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点简支正交各向异性矩形薄板弯曲的精确解*

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摘　要

本文利用迭加原理，给出了点简支正交各向异性矩形薄板弯曲问题的封闭的级数式解答．简支点的位置和横向载荷的分布均可任意．用本文的级数解给出的算例与以往的数值解是十分一致的．

关键词　精确解　点支承　正交各向异性矩形薄板

一、引　言

用铰或柱支承的矩形板在横向载荷作用下的弯曲问题，在土建、机械等工程问题中是十分常见的．由于点支处的边界条件与其余自由边处截然不同，使得求解此类问题在数学上存在很大的困难．从历史上说，最早提出此类问题并试图解决的可能是Thorne[1]，可惜他得到的实际上是周边固支板的配点解，而非点简支板的解．Yoshihi和Kawai[2]提出了用Raileigh-Ritz法求解点简支板所适用的一些函数，但没有给出任何数值结果．1978年，Rajaiah和Rao[3]首次给出了等间距点简支方板的配位解．而后，吴兹潜、张佑啟和范寿昌[4]用样条有限条法给出了此类问题的数值解．但是，在寻求问题的解析解方面至今仍无结果．

本文讨论了点简支正交各向异性薄板的小挠度弯曲问题，简支点的位置和横向载荷均可任意．给出了精确满足微分方程和全部边界条件的封闭的级数解．用本文的级数解给出的算例与文献[3]、[4]的数值解是十分一致的．

二、微分方程及定解条件

考虑图1所示的等厚度正交各向异性薄板，a，b和h为板的几何尺寸．整块板简支在若干点上，其余边界都是自由的．计算时假定：（1）简支点的位置是任意的；（2）横向载荷是任意的；（3）弹性主向与所选坐标轴方向一致；（4）采用Kirchhoff直法线假设，只考虑小挠度弯曲变形．

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2.1 微分方程

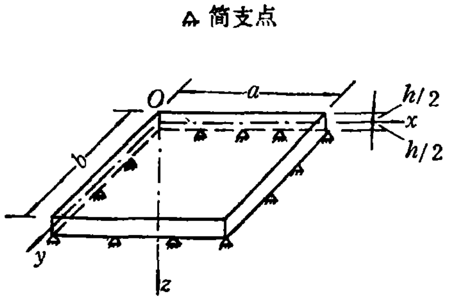


图 1

在直角坐标系中，板挠曲面的微分方程为[5]

$$D_x\frac{\partial^4 w}{\partial x^4}+2H\frac{\partial^4 w}{\partial x^2\partial y^2}+D_y\frac{\partial^4 w}{\partial y^4}=q(x,\ y)\qquad(2.1)$$

式中 $D_x=\frac{E_x h^3}{12(1-\mu_x\mu_y)}$，$D_y=\frac{E_y h^3}{12(1-\mu_x\mu_y)}$，$H=D+2D_{xy}$

$D=D_x\mu_y=D_y\mu_x$，$D_{xy}=Gh^3/12$

E_x，E_y，μ_x，μ_y，G分别是弹性主向上的杨氏模量、泊松比和剪切弹性模量。

板的内力可用挠度函数表示如下

$$\left.\begin{aligned}&M_x=-D_x\left(\frac{\partial^2 w}{\partial x^2}+\mu_y\frac{\partial^2 w}{\partial y^2}\right),\ M_y=-D_y\left(\frac{\partial^2 w}{\partial y^2}+\mu_x\frac{\partial^2 w}{\partial x^2}\right)\\&M_{xy}=-2D_{xy}\frac{\partial^2 w}{\partial x\partial y}\\&Q_x=-\left(D_x\frac{\partial^3 w}{\partial x^3}+H\frac{\partial^3 w}{\partial x\partial y^2}\right),\ Q_y=-\left(D_y\frac{\partial^3 w}{\partial y^3}+H\frac{\partial^3 w}{\partial y\partial x^2}\right)\end{aligned}\right\}\qquad(2.2)$$

为了利用边界条件得到方程(2.1)的解答，导出等效边界剪力的表达式如下：

$$\left.\begin{aligned}&V_x=-\left[D_x\frac{\partial^3 w}{\partial x^3}+(H+2D_{xy})\frac{\partial^3 w}{\partial x\partial y^2}\right]\\&V_y=-\left[D_y\frac{\partial^3 w}{\partial y^3}+(H+2D_{xy})\frac{\partial^3 w}{\partial y\partial x^2}\right]\end{aligned}\right\}\qquad(2.3)$$

