求解．显然，该问题的解可以分解为如下几个问题的解的适当组合．

问题(Ⅰ)　在均布载荷q的作用下的四边简支矩形板，此时，边值问题为

$$\Delta^2 w_1=\frac{q}{D}-\frac{(2-\nu)h^2}{10(1-\nu)D}\Delta q,\quad \Delta\Phi_1-\frac{1}{h^2}\Phi_1=0 \tag{2.4a}$$

$$\left.\begin{aligned} &w_1=M_{1x}=\beta_{1y}=0 \qquad (\text{当}x=0,a\text{时})\\ &w_1=M_{1y}=\beta_{1x}=0 \qquad (\text{当}y=0,b\text{时})\end{aligned}\right\} \tag{2.4b}$$

显然，(2.4)的解为

$$\left.\begin{aligned} w_1(x,y)=A_1\sum_{n=1}^{\infty}\Bigg[&Z_{0n}-\frac{\operatorname{sh}\alpha_n\frac{x}{b}-\operatorname{sh}\alpha_n\frac{a-x}{b}}{\operatorname{sh}\alpha_n\frac{a}{b}}A_{1n}+\alpha_n\frac{x}{b}\frac{\operatorname{ch}\alpha_n\frac{x}{b}-\operatorname{ch}\alpha_n\frac{a-x}{b}}{\operatorname{sh}\alpha_n\frac{a}{b}}D_{1n}\\ &-\alpha_n\frac{a}{b}\frac{\operatorname{ch}\alpha_n\frac{a}{b}-1}{\operatorname{sh}\alpha_n\frac{a}{b}}\frac{\operatorname{sh}\alpha_n\frac{x}{b}}{\operatorname{sh}\alpha_n\frac{a}{b}}D_{1n}\Bigg]\sin\alpha_n\frac{y}{b},\quad \Phi_1(x,y)=0\end{aligned}\right\} \tag{2.5}$$

其中　$Z_{0n}=\dfrac{(b/a)^4}{\alpha_n^4}\left(1+\dfrac{(2-\nu)h^2}{10(1-\nu)b^2}\alpha_n^2\right)Q_n$，$A_{1n}=-Z_{0n}$，

$D_{1n}=\dfrac{(b/a)^4}{2\alpha_n^4}Q_n$，$Q_n=2(1-(-1)^n)/\alpha_n$，$\alpha_n=n\pi$，

$$\gamma_n=\left(\alpha_n^2+\frac{10b^2}{h^2}\right)^{1/2},\ \Lambda_1=\frac{qa^4}{D}.$$

问题(Ⅰ) 设矩形板的$x=a$，$y=0$，b三边为简支，而$x=0$的边上受分布弯矩的作用，则边值问题为：

$$\Delta^2 w_2=0,\quad \Delta\Phi_2-\frac{10}{h^2}\Phi_2=0 \tag{2.6a}$$

$$\left.\begin{aligned}&w_2=0,\ M_{2x}=\Lambda_2\sum_{n=1}^{\infty}a_n\sin\alpha_n\frac{y}{y},\ \beta_{2y}=0 \quad (\text{当}x=0\text{时})\\&w_2=M_{2x}=\beta_{2y}=0 \quad (\text{当}x=a\text{时})\\&w_2=M_{2y}=\beta_{2x}=0 \quad (\text{当}y=0,b\text{时})\end{aligned}\right\} \tag{2.6b}$$

其中 $\Lambda_2=qa^2$，a_n是待定常数。

于是，(2.6)的解为

$$\left.\begin{aligned}&w_2(x,y)=\Lambda_1\sum_{n=1}^{\infty}\frac{1}{2\alpha_n}\frac{b}{a}\left[\frac{x}{a}\frac{\mathrm{ch}\alpha_n\frac{a-x}{b}}{\mathrm{sh}\alpha_n\frac{a}{b}}-\frac{\mathrm{sh}\alpha_n\frac{x}{b}}{\left(\mathrm{sh}\alpha_n\frac{a}{b}\right)^2}\right]a_n\sin\alpha_n\frac{y}{b}\\&\Phi_2(x,y)=\Lambda_2\sum_{n=1}^{\infty}-\frac{\alpha_n}{\gamma_n}\frac{\mathrm{ch}\gamma_n\frac{a-x}{b}}{\mathrm{sh}\gamma_n\frac{a}{b}}a_n\cos\alpha_n\frac{y}{b}\end{aligned}\right\} \tag{2.7}$$

问题(Ⅱ) 设矩形板的三边$x=0$，$y=0$，b为简支边，而$x=a$为广义简支边，则边值问题为

$$\Delta^2 w_3=0,\ \Delta\Phi_3-\frac{10}{h^2}\Phi_3=0 \tag{2.8a}$$

$$\left.\begin{aligned}&w_3=M_{3x}=\beta_{3y}=0 \quad (\text{当}x=0\text{时})\\&w_3=\Lambda_1\sum_{n=1}^{\infty}b_n\sin\alpha_n\frac{y}{b},\ M_{3x}=0,\ \beta_{3y}=\Lambda_3\sum_{n=0}^{\infty}c_n\cos\alpha_n\frac{y}{b} \quad (\text{当}x=a\text{时})\\&w_3=M_{3y}=\beta_{3x}=0 \quad (\text{当}y=0,b\text{时})\end{aligned}\right\} \tag{2.8b}$$

其中 $\Lambda_3=qa^3/D$，b_n，c_n为待定常数。

(2.8)的解为

$$\left.\begin{aligned}&w_3(x,y)=\Lambda_1\sum_{n=1}^{\infty}\left[\frac{\mathrm{sh}\alpha_n\frac{x}{b}}{\mathrm{sh}\alpha_n\frac{a}{b}}b_n+\frac{1-\nu}{2}\left(\frac{x}{a}\frac{\mathrm{ch}\alpha_n\frac{x}{b}}{\mathrm{sh}\alpha_n\frac{a}{b}}\right.\right.\\&\qquad\left.\left.-\mathrm{cth}\alpha_n\frac{a}{b}\cdot\frac{\mathrm{sh}\alpha_n\frac{x}{b}}{\mathrm{sh}\alpha_n\frac{a}{b}}\right)c_n\right]\sin\alpha_n\frac{y}{b}\\&\Phi_3(x,y)=\Lambda_2\left\{-\frac{5(1-\nu)ab}{\gamma_0h^2}\frac{\mathrm{ch}\gamma_0\frac{x}{b}}{\mathrm{sh}\gamma_0\frac{a}{b}}c_0+\left[\sum_{n=1}^{\infty}-\frac{5(1-\nu)\alpha_na^2}{h^2\gamma_n}\frac{\mathrm{ch}\gamma_n\frac{x}{b}}{\mathrm{sh}\gamma_n\frac{a}{b}}b_n\right.\right.\end{aligned}\right\} \tag{2.9}$$

$$-\frac{5(1-\nu)ab}{h^2\gamma_n}\left(1+\frac{h^2\alpha_n^2}{5b^2}\right)\frac{\mathrm{ch}\gamma_n\frac{x}{b}}{\mathrm{sh}\gamma_n\frac{a}{b}}c_n\Bigg]\cos\alpha_n\frac{y}{b}\Bigg\}$$