2.2 定解条件

挠度函数w除满足方程(2.1)外，还必须满足如下条件：

在板的四边，有

$$\left.\begin{aligned}&(M_x)_{x=0}=0,\ (V_x)_{x=0}=V_x(0,y),\ (M_x)_{x=a}=0,\ (V_x)_{x=a}=V_x(a,y)\\&(M_y)_{y=0}=0,\ (V_y)_{y=0}=V_y(x,0),\ (M_y)_{y=b}=0,\ (V_y)_{y=b}=V_y(x,b)\end{aligned}\right\}\qquad(2.4a)$$

在简支点处，有

$$w=0\qquad(2.4b)$$

在自由角点处，有

$$\partial^2 w/\partial x\partial y=0\qquad(2.4c)$$

在简支点处，支承点的反力是突变力，假定在$x=0$边上有N_{1x}个简支点(不包括角点)，则该边上的剪力可表示为

$$V_x(0,y)=\sum_{i=1}^{N_{1x}}R_{ix}\delta(y-y_i)\qquad(2.5a)$$

式中R_{ix}是支承点的反力；y_i是支承点的坐标；δ是Dirichlet函数。

同理，$V_x(a,y)$，$V_y(x,0)$，$V_y(x,b)$可表示如下：

$$V_x(a,\ y)=\sum_{i=1+N_{1x}}^{N_{1x}+N_{2x}} R_{ix}\delta(y-y_i) \tag{2.5b}$$

$$V_y(x,\ 0)=\sum_{i=1}^{N_{1y}} R_{iy}\delta(x-x_i) \tag{2.5c}$$

$$V_y(x,\ b)=\sum_{i=1+N_{1y}}^{N_{1y}+N_{2y}} R_{iy}\delta(x-x_i) \tag{2.5d}$$

式中N_{2x},N_{1y},N_{2y}分别为$x=a$，$y=0$，$y=b$边上的简支点数(均不包括角点)。

利用Fourier级数，可将$\delta(x-x_i)$，$\delta(y-y_i)$展为级数如下：

$$\left.\begin{aligned}\delta(x-x_i)&=\frac{2}{a}\sum_{m=1}^{\infty}\sin\frac{m\pi x_i}{a}\sin\frac{m\pi x}{a}\\ \delta(y-y_i)&=\frac{2}{b}\sum_{n=1}^{\infty}\sin\frac{n\pi y_i}{a}\sin\frac{n\pi y}{a}\end{aligned}\right\} \tag{2.6}$$

三、问 题 的 解 答

3.1 解的形式

本文用迭加法求解满足方程(2.1)和定解条件(2.4)的挠度函数w，其形式取为

$$w=\sum_{i=1}^{4} w_i \tag{3.1}$$

式中，w_1为四边简支矩形板受分布和集中载荷作用的Navier解；w_2为$x=0$，$x=a$边简支，另两边任意的Levy解；w_3为$y=0$，$y=b$边简支，另两边任意的Levy解；w_4为角点支座沉陷时的解。

以下就解$w_i(i=1,2,3,4)$的具体形式进行讨论。

分布载荷和集中载荷作用下的四边简支矩形板，其挠度函数

$$w_1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(C_{mn}+\sum_{k=1}^{K}C_{mn}^{k}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{3.2}$$

式中
$$C_{mn}=\frac{4}{D_{mn}}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$C_{mn}^{k}=\frac{4P_k}{D_{mn}}\sin\frac{m\pi x_k}{a}\sin\frac{n\pi y_k}{b}$$

$$D_{mn}=\pi^4ab[D_x(m/a)^4+2H(mn/ab)^2+D_y(n/b)^4]$$

$q(x,y)$为分布载荷的集度；P_k为第k个集中力($k=1,2,\cdots,K$)；(x_k,y_k)为P_k作用点的坐标。

$x=0$，$x=a$边简支，另两边任意的Levy解取为

$$w_2=\sum_{m=1}^{\infty}\sum_{j=1}^{4}A_{jm}Y_{jm}(y)\sin\frac{m\pi x}{a} \tag{3.3}$$

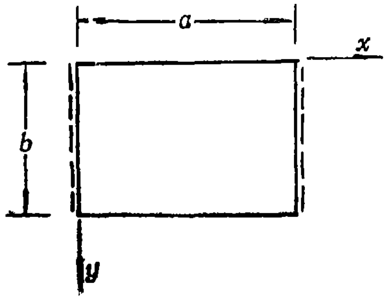


图 2

根据H^2与D_xD_y之间的三种不同的关系，$Y_{jm}(y)$分别为

（1） $H^2>D_xD_y$

$$\left.\begin{aligned}&Y_{1m}(y)=\mathrm{ch}\beta_{1m}(2y/b-1),\ Y_{2m}(y)=\mathrm{ch}\beta_{2m}(2y/b-1)\\&Y_{3m}(y)=\mathrm{sh}\beta_{1m}(2y/b-1),\ Y_{4m}(y)=\mathrm{sh}\beta_{2m}(2y/b-1)\end{aligned}\right\} \tag{3.4a}$$

式中

$$\beta_{im}=\frac{m\pi b}{2a}r_{iy},\quad r_{iy}=\sqrt{\frac{H}{D_y}+(-1)^{i-1}\sqrt{\left(\frac{H}{D_y}\right)^2-\frac{D_x}{D_y}}}\quad (i=1,2) \tag{3.4b}$$

（2） $H^2=D_xD_y$

$$\left.\begin{aligned}&Y_{1m}(y)=\mathrm{ch}\beta_m(2y/b-1),\ Y_{2m}(y)=\beta_m(2y/b-1)\mathrm{sh}\beta_m(2y/b-1)\\&Y_{3m}(y)=\mathrm{sh}\beta_m(2y/b-1),\ Y_{4m}(y)=\beta_m(2y/b-1)\mathrm{ch}\beta_m(2y/b-1)\end{aligned}\right\} \tag{3.4c}$$

式中 $$\beta_m=\frac{m\pi b}{2a}r_y,\quad r_y=\sqrt{\frac{H}{D_y}} \tag{3.4d}$$

（3） $H^2<D_xD_y$

$$\left.\begin{aligned}&Y_{1m}(y)=\mathrm{ch}\beta_{1m}(2y/b-1)\cos\beta_{2m}(2y/b-1)\\&Y_{2m}(y)=\mathrm{sh}\beta_{1m}(2y/b-1)\sin\beta_{2m}(2y/b-1)\\&Y_{3m}(y)=\mathrm{ch}\beta_{1m}(2y/b-1)\sin\beta_{2m}(2y/b-1)\\&Y_{4m}(y)=\mathrm{sh}\beta_{1m}(2y/b-1)\cos\beta_{2m}(2y/b-1)\end{aligned}\right\} \tag{3.4e}$$

式中 $$\beta_{im}=\frac{m\pi b}{2a}r_{iy},\quad r_{iy}=\sqrt{\frac{1}{2}\left(\sqrt{\frac{D_x}{D_y}}+(-1)^{i-1}\frac{H}{D_y}\right)}\quad (i=1,2) \tag{3.4f}$$

$y=0$，$y=b$边简支，另两边任意的Levy解取为

$$w_3=\sum_{n=1}^{\infty}\sum_{j=1}^{4}B_{jn}X_{jn}(x)\sin\frac{n\pi y}{b} \tag{3.5}$$

上式中$X_{jn}(x)$的具体形式与$Y_{jm}(y)$相似，将(3.4)诸式中的$Y_{jm}(y)$，β_{im}，β_m，r_{iy}，r_y，D_x，D_y，b，y分别换为$X_{jn}(x)$，α_{in}，α_n，r_{ix}，r_x，D_y，D_x，a，x，即得到$X_{jn}(x)$，α_{in}，α_n，r_{ix}，r_x；$j=1,2,3,4$；$i=1,2$。