问题(Ⅳ) 设矩形板的三边$x=0,a,y=0$为简支边，而$y=b$为广义简支边，则边值问题为

$$\Delta^2 w_4=0,\ \Delta\Phi_4-\frac{10}{h^2}\Phi_4=0 \tag{2.10a}$$

$$\left.\begin{aligned}&w_4=M_{4x}=\beta_{4y}=0 \quad (\text{当}x=0,a\text{时})\\&w_4=M_{4y}=\beta_{4x}=0 \quad (\text{当}y=0\text{时})\\&w_4=A_1\sum_{m=1}^{\infty}d_m\sin\lambda_m\frac{x}{a},\ M_{4y}=0,\ \beta_{4x}=A_3\sum_{m=0}^{\infty}e_m\cos\lambda_m\frac{x}{a} \quad (\text{当}y=b\text{时})\end{aligned}\right\} \tag{2.10b}$$

其中 $\lambda_m=m\pi$，d_m，e_m为待定常数。

于是，(2.10)的解可表示为

$$\left.\begin{aligned}w_4(x,y)=&A_1\sum_{m=1}^{\infty}\left[\frac{\mathrm{sh}\lambda_m\frac{y}{a}}{\mathrm{sh}\lambda_m\frac{b}{a}}d_m-\frac{1-\nu}{2}\left(\frac{y}{a}\frac{\mathrm{ch}\lambda_m\frac{y}{a}}{\mathrm{sh}\lambda_m\frac{b}{a}}\right.\right.\\&\left.\left.-\frac{b}{a}\mathrm{cth}\,\lambda_m\frac{b}{a}\frac{\mathrm{sh}\lambda_m\frac{y}{a}}{\mathrm{sh}\lambda_m\frac{b}{a}}\right)e_m\right]\sin\lambda_m\frac{x}{a}\\\Phi_4(x,y)=&A_2\left\{\frac{5(1-\nu)a^2}{h^2\beta_0}\frac{\mathrm{ch}\beta_0\frac{y}{a}}{\mathrm{sh}\beta_0\frac{b}{a}}e_0+\sum_{m=1}^{\infty}\left[\frac{5(1-\nu)a^2\lambda_m}{h^2\beta_m}\frac{\mathrm{ch}\beta_m\frac{y}{a}}{\mathrm{sh}\beta_m\frac{b}{a}}d_m\right.\right.\\&\left.\left.+\frac{5(1-\nu)a^2}{h^2\beta_m}\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\mathrm{ch}\beta_m\frac{y}{a}}{\mathrm{sh}\beta_m\frac{b}{a}}e_m\right]\cos\lambda_m\frac{x}{a}\right\}\end{aligned}\right\} \tag{2.11}$$

其中 $\beta_m=(\lambda_m^2+10a^2/h^2)^{1/2}$。

问题(Ⅴ) 设矩形板三边$x=0,a,y=b$为简支边，而$y=0$为广义简支边，则边值问题为：

$$\Delta^2 w_5=0,\ \Delta\Phi_5-\frac{10}{h^2}\Phi_5=0 \tag{2.12a}$$

$$\left.\begin{aligned}&w_5=w_{5x}=\beta_{5y}=0 \quad (\text{当}x=0,a\text{时})\\&w_5=A_1\sum_{m=1}^{\infty}f_m\sin\lambda_m\frac{x}{a},\ M_{5y}=0,\ \beta_{5x}=A_3\sum_{m=0}^{\infty}g_m\cos\lambda_m\frac{x}{a} \quad (\text{当}y=0\text{时})\\&w_5=M_{5y}=\beta_{5x}=0 \quad (\text{当}y=b\text{时})\end{aligned}\right\} \tag{2.12b}$$

其中 f_m，g_m为待定常数。

于是，(2.12)的解为

$$\left.\begin{aligned} w_5(x,y)=&A_1\sum_{m=1}^{\infty}\left\{\frac{\operatorname{sh}\lambda_m\frac{b-y}{a}}{\operatorname{sh}\lambda_m\frac{b}{a}}f_m+\frac{1-\nu}{2}\left[\frac{b}{a}\frac{\operatorname{sh}\lambda_m\frac{y}{a}}{\left(\operatorname{sh}\lambda_m\frac{b}{a}\right)^2}\right.\right.\\ &\left.\left.-\frac{y}{a}\frac{\operatorname{ch}\lambda_m\frac{b-y}{a}}{\operatorname{sh}\lambda_m\frac{b}{a}}\right]g_m\right\}\sin\lambda_m\frac{x}{a}\\ \Phi_5(x,y)=&A_2\left\{-\frac{5(1-\nu)a^2}{h^2\beta_0}\frac{\operatorname{ch}\beta_0\frac{b-y}{a}}{\operatorname{sh}\beta_0\frac{b}{a}}g_0\right.\\ &+\sum_{m=1}^{\infty}\left[-\frac{5(1-\nu)a^2\lambda_m}{h^2\beta_m}\frac{\operatorname{ch}\beta_m\frac{b-y}{a}}{\operatorname{sh}\beta_m\frac{b}{a}}f_m\right.\\ &\left.\left.-\frac{5(1-\nu)a^2}{h^2\beta_m}\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\operatorname{ch}\beta_m\frac{b-y}{a}}{\operatorname{sh}\beta_m\frac{b}{a}}g_m\right]\cos\lambda_m\frac{x}{a}\right\} \end{aligned}\right\}\tag{2.13}$$

问题(Ⅵ) 设矩形板具有刚体位移和纯扭转，

$$w_6(x,y)=A_1\left(k_1\frac{x}{a}+k_2\frac{x}{a}\frac{y}{b}\right),\quad \Phi_6(x,y)=0\tag{2.14}$$

其中 k_1，k_2为待定常数．显然，w_6和Φ_6满足

$$\Delta^2 w_6=0,\quad \Delta\Phi_6-\frac{10}{h^2}\Phi_6=0\tag{2.15}$$

现在，令 $w=\sum_{i=1}^{6}w_i(x,y)$，$\Phi=\sum_{i=1}^{6}\Phi_i(x,y)$ (2.16)

显然，(2.16)满足方程(2.1)以及边界条件

$$w|_{x=0}=\beta_y|_{x=0}=M_x|_{x=a}=M_y|_{y=0,b}=0\tag{2.17}$$

因此，(2.16)为(2.1)～(2.3)的解的充要条件是(2.16)满足

$$\left.\begin{aligned}&\beta_x|_{x=0}=0,\quad M_{xy}|_{x=a}=V_x|_{x=a}=0\\ &M_{xy}|_{y=0,b}=V_y|_{y=0,b}=0\end{aligned}\right\}\tag{2.18}$$