角点支座沉陷时的解为

$$w_4=k_1+k_2x+k_3y+k_4xy \tag{3.6}$$

3.2 定解条件的满足

将(3.1)式中的挠度函数w代入定解条件(2.4)，并把一些非正弦函数展为正弦级数，整理后得

$$
\left.\begin{aligned}
&\sum_{j=1}^{2} a_{jm}^{(1)} A_{jm}=0,\ \sum_{j=3}^{4} a_{jm}^{(2)} A_{jm}=0 \qquad (m=1,2,\cdots,\infty)\\
&\sum_{j=1}^{2} b_{jn}^{(3)} B_{jn}=0,\ \sum_{j=3}^{4} b_{jn}^{(4)} B_{jn}=0 \qquad (n=1,2,\cdots,\infty)\\
&\sum_{j=1}^{4} a_{jm}^{(5)} A_{jm}+\sum_{n=1}^{\infty}\sum_{j=1}^{4} b_{jn}^{(5)} B_{jn}+\sum_{k=1}^{K} p_{km}^{(5)} P_k\\
&\qquad +\frac{2}{a}\left(\frac{b}{\pi}\right)^3\cdot\sum_{i=1}^{N_{1y}}\sin\frac{m\pi x_i}{a}R_{iy}+U_m^{(5)}=0 \qquad (m=1,2,\cdots,\infty)\\
&\sum_{j=1}^{4} a_{jm}^{(6)} A_{jm}+\sum_{n=1}^{\infty}\sum_{j=1}^{4} b_{jn}^{(6)} B_{jn}+\sum_{k=1}^{K} p_{km}^{(6)} P_k\\
&\qquad +\frac{2}{a}\left(\frac{b}{\pi}\right)^3\sum_{i=1+N_{1y}}^{N_{1y}+N_{2y}}\sin\frac{m\pi x_i}{a}R_{iy}+U_m^{(6)}=0 \qquad (m=1,2,\cdots,\infty)\\
&\sum_{m=1}^{\infty}\sum_{j=1}^{4} a_{jm}^{(7)} A_{jm}+\sum_{j=1}^{4} b_{jn}^{(7)} B_{jn}+\sum_{k=1}^{K} p_{kn}^{(7)} P_k\\
&\qquad +\frac{2}{b}\left(\frac{a}{\pi}\right)^3\sum_{i=1}^{N_{1x}}\sin\frac{n\pi y_i}{b}R_{ix}+U_n^{(7)}=0 \qquad (n=1,2,\cdots,\infty)\\
&\sum_{m=1}^{\infty}\sum_{j=1}^{4} a_{jm}^{(8)} A_{jm}+\sum_{j=1}^{4} b_{jn}^{(8)} B_{jn}+\sum_{k=1}^{K} p_{kn}^{(8)} P_k\\
&\qquad +\frac{2}{b}\left(\frac{a}{\pi}\right)^3\sum_{i=1+N_{1x}}^{N_{1x}+N_{2x}}\sin\frac{n\pi y_i}{b}R_{ix}+U_n^{(8)}=0 \qquad (n=1,2,\cdots,\infty)\\
&\sum_{m=1}^{\infty}\sum_{j=1}^{4} a_{jm}^{(s)} A_{jm}+\sum_{n=1}^{\infty}\sum_{j=1}^{4} b_{jn}^{(s)} B_{jn}+\sum_{k=1}^{K} p_k^{(s)} P_k+w_4(x_l,y_l)+U^{(s)}=0\\
&\qquad\qquad (s=l+8,\ l=1,2,\cdots,L)\\
&\sum_{m=1}^{\infty}\sum_{j=1}^{4} a_{jm}^{(t)} A_{jm}+\sum_{n=1}^{\infty}\sum_{j=1}^{4} b_{jn}^{(t)} B_{jn}+\sum_{k=1}^{K} p_k^{(s)} P_k+k_4+U^{(t)}=0\\
&\qquad\qquad (t=f+L+8,\ f=1,2,\cdots,N_f)
\end{aligned}\right\} \tag{3.7}
$$

式中，L为简支点总数；N_f为自由角点数。

(3.7)式为一无穷维的代数方程组,为确定挠度函数w中的待定系数($A_{jm},B_{jn},k_j,j=1,2,3,4$)和简支点的反力，对$m$和$n$分别取$M$和$N$项，则可得$4M+4N+N_f+L$个方程，共有$4M+4N+N_f+L$个待定常数，构成封闭解。

根据H^2与D_xD_y的三种关系，(3.7)式中的系数可分别确定如下：

(1) $H^2>D_xD_y$

$$a_{1m}^{(1)}=\lambda_{1y}Y_{1m}(b),\ \ a_{2m}^{(1)}=\lambda_{2y}Y_{2m}(b) \tag{3.8a}$$

$$a_{3m}^{(2)}=\lambda_{1y}Y_{3m}(b),\ \ a_{4m}^{(2)}=\lambda_{2y}Y_{4m}(b) \tag{3.8b}$$

$$b_{1n}^{(3)}=\lambda_{1x}X_{1n}(a),\ \ b_{2n}^{(3)}=\lambda_{2x}X_{2n}(a) \tag{3.8c}$$

$$b_{3n}^{(4)}=\lambda_{1x}X_{3n}(a),\ \ b_{4n}^{(4)}=\lambda_{2x}X_{4n}(a) \tag{3.8d}$$

式中 $\lambda_{iy}=r_{iy}^2-\mu_x,\ \lambda_{ix}=r_{ix}^2-\mu_y \quad (i=1,2)$

$$\left.\begin{aligned}
&a_{1m}^{(5)}=-\xi_{1y}Y_{3m}(b)(mb/a)^3,\ \ a_{2m}^{(5)}=-\xi_{2y}Y_{4m}(b)(mb/a)^3\\
&a_{3m}^{(5)}=\xi_{1y}Y_{1m}(b)(mb/a)^3,\ \ a_{4m}^{(5)}=\xi_{2y}Y_{2m}(b)(mb/a)^3\\
&b_{1n}^{(5)}=\eta_{1x}S_{1m}n^3,\ \ b_{2n}^{(5)}=\eta_{2x}S_{2m}n^3\\
&b_{3n}^{(5)}=\eta_{1x}T_{1m}n^3,\ \ b_{4n}^{(5)}=\eta_{2x}T_{2m}n^3\\
&p_{km}^{(5)}=-\sum_{n=1}^{\infty}\frac{4E_{mn}}{D_{mn}}\sin\frac{m\pi x_k}{a}\sin\frac{n\pi y_k}{b}\\
&U_m^{(5)}=-\sum_{n=1}^{\infty}E_{mn}C_{mn}
\end{aligned}\right\} \tag{3.8e}$$

式中
$$\left.\begin{aligned}
&\xi_{iy}=r_{iy}[D_y r_{iy}^2-(H+2D_{xy})]\\
&\eta_{ix}=r_{ix}^2(H+2D_{xy})-D_y\\
&S_{im}=-\frac{2m\pi}{4\alpha_{in}^2+(m\pi)^2}X_{in}(a)(\cos m\pi-1)\\
&T_{im}=-\frac{2m\pi}{4\alpha_{in}^2+(m\pi)^2}X_{i+2\,n}(a)(\cos m\pi+1)\\
&E_{mn}=n[D_y n^2+(H+2D_{xy})(mb/a)^2]\\
&\qquad\qquad (i=1,\ 2)
\end{aligned}\right\} \tag{3.8f}$$

$$\left.\begin{aligned}
&a_{jm}^{(6)}=-a_{jm}^{(5)}\quad (j=1,\ 2),\ \ a_{jm}^{(6)}=a_{jm}^{(5)}\quad (j=3,\ 4)\\
&b_{jn}^{(6)}=b_{jn}^{(5)}\cos n\pi\qquad (j=1,2,3,4)\\
&p_{km}^{(6)}=-\sum_{n=1}^{\infty}\frac{4E_{mn}}{D_{mn}}\sin\frac{m\pi x_k}{a}\sin\frac{n\pi y_k}{b}\cos n\pi\\
&U_m^{(6)}=-\sum_{n=1}^{\infty}E_{mn}C_{mn}\cos n\pi
\end{aligned}\right\} \tag{3.8g}$$

在(3.8e)和(3.8g)两式中，分别用 $a_{jm}^{(7)}$，$b_{jn}^{(7)}$，$p_{kn}^{(7)}$，$U_n^{(7)}$，$X_{jn}(a)$，$Y_{jm}(b)$，ξ_{ix}，η_{iy}，S_{in}，T_{in}，F_{mn}，β_{im}，r_{ix}，r_{iy}，a/b，D_x，n和m代替$b_{jn}^{(5)}$，$a_{jm}^{(5)}$，$p_{km}^{(5)}$，$U_m^{(5)}$，$Y_{jm}(b)$，$X_{jn}(a)$，ξ_{iy}，η_{ix}，S_{im}，T_{im}，E_{mn}，α_{in}，r_{iy}，r_{ix}，b/a，D_y，m和n，即得到$a_{jm}^{(7)}$，$b_{jn}^{(7)}$，$p_{kn}^{(7)}$，$U_n^{(7)}$，ξ_{ix}，η_{iy}，S_{in}，T_{in}和F_{mn}，$j=1,2,3,4$，$i=1,2$.