将w，Φ代入(2.18)并注意到剪力，弯矩和转角的表达式，我们得到如下方程．由$\beta_x|_{x=0}=0$得

$$\left[\frac{1}{2\left(2\operatorname{sh}\alpha_n\frac{a}{b}\right)^2}-\left(\frac{b}{a}+\frac{2h^2\alpha_n^2}{5(1-\nu)a^2}\right)\frac{\operatorname{cth}\alpha_n\frac{a}{b}}{2\alpha_n}+\frac{h^2\alpha_n^2}{5(1-\nu)ab\gamma_n}\operatorname{cth}\gamma_n\frac{a}{b}\right]a_n$$

$$+\frac{a}{b}\alpha_n\left[\frac{\alpha_n}{\gamma_n\operatorname{sh}\gamma_n\frac{a}{b}}-\frac{1}{\operatorname{sh}\alpha_n\frac{a}{b}}\right]b_n$$

$$+\left[\frac{a}{b}\alpha_n(1-\nu)\frac{\operatorname{cth}\alpha_n\frac{a}{b}}{2\operatorname{sh}\alpha_n\frac{a}{b}}-\left(1+\frac{2h^2\alpha_n^2}{5(1-\nu)b^2}\right)\frac{(1-\nu)}{2\operatorname{sh}\alpha_n\frac{a}{b}}\right.$$

$$\left.+\left(1+\frac{h^2\alpha_n^2}{5b^2}\right)\frac{\alpha_n}{\gamma_n\operatorname{sh}\gamma_n\frac{a}{b}}\right]c_n+\sum_{m=1}^{\infty}2(-1)^{n+1}\frac{\lambda_m}{\alpha_n}\left(\frac{1}{1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2}\right.$$

$$\left.-\frac{1}{1+\left(\frac{\lambda_m}{\alpha_n}\frac{b}{a}\right)^2}\right)d_m+\frac{2(-1)^{n+1}}{\alpha_n\left(1+\left(\frac{\beta_0}{\alpha_n}\frac{b}{a}\right)^2\right)}e_0$$

$$+\sum_{m=1}^{\infty}2(-1)^{n+1}\left[\frac{\lambda_m^2}{\alpha_n\left(1+\left(\frac{\lambda_m}{\alpha_n}\frac{b}{a}\right)^2\right)}\left(\frac{1-\nu}{\left(\alpha_n\frac{a}{b}\right)^2+\lambda_m^2}-\frac{h^2}{5a^2}\right)\right.$$

$$\left.+\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{1}{\alpha_n\left(1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2\right)}\right]e_m$$

$$+\sum_{m=1}^{\infty}2\frac{\lambda_m}{\alpha_n}\left(\frac{1}{1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2}-\frac{1}{1+\left(\frac{\lambda_m}{\alpha_n}\frac{b}{a}\right)^2}\right)f_m$$

$$+\frac{2}{\alpha_n\left(1+\left(\frac{\beta_0}{\alpha_n}\frac{b}{a}\right)^2\right)}g_0+\sum_{m=1}^{\infty}2\left[\frac{\lambda_m^2}{\alpha_n\left(1+\left(\frac{\lambda_m}{\alpha_n}\frac{b}{a}\right)^2\right)}\right.$$

$$\left.\cdot\left(\frac{1-\nu}{\left(\alpha_n\frac{a}{b}\right)^2+\lambda_m^2}-\frac{h^2}{5a^2}\right)+\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{1}{\alpha_n\left(1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2\right)}\right]g_m$$

$$-\frac{2(1+(-1)^{n+1})}{\alpha_n}k_1+\frac{2(-1)^n}{\alpha_n}k_2$$

$$=\frac{a}{b}\alpha_n\left[B_{1n}+\left(1+\frac{2h^2\alpha_n^2}{5(1-\nu)b^2}\right)C_{1n}\right]\qquad(n=1,2,3,\cdots)\qquad(2.19)$$

其中
$$B_{1n}=\frac{1-\operatorname{ch}\alpha_n\frac{a}{b}}{\operatorname{sh}\alpha_n\frac{a}{b}}A_{1n}-\alpha_n\frac{a}{b}\frac{\operatorname{ch}\alpha_n\frac{a}{b}-1}{\left(\operatorname{sh}\alpha_n\frac{a}{b}\right)^2}D_{1n},$$

$$C_{1n}=\frac{1-\operatorname{ch}\alpha_n\frac{a}{b}}{\operatorname{sh}\alpha_n\frac{a}{b}}D_{1n}.$$

由$V_x|_{x=a}=0$得

$$\alpha_n\frac{a}{b}\left(\frac{\alpha_n}{\gamma_n\operatorname{sh}\gamma_n\frac{a}{b}}-\frac{1}{\operatorname{sh}\alpha_n\frac{a}{b}}\right)a_n+5(1-\nu)\frac{a}{b}\left(\frac{a}{h}\right)^2\frac{\alpha_n^2}{\gamma_n\operatorname{th}\gamma_n\frac{a}{b}}b_n$$

$$+(1-\nu)\alpha_n^2\left(\frac{a}{b}\right)^2\left[\left(1+\frac{5b^2}{h^2\alpha_n^2}\right)\frac{\alpha_n}{\gamma_n}\operatorname{cth}\gamma_n\frac{a}{b}-\operatorname{cth}\alpha_n\frac{a}{b}\right]c_n$$

$$
+\sum_{m=1}^{\infty}(-1)^{m+n+1}\frac{10(1-\nu)a^2\lambda_m}{h^2\alpha_n}\cdot\frac{1}{1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2}d_m
$$

$$
+(-1)^{n+1}\frac{10(1-\nu)a^2}{h^2\alpha_n}\frac{1}{1+\left(\frac{\beta_0}{\alpha_n}\frac{b}{a}\right)^2}e_0
$$

$$
+\sum_{m=1}^{\infty}2(-1)^{m+n+1}(1-\nu)\frac{\lambda_m^2}{\alpha_n}\left[\left(1+\frac{5a^2}{\lambda_m^2h^2}\right)\frac{1}{1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2}\right.
$$

$$
\left.-\frac{1}{1+\left(\frac{\lambda_m}{\alpha_n}\frac{b}{a}\right)^2}\right]e_m+\sum_{m=1}^{\infty}(-1)^m\frac{10(1-\nu)a^2\lambda_m}{h^2\alpha_n}\frac{1}{1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2}f_m
$$

$$
+\frac{10(1-\nu)a^2}{h^2\alpha_n}\frac{1}{1+\left(\frac{\beta_0}{\alpha_n}\frac{b}{a}\right)^2}g_0+\sum_{m=1}^{\infty}2(-1)^m(1-\nu)\frac{\lambda_m^2}{\alpha_n}
$$

$$
\cdot\left[\left(1+\frac{5a^2}{h^2\lambda_m^2}\right)\frac{1}{1+\left(\frac{\beta_m}{\alpha_n}\frac{b}{a}\right)^2}-\frac{1}{1+\left(\frac{\lambda_m}{\alpha_n}\frac{b}{a}\right)^2}\right]g_m
$$

$$
=-2\left(\frac{a}{b}\alpha_n\right)^3C_{1n}\qquad(n=1,2,3,\cdots)\tag{2.20}
$$

由$M_{xy}|_{x=a}=0$得

$$
\frac{1}{2}(1-\nu)\frac{a}{b}\gamma_0\mathrm{cth}\gamma_0\frac{a}{b}c_0+\sum_{m=1}^{\infty}(-1)^m(1-\nu)\lambda_m\frac{a}{b}\frac{\lambda_m^2-\beta_m^2}{2\beta_m^2}d_m
$$

$$
+\frac{1}{2}(1-\nu)\frac{a}{b}e_0+\sum_{m=1}^{\infty}(-1)^m(1-\nu)\frac{a}{b}\frac{1}{2\beta_m^2}\left[\lambda_m^2+\beta_m^2+\frac{h^2\lambda_m^2}{5a^2}(\lambda_m^2-\beta_m^2)\right]e_m
$$

$$
+\sum_{m=1}^{\infty}(-1)^{m+1}(1-\nu)\lambda_m\frac{a}{b}\frac{\lambda_m^2-\beta_m^2}{2\beta_m^2}f_m-\frac{1}{2}(1-\nu)\frac{a}{b}g_0
$$

$$
+\sum_{m=1}^{\infty}(-1)^{m+1}(1-\nu)\frac{a}{b}\frac{1}{2\beta_m^2}\left[\lambda_m^2+\beta_m^2+\frac{h^2\lambda_m^2}{5a^2}(\lambda_m^2-\beta_m^2)\right]g_m
$$

$$
-(1-\nu)\frac{a}{b}k_2=0\tag{2.21a}
$$

$$
\left[(1-\nu)\frac{a}{b}\alpha_n\frac{\mathrm{cth}\alpha_n\frac{a}{b}}{2\mathrm{sh}\alpha_n\frac{a}{b}}-\left(1-\nu+\frac{2h^2\alpha_n^2}{5b^2}\right)\frac{1}{2\mathrm{sh}\alpha_n\frac{a}{b}}+\frac{h^2}{10b^2}(\alpha_n+\gamma_n)^2\frac{\alpha_n}{\gamma_n\mathrm{sh}\gamma_n\frac{a}{b}}\right]a_n
$$

$$
+(1-\nu)\left(\frac{a}{b}\right)^2\alpha_n\left(\frac{\alpha_n^2+\gamma_n^2}{2\gamma_n}\mathrm{cth}\,\gamma_n\frac{a}{b}-\alpha_n\mathrm{cth}\,\alpha_n\frac{a}{b}\right)b_n
$$

$$
+(1-\nu)\frac{a}{b}\left[\alpha_n^2\frac{a}{b}\left(\frac{1-\nu}{2\left(\mathrm{sh}\alpha_n\frac{a}{b}\right)^2}-\left(1-\nu+\frac{2h^2\alpha_n^2}{5b^2}\right)\frac{b}{2\alpha_na}\mathrm{cth}\,\alpha_n\frac{a}{b}\right)\right.
$$

$$+\left(1+\frac{h^2\alpha_n^2}{5b^2}\right)\frac{\alpha_n^2+\gamma_n^2}{2\gamma_n}\operatorname{cth}\gamma_n\frac{a}{b}\Bigg]c_n+\sum_{m=1}^{\infty}(-1)^{m+n}(1-\nu)\frac{b}{a}\lambda_m$$

$$\cdot\left[\frac{\lambda_m^2+\beta_m^2}{\alpha_n^2+\left(\beta_m\frac{b}{a}\right)^2}-\frac{2\lambda_m^2}{\alpha_n^2+\left(\lambda_m\frac{b}{a}\right)^2}\right]d_m+(-1)^n\frac{b}{a}\frac{(1-\nu)\beta_0^2}{\alpha_n^2+\left(\beta_0\frac{b}{a}\right)^2}e_0$$

$$+\sum_{m=1}^{\infty}(-1)^{m+n}(1-\nu)\frac{b}{a}\Bigg[\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\lambda_m^2+\beta_m^2}{\alpha_n^2+\left(\beta_m\frac{b}{a}\right)^2}-\frac{2h^2}{5a^2}\frac{\lambda_m^4}{\alpha_n^2+\left(\lambda_m\frac{b}{a}\right)^2}$$

$$-\frac{2(1-\nu)\lambda_m^2\alpha_n^2}{\left(\alpha_n^2+\left(\lambda_m\frac{b}{a}\right)^2\right)^2}\Bigg]e_m+\sum_{m=1}^{\infty}(-1)^{m+1}(1-\nu)\frac{b}{a}\lambda_m$$

$$\cdot\left[\frac{\lambda_m^2+\beta_m^2}{\alpha_n^2+\left(\beta_m\frac{b}{a}\right)^2}-\frac{2\lambda_m^2}{\alpha_n^2+\left(\lambda_m\frac{b}{a}\right)^2}\right]f_m-(1-\nu)\frac{b}{a}\frac{\beta_0^2}{\alpha_n^2+\left(\beta_0\frac{b}{a}\right)^2}g_0$$

$$+\sum_{m=1}^{\infty}(-1)^{m+1}(1-\nu)\frac{b}{a}\Bigg[\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\lambda_m^2+\beta_m^2}{\alpha_n^2+\left(\beta_m\frac{b}{a}\right)^2}$$

$$-\frac{2h^2}{5a^2}\frac{\lambda_m^4}{\alpha_n^2+\left(\lambda_m\frac{b}{a}\right)^2}-\frac{2(1-\nu)\lambda_m^2\alpha_n^2}{\left(\alpha_n^2+\left(\lambda_m\frac{b}{a}\right)^2\right)^2}\Bigg]g_m$$

$$=-\left(\frac{a}{b}\right)^2\alpha_n^2\left[(1-\nu)B_{1n}+\left(1-\nu+\frac{2h^2\alpha_n^2}{5b^2}\right)C_{1n}\right]\quad(n=1,2,3,\cdots)\qquad(2.21\text{b})$$