$$\left.\begin{aligned}
&a_{jm}^{(8)}=a_{jm}^{(7)}\cos m\pi \qquad (j=1,2,3,4)\\
&b_{jn}^{(8)}=-b_{jn}^{(7)} \quad (j=1,2),\quad b_{jn}^{(8)}=b_{jn}^{(7)} \quad (j=3,4)\\
&p_{kn}^{(8)}=-\sum_{m=1}^{\infty}\frac{4F_{mn}}{D_{mn}}\sin\frac{m\pi x_k}{a}\sin\frac{n\pi y_k}{b}\cos m\pi\\
&U_n^{(8)}=-\sum_{m=1}^{\infty}F_{mn}C_{mn}\cos m\pi
\end{aligned}\right\} \tag{3.8h}$$

$$\left.\begin{aligned}
&a_{jm}^{(s)}=Y_{jm}(y_l)\sin\frac{m\pi x_l}{a} \qquad\qquad (j=1,2,3,4)\\
&b_{jn}^{(s)}=X_{jn}(x_l)\sin\frac{n\pi y_l}{b}\\
&p_k^{(s)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{D_{mn}}\sin\frac{m\pi x_k}{a}\sin\frac{n\pi y_k}{b}\\
&\qquad\cdot\sin\frac{m\pi x_l}{a}\sin\frac{n\pi y_l}{b}\\
&U^{(s)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x_l}{a}\sin\frac{n\pi y_l}{b}\\
&\qquad\qquad (l=1,2,\cdots,L)
\end{aligned}\right\} \tag{3.8i}$$

$$\left.\begin{aligned}
&a_{jm}^{(t)}=Y'_{jm}(y_f)\frac{m\pi}{a}\cos\frac{m\pi x_f}{a} \qquad\qquad (j=1,2,3,4)\\
&b_{jn}^{(t)}=X'_{jn}(x_f)\frac{n\pi}{b}\cos\frac{n\pi y_f}{b}\\
&p_k^{(t)}=\frac{4\pi^2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn}{D_{mn}}\sin\frac{m\pi x_k}{a}\sin\frac{n\pi y_k}{b}\\
&\qquad\cdot\cos\frac{m\pi x_f}{a}\cos\frac{n\pi y_f}{b}\\
&U^{(t)}=\frac{\pi^2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}mn\cos\frac{m\pi x_f}{a}\cos\frac{n\pi y_f}{b}\\
&\qquad\qquad (f=1,2,\cdots,N_f)
\end{aligned}\right\} \tag{3.8j}$$

(2) $H^2=D_xD_y$

$$a_{1m}^{(1)}=\lambda_yY_{1m}(b),\quad a_{2m}^{(1)}=2r_y^2Y_{1m}(b)+\lambda_yY_{2m}(b) \tag{3.9a}$$

$$a_{3m}^{(2)}=\lambda_yY_{3m}(b),\quad a_{4m}^{(2)}=2r_y^2Y_{3m}(b)+\lambda_yY_{4m}(b) \tag{3.9b}$$

$$b_{1n}^{(3)}=\lambda_xX_{1n}(a),\quad b_{2n}^{(3)}=2r_x^2X_{1n}(a)+\lambda_xX_{2n}(a) \tag{3.9c}$$

$$b_{3n}^{(4)}=\lambda_xX_{3n}(a),\quad b_{4n}^{(4)}=2r_x^2X_{3n}(a)+\lambda_xX_{4n}(a) \tag{3.9d}$$