由$V_y|_{y=0}=0$得

$$\sum_{n=1}^{\infty}2\frac{a}{b}\frac{\alpha_n}{\lambda_m}\left[\frac{1}{1+\left(\frac{\alpha_n}{\lambda_m}\frac{a}{b}\right)^2}-\frac{1}{1+\left(\frac{\gamma_n}{\lambda_m}\frac{a}{b}\right)^2}\right]a_n$$

$$+\sum_{n=1}^{\infty}(-1)^{m+1}10(1-\nu)\frac{a}{b}\left(\frac{a}{h}\right)^2\frac{\alpha_n}{\lambda_m\left(1+\left(\frac{\gamma_n}{\lambda_m}\frac{a}{b}\right)^2\right)}b_n$$

$$+(-1)^{m+1}\left(\frac{a}{h}\right)^2\frac{10(1-\nu)}{\lambda_m\left(1+\left(\frac{\gamma_0}{\lambda_m}\frac{a}{b}\right)^2\right)}c_0$$

$$+\sum_{n=1}^{\infty}(-1)^{m+1}2(1-\nu)\left[\frac{5\left(\frac{a}{h}\right)^2+\left(\frac{a}{b}\alpha_n\right)^2}{\lambda_m\left(1+\left(\frac{\gamma_n}{\lambda_m}\frac{a}{b}\right)^2\right)}-\frac{\left(\frac{a}{b}\alpha_n\right)^2}{\lambda_m\left(1+\left(\frac{\alpha_n}{\lambda_m}\frac{a}{b}\right)^2\right)}\right]c_n$$

$$+\frac{5(1-\nu)a^2\lambda_m^2}{\beta_m h^2\operatorname{sh}\beta_m\frac{b}{a}}d_m+(1-\nu)\lambda_m\left(\frac{\lambda_m^2+5\left(\frac{a}{h}\right)^2}{\beta_m\operatorname{sh}\beta_m\frac{b}{a}}-\frac{\lambda_m}{\operatorname{sh}\lambda_m\frac{b}{a}}\right)e_m$$

$$-\frac{5(1-\nu)a^2\lambda_m^2}{\beta_m h^2}\operatorname{cth}\beta_m\frac{b}{a}f_m+(1-\nu)\lambda_m\Big[\lambda_m\operatorname{cth}\lambda_m\frac{b}{a}$$

$$-\left(\lambda_m^2+\frac{5a^2}{h^2}\right)\frac{1}{\beta_m}\operatorname{cth}\beta_m\frac{b}{a}\Big]g_m=2\lambda_m^3\tilde{C}_{1m}\qquad(m=1,2,3,\cdots)\qquad(2.22)$$

其中 $\bar{C}_{1m}=\dfrac{Q_m}{2\lambda_m^4}\dfrac{1-\mathrm{ch}\lambda_m\frac{b}{a}}{\mathrm{sh}\lambda_m\frac{b}{a}}$.

由$M_{xy}|_{y=0}=0$得

$$\sum_{n=1}^{\infty}\frac{h^2\alpha_n}{10ab}\left(\left(\frac{\alpha_n}{\gamma_n}\right)^2-1\right)a_n+\sum_{n=1}^{\infty}\frac{1-\nu}{2}\frac{a}{b}\alpha_n\left(\left(\frac{\alpha_n}{\gamma_n}\right)^2-1\right)b_n$$

$$+\frac{1-\nu}{2}c_0+\sum_{n=1}^{\infty}\frac{1-\nu}{2}\left[1+\left(\frac{\alpha_n}{\gamma_n}\right)^2+\frac{h^2\alpha_n^2}{5b^2}\left(\left(\frac{\alpha_n}{\gamma_n}\right)^2-1\right)\right]c_n$$

$$+\frac{1-\nu}{2}\frac{\beta_0}{\mathrm{sh}\beta_0\frac{b}{a}}e_0-\frac{1-\nu}{2}\beta_0\mathrm{cth}\,\beta_0\frac{b}{a}g_0-\left(\frac{a}{b}\right)(1-\nu)k_2=0 \qquad (2.23a)$$

$$\sum_{n=1}^{\infty}-\frac{a}{b}\alpha_n\left[\frac{2\lambda_m^2(1-\nu)}{\left(\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2\right)^2}+\frac{h^2}{5b^2}\left(\frac{2\alpha_n^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}-\frac{\alpha_n^2+\gamma_n^2}{\lambda_m^2+\left(\gamma_n\frac{a}{b}\right)^2}\right)\right]a_n$$

$$+\sum_{n=1}^{\infty}(-1)^m(1-\nu)\alpha_n\left(\frac{a}{b}\right)^3\left(\frac{\alpha_n^2+\gamma_n^2}{\lambda_m^2+\left(\gamma_n\frac{a}{b}\right)^2}-\frac{2\alpha_n^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}\right)b_n$$

$$+(-1)^m(1-\nu)\left(\frac{a}{b}\right)^2\frac{\gamma_0^2}{\lambda_m^2+\left(\gamma_0\frac{a}{b}\right)^2}c_0+\sum_{n=1}^{\infty}(-1)^m\left(\frac{a}{b}\right)^2(1-\nu)$$

$$\cdot\left[\left(1+\frac{h^2\alpha_n^2}{5b^2}\right)\frac{\alpha_n^2+\gamma_n^2}{\lambda_m^2+\left(\gamma_n\frac{a}{b}\right)^2}-\frac{2\alpha_n^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}\left(\frac{(1-\nu)\lambda_m^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}\right.\right.$$

$$\left.\left.+\frac{h^2\alpha_n^2}{5b^2}\right)\right]c_n+(1-\nu)\lambda_m\left[\frac{\lambda_m^2+\beta_m^2}{2\beta_m}\frac{1}{\mathrm{sh}\beta_m\frac{b}{a}}-\frac{\lambda_m}{\mathrm{sh}\lambda_m\frac{b}{a}}\right]d_m$$

$$+\frac{1-\nu}{2}\left[(1-\nu)\frac{b}{a}\lambda_m^2\frac{\mathrm{cth}\,\lambda_m\frac{b}{a}}{\mathrm{sh}\lambda_m\frac{b}{a}}-\lambda_m\left(1-\nu+\frac{2h^2\lambda_m^2}{5a^2}\right)\frac{1}{\mathrm{sh}\lambda_m\frac{b}{a}}\right.$$

$$\left.+\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\lambda_m^2+\beta_m^2}{\beta_m\mathrm{sh}\beta_m\frac{b}{a}}\right]e_m+(1-\nu)\lambda_m\left[\lambda_m\mathrm{cth}\lambda_m\frac{b}{a}\right.$$

$$\left.-\frac{\lambda_m^2+\beta_m^2}{2\beta_m}\mathrm{cth}\beta_m\frac{b}{a}\right]f_m-\frac{1-\nu}{2}\left[(1-\nu)\frac{b}{a}\lambda_m^2\frac{1}{\left(\mathrm{sh}\lambda_m\frac{b}{a}\right)^2}\right.$$

$$\left.-\lambda_m\left(1-\nu+\frac{2h^2\lambda_m^2}{5a^2}\right)\mathrm{cth}\,\lambda_m\frac{b}{a}+\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\lambda_m^2+\beta_m^2}{\beta_m}\mathrm{cth}\beta_m\frac{b}{a}\right]g_m$$

$$=\lambda_m^2\left[(1-\nu)\bar{B}_{1m}+\left(1-\nu+\frac{2h^2\lambda_m^2}{5a^2}\right)\bar{C}_{1m}\right]\qquad(m=1,2,3,\cdots) \qquad (2.23b)$$

其中 $$\tilde{B}_{1m}=\left[\frac{Q_m}{\lambda_m^4}\left(1+\frac{(2-\nu)h^2\lambda_m^2}{10(1-\nu)a^2}\right)\frac{\operatorname{ch}\lambda_m\frac{b}{a}-1}{\operatorname{sh}\lambda_m\frac{b}{a}}+\lambda_m\frac{b}{a}\frac{\tilde{C}_{1m}}{\operatorname{sh}\lambda_m\frac{b}{a}}\right].$$

由$V_y|_{y=b}=0$得

$$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^n 2\frac{a}{b}\frac{\alpha_n}{\lambda_m}\left[\frac{1}{1+\left(\frac{\alpha_n}{\lambda_m}\frac{a}{b}\right)^2}-\frac{1}{1+\left(\frac{\gamma_n}{\lambda_m}\frac{a}{b}\right)^2}\right]a_n\\
&+\sum_{n=1}^{\infty}(-1)^{m+n+1}10(1-\nu)\frac{a}{b}\left(\frac{a}{h}\right)^2\frac{\alpha_n}{\lambda_m\left(1+\left(\frac{\gamma_n}{\lambda_m}\frac{b}{a}\right)^2\right)}b_n\\
&+(-1)^{m+1}\left(\frac{a}{h}\right)^2\frac{10(1-\nu)}{\lambda_m\left(1+\left(\frac{\gamma_0}{\lambda_m}\frac{a}{b}\right)^2\right)}c_0+\sum_{n=1}^{\infty}(-1)^{n+m+1}2(1-\nu)\\
&\cdot\left[\frac{5\left(\frac{a}{h}\right)^2+\left(\frac{a}{b}\alpha_n\right)^2}{\lambda_m\left(1+\left(\frac{\gamma_n}{\lambda_m}\frac{a}{b}\right)^2\right)}-\frac{\left(\frac{a}{b}\alpha_n\right)^2}{\lambda_m\left(1+\left(\frac{\alpha_n}{\lambda_m}\frac{a}{b}\right)^2\right)}\right]c_n\\
&+\frac{5(1-\nu)a^2\lambda_m^2}{\beta_m h^2}\operatorname{cth}\beta_m\frac{b}{a}d_m\\
&+(1-\nu)\lambda_m\left[\left(\lambda_m^2+5\left(\frac{a}{h}\right)^2\right)\frac{\operatorname{cth}\beta_m\frac{b}{a}}{\beta_m}-\lambda_m\operatorname{cth}\lambda_m\frac{b}{a}\right]e_m\\
&-\frac{5(1-\nu)a^2\lambda_m^2}{\beta_m h^2\operatorname{sh}\beta_m\frac{b}{a}}f_m-(1-\nu)\lambda_m\left[\frac{\lambda_m^2+5\left(\frac{a}{h}\right)^2}{\beta_m\operatorname{sh}\beta_m\frac{b}{a}}-\frac{\lambda_m}{\operatorname{sh}\lambda_m\frac{b}{a}}\right]g_m\\
&=-2\lambda_m^3\tilde{C}_{1m}\qquad(m=1,2,3,\cdots)\end{aligned}\tag{2.24}$$

由$M_{xy}|_{y=b}=0$得

$$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^n\frac{h^2\alpha_n}{10ab}\left(\left(\frac{\alpha_n}{\gamma_n}\right)^2-1\right)a_n+\sum_{n=1}^{\infty}(-1)^n\frac{1-\nu}{2}\frac{a}{b}\alpha_n\left(\left(\frac{\alpha_n}{\gamma_n}\right)^2-1\right)b_n\\
&+\frac{1-\nu}{2}c_0+\sum_{n=1}^{\infty}(-1)^n\frac{1-\nu}{2}\left[1+\left(\frac{\alpha_n}{\gamma_n}\right)^2+\frac{h^2\alpha_n^2}{5b^2}\left(\left(\frac{\alpha_n}{\gamma_n}\right)^2-1\right)\right]c_n\\
&+\frac{1-\nu}{2}\beta_0\operatorname{cth}\beta_0\frac{b}{a}e_0-\frac{1-\nu}{2}\frac{\beta_0}{\operatorname{sh}\beta_0\frac{b}{a}}g_0-\frac{a}{b}(1-\nu)k_2=0\end{aligned}\tag{2.25a}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^{n+1}\alpha_n\frac{a}{b}\left[\frac{2\lambda_m^2(1-\nu)}{\left(\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2\right)^2}-\frac{h^2}{5b^2}\left(\frac{2\alpha_n^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}\right.\right.\\
&\left.\left.-\frac{\alpha_n^2+\gamma_n^2}{\lambda_m^2+\left(\gamma_n\frac{a}{b}\right)^2}\right)\right]a_n+\sum_{n=1}^{\infty}(-1)^{m+n}(1-\nu)\alpha_n\left(\frac{a}{b}\right)^3\left(\frac{\alpha_n^2+\gamma_n^2}{\lambda_m^2+\left(\gamma_n\frac{a}{b}\right)^2}\right.\end{aligned}$$

$$-\frac{2\alpha_n^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}\Bigg)b_n+(-1)^m(1-\nu)\left(\frac{a}{b}\right)^2\frac{\gamma_0^2}{\lambda_m^2+\left(\gamma_0\frac{a}{b}\right)^2}c_0$$

$$+\sum_{n=1}^{\infty}(-1)^{m+n}\left(\frac{a}{b}\right)^2(1-\nu)\cdot\Bigg[\left(1+\frac{h^2\alpha_n^2}{5b^2}\right)\frac{\alpha_n^2+\gamma_n^2}{\lambda_m^2+\left(\gamma_n\frac{a}{b}\right)^2}$$

$$-\frac{2\alpha_n^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}\left(\frac{(1-\nu)\lambda_m^2}{\lambda_m^2+\left(\alpha_n\frac{a}{b}\right)^2}+\frac{h^2\alpha_n^2}{5b^2}\right)\Bigg]c_n+(1-\nu)\lambda_m\Bigg[\lambda_m\mathrm{cth}\lambda_m\frac{b}{a}$$

$$-\frac{\lambda_m^2+\beta_m^2}{2\beta_m}\mathrm{cth}\beta_m\frac{b}{a}\Bigg]d_m-\frac{1-\nu}{2}\Bigg[(1-\nu)\frac{b}{a}\lambda_m^2\frac{1}{\left(\mathrm{sh}\lambda_m\frac{b}{a}\right)^2}$$

$$-\lambda_m\left(1-\nu+\frac{2h^2\lambda_m^2}{5a^2}\right)\mathrm{cth}\,\lambda_m\frac{b}{a}+\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\lambda_m^2+\beta_m^2}{\beta_m}$$

$$\cdot\,\mathrm{cth}\beta_m\frac{b}{a}\Bigg]e_m-(1-\nu)\lambda_m\Bigg[\frac{\lambda_m^2+\beta_m^2}{2\beta_m}\frac{1}{\mathrm{sh}\beta_m\frac{b}{a}}$$

$$-\frac{\lambda_m}{\mathrm{sh}\lambda_m\frac{b}{a}}\Bigg]f_m-\frac{1-\nu}{2}\Bigg[(1-\nu)\frac{b}{a}\lambda_m^2\frac{\mathrm{cth}\,\lambda_m\frac{b}{a}}{\mathrm{sh}\lambda_m\frac{b}{a}}$$

$$-\lambda_m\left(1-\nu+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{1}{\mathrm{sh}\,\lambda_m\frac{b}{a}}+\left(1+\frac{h^2\lambda_m^2}{5a^2}\right)\frac{\lambda_m^2+\beta_m^2}{\beta_m\mathrm{sh}\beta_m\frac{b}{a}}\Bigg]g_m$$

$$=-\lambda_m^2\left[(1-\nu)\tilde{B}_{1m}+\left(1-\nu+\frac{2h^2\lambda_m^2}{5a^2}\right)\tilde{C}_{1m}\right]\qquad(m=1,2,3,\cdots)\tag{2.25b}$$

至此，由(2.18)，我们得到了待定常数 a_n，b_n，c_n，d_m，e_m，f_m，g_m 所满足的方程(2.19)～(2.25)。注意到在由$V_x|_{x=a}=V_y|_{y=0,b}=0$得到(2.20)，(2.22)，(2.24)时采用了正弦展开方法，因此，需要在角点$(a,0)$，(a,b)处引入两个补充方程使得角点处的剪力为零。由此得如下两个方程．

$$\left.\begin{aligned}&\sum_{n=1}^{\infty}-\left(\frac{a}{b}\right)\alpha_n b_n-c_0-\sum_{n=1}^{\infty}c_n+\sum_{m=1}^{\infty}(-1)^m\lambda_m f_m+g_0+\sum_{m=1}^{\infty}(-1)^m g_m=0\\&\sum_{n=1}^{\infty}(-1)^n\left(\frac{a}{b}\right)\alpha_n b_n+c_0+\sum_{n=1}^{\infty}(-1)^n c_n+\sum_{m=1}^{\infty}(-1)^m\lambda_m d_m+e_0+\sum_{m=1}^{\infty}(-1)^m e_m=0\end{aligned}\right\}\tag{2.26}$$

(2.19)～(2.26)构成了 $a_n,b_n,c_n,d_m,e_m,f_m,g_m$ 所满足的无穷维线性代数方程组．求解这个方程组，得到$a_n,b_n,c_n,c_m,d_m,f_m,g_m$，从而求得边值问题(2.1)～(2.3)的解。

另外，为了考察悬臂板的非对称弯曲，我们分析上述矩形板在点(a,y_0)处受集中载荷P作用下的弯曲．如图 2 所示．此时，$V_x|_{x=a}=P\delta(y-y_0)$，利用三角级数展开方法，$V_x|_{x=a}$可表示为

$$V_x|_{x=a}=\frac{P}{a}\sum_{n=1}^{\infty}\left(2\frac{a}{b}\sin\alpha_n\frac{y_0}{b}\right)\sin\alpha_n\frac{y}{b}$$

显然，该问题的解可表示为问题(Ⅱ)～(Ⅵ)的适当组合。此时取$A_1=Pa^2/D$，$A_2=P$，$A_3=Pa/D$。由上面相同的方法，可得$a_n, b_n, c_n, d_m, e_m, f_m, g_m$所满足的方程，从而得到该问题的精确解。

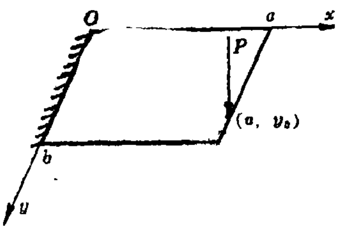

图 2

三、数值结果及分析

实际求解(2.19)～(2.26)时，待定常数只能取有限项，为此，首先要确定取多少项才能得到满意的结果，在薄板范围内，对$\nu=0.3$的正方形板($a/b=1.0$)和长方形板($a/b=0.5$)，待定常数$a_n, b_n, c_n, e_m, d_m, f_m, g_m$均取25项，所得结果列于表1。由此表可见，取25项时，所得结果和已有结果吻合很好，故在以后的计算中均取25项为标准。

表 1 均布载荷作用下薄板范围内的解 ($q=\text{const}$, $P=0$)

$\frac{a}{b}$	$\frac{h}{a}$	$w(a,0)/A_1$	$w\left(a,\frac{b}{4}\right)/A_1$	$w\left(a,\frac{b}{2}\right)/A_1$	$M_x\left(0,\frac{b}{4}\right)/A_2$	$M_x\left(0,\frac{b}{2}\right)/A_2$
1.0	0.01	0.1259	0.1272	0.1278	−0.5250	−0.5319
	0.05	0.1275	0.1288	0.1293	−0.5305	−0.5419
	0.10	0.1297	0.1310	0.1315	−0.5278	−0.5522
E. S.*		0.1293	0.1306	0.1310	−0.5335	−0.5356
F. E. S.		0.1271	0.1285	0.1291	−0.5276	−0.5309
0.5	0.01	0.1238	0.1267	0.1274	−0.5141	−0.5205
	0.05	0.1260	0.1285	0.1290	−0.5194	−0.5205
	0.10	0.1311	0.1328	0.1330	−0.5296	−0.5191
E. S.		0.1254	0.1278	0.1284	−0.5139	−0.5105

其中，E. S. 和 F. E. S. 分别表示薄板理论的精确解和有限元解[5]。

图3，图4示出了均布载荷作用下悬臂矩形板自由边$y=0$的挠度分布，而最大挠度$w(a, b/2)$随厚度比h/a的变化列于图5中，表2，表3分别列出了自由边$x=a$的挠度和固定边$x=0$的弯矩分布。可见，随着厚度的增加，剪切效应越来越显著，当$h/a>0.1$时，Kirchhoff-Love薄板理论所得结果将引起较大误差，此时应考虑剪切变形对弯曲的影响。

表 2 均布载荷作用下自由边$x=a$的挠度$w(a,y)/A_1$ ($q=\text{const}$, $P=0$)

$\frac{a}{b}$	$\frac{h}{a}$ \ $\frac{y}{b}$	0.0	0.1	0.2	0.3	0.4	0.5
1.0	0.01	0.1259	0.1265	0.1270	0.1274	0.1277	0.1278
	0.10	0.1297	0.1302	0.1307	0.1311	0.1314	0.1315
	0.20	0.1357	0.1363	0.1368	0.1372	0.1374	0.1375
	0.30	0.1446	0.1451	0.1456	0.1460	0.1462	0.1463
	0.40	0.1565	0.1569	0.1573	0.1577	0.1579	0.1580
0.5	0.01	0.1238	0.1253	0.1263	0.1270	0.1273	0.1274
	0.10	0.1311	0.1320	0.1326	0.1329	0.1330	0.1330
	0.20	0.1483	0.1486	0.1486	0.1484	0.1482	0.1481
	0.30	0.1744	0.1742	0.1738	0.1732	0.1727	0.1725
	0.40	0.2091	0.2087	0.2080	0.2072	0.2065	0.2062

表 3 均布载荷作用下固定边$x=0$的弯矩$M_x(0,y)/\Lambda_2$ ($q=$const, $P=0$)

a/b	h/a \ y/b	0.0	0.1	0.2	0.3	0.4	0.5	总弯矩
1.0	0.01	0.0	−0.4981	−0.5289	−0.5221	−0.5256	−0.5319	−0.4958
	0.10	0.0	−0.4767	−0.5388	−0.5181	−0.5317	−0.5522	−0.4992
	0.20	0.0	−0.4764	−0.5304	−0.5008	−0.5240	−0.5548	−0.4996
	0.30	0.0	−0.4869	−0.5277	−0.4875	−0.5127	−0.5479	−0.4997
	0.40	0.0	−0.4958	−0.5273	−0.4792	−0.5041	−0.5411	−0.4998
0.5	0.01	0.0	−0.5153	−0.5269	−0.5061	−0.5100	−0.5205	−0.4985
	0.10	0.0	−0.5153	−0.5388	−0.5227	−0.5211	−0.5257	−0.4999
	0.20	0.0	−0.4851	−0.5427	−0.5428	−0.5427	−0.5454	−0.5000
	0.30	0.0	−0.4604	−0.5395	−0.5562	−0.5629	−0.5666	−0.5000
	0.40	0.0	−0.4419	−0.5365	−0.5651	−0.5780	−0.5832	−0.5000

图6～图8列出了正方形板在点(a, $3a/4$)处受集中力P作用下的挠度分布。其中，实线，虚线和点线分别表示$h/a=0.1$，0.3和0.5时的挠度曲线，而"·"为Kirchhoff-Love 薄板理论的解[6]。

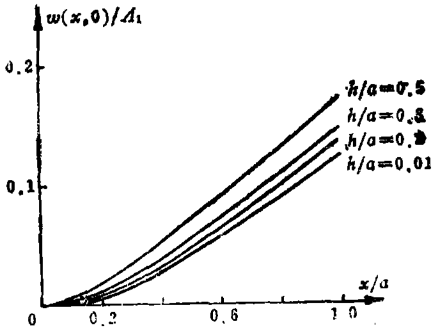

图3 $y=0$边的挠度w分布($a/b=1.0$, $q=$const, $P=0$)

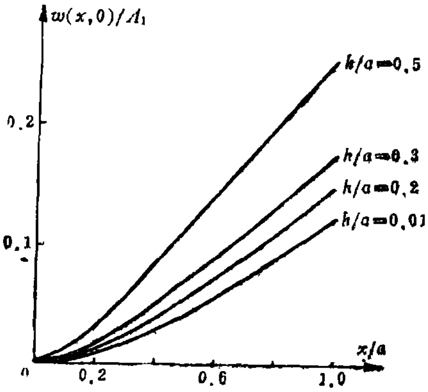

图4 $y=0$边的挠度w分布($a/b=0.5$, $q=$const, $P=0$)

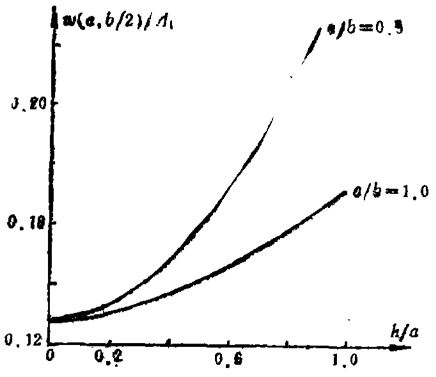

图5 w随h/a的变化($q=$const, $P=0$)

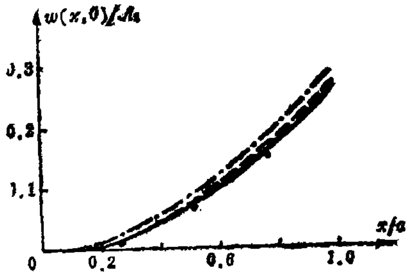

图6 $y=0$边的挠度w分布($a/b=1.0$, $P=$const, $q=0$)

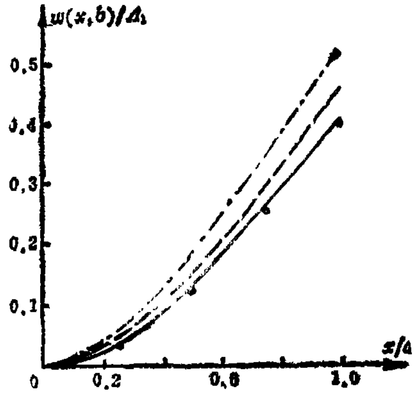

图7 $y=b$边的挠度w分布 ($a/b=1.0$, $P=\text{const}$, $q=0$)

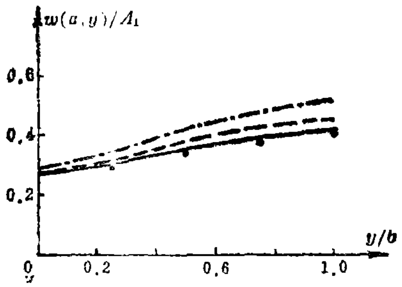

图8 $x=a$边的挠度w分布 ($a/b=1.0$, $P=\text{const}$, $q=0$)

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Bending of Cantilever Rectaugular Plates with the Effect of Transverse Shear Deformation

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Abstract

On the basis of Reissner's theory, the exact solutions of bending of cantilever rectangular plates are obtained by means of the concept of generalized simply-supported boundary. From the results obtained, it can be found that the method is valid.

Key words Reissner's theory, cantilever rectangular plate, symmetrical and asymmetrical bending, generalized simply-supported boundary, exact solution