式中 $\lambda_y=r_y^2-\mu_x$, $\lambda_x=r_x^2-\mu_y$

$$\left.\begin{aligned}
&a_{1m}^{(5)}=2r_yD_{xy}Y_{3m}(b)(mb/a)^3\\
&a_{2m}^{(5)}=-2r_y[(H-D_{xy})Y_{3m}(b)-D_{xy}Y_{4m}(b)](mb/a)^3\\
&a_{3m}^{(5)}=-2r_yD_{xy}Y_{1m}(b)(mb/a)^3\\
&a_{4m}^{(5)}=2r_y[(H-D_{xy})Y_{1m}(b)-D_{xy}Y_{2m}(b)](mb/a)^3\\
&b_{1n}^{(5)}=(\eta_x-D_y)S_m\cdot n^3\\
&b_{2n}^{(5)}=[2\eta_xS_m+(\eta_x-D_y)R_m]n^3\\
&b_{3n}^{(5)}=(\eta_x-D_y)T_mn^3\\
&b_{4n}^{(5)}=[2\eta_xT_m+(\eta_x-D_y)Q_m]n^3
\end{aligned}\right\}\tag{3.9e}$$

式中

$$\left.\begin{aligned}
&\eta_x=(H+2D_{xy})r_x^2\\
&S_m=-2m\pi\Phi_mX_{1n}(a)(\cos m\pi-1)\\
&T_m=-2m\pi\Phi_mX_{3n}(a)(\cos m\pi+1)\\
&R_m=-2m\pi\alpha_n\Phi_m[8\alpha_n\Phi_mX_{4n}(a)-X_{3n}(a)](\cos m\pi-1)\\
&Q_m=-2m\pi\alpha_n\Phi_m[8\alpha_n\Phi_mX_{2n}(a)-X_{1n}(a)](\cos m\pi+1)\\
&\Phi_m=[4\alpha_n^2+(mn)^2]^{-1}
\end{aligned}\right\}\tag{3.9f}$$

在(3.9e,f)两式中,用$a_{jm}^{(7)}$, $b_{jn}^{(7)}$, S_n, T_n, R_n, Q_n, Φ_n, $X_{jn}(a)$, $Y_{jm}(b)$, β_m, η_y, r_x,r_y,D_x,a/b, n和m代替$b_{jn}^{(6)}$, $a_{jm}^{(6)}$, S_m, T_m, R_m, Q_m, Φ_m, $Y_{jm}(b)$, $X_{jn}(a)$, D_y, α_n, η_x, r_y, r_x, b/a, m和n, 即得到$a_{jm}^{(7)}$, $b_{jn}^{(7)}$, S_n, T_n, R_n, Q_n, Φ_n, η_y, $j=1,2,3,4$.

（3） $H^2<D_xD_y$

$$a_{1m}^{(1)}=\lambda_{1y}Y_{1m}(b)-\lambda_{2y}Y_{2m}(b),\quad a_{2m}^{(1)}=\lambda_{1y}Y_{2m}(b)+\lambda_{2y}Y_{1m}(b)\tag{3.10a}$$

$$a_{3m}^{(2)}=\lambda_{1y}Y_{3m}(b)+\lambda_{2y}Y_{4m}(b),\quad a_{4m}^{(2)}=\lambda_{1y}Y_{4m}(b)-\lambda_{2y}Y_{3m}(b)\tag{3.10b}$$

式中 $\lambda_{1y}=r_{1y}^2-r_{2y}^2-\mu_x$, $\lambda_{2y}=2r_{1y}r_{2y}$

$$b_{1n}^{(3)}=\lambda_{1x}X_{1n}(a)-\lambda_{2x}X_{2n}(a),\quad b_{2n}^{(3)}=\lambda_{1x}X_{2n}(a)+\lambda_{2x}X_{1n}(a)\tag{3.10c}$$

$$b_{3n}^{(4)}=\lambda_{1x}X_{3n}(a)+\lambda_{2x}X_{4n}(a),\quad b_{4n}^{(4)}=\lambda_{1x}X_{4n}(a)-\lambda_{2x}X_{3n}(a)\tag{3.10d}$$

式中 $\lambda_{1x}=r_{1x}^2-r_{2x}^2-\mu_y$, $\lambda_{2x}=2r_{1x}r_{2x}$

$$\left.\begin{aligned}
&a_{1m}^{(5)}=-[\xi_{1y}Y_{4m}(b)+\xi_{2y}Y_{3m}(b)](mb/a)^3\\
&a_{2m}^{(5)}=-[\xi_{1y}Y_{3m}(b)-\xi_{2y}Y_{4m}(b)](mb/a)^3\\
&a_{3m}^{(5)}=[\xi_{1y}Y_{2m}(b)-\xi_{2y}Y_{1m}(b)](mb/a)^3\\
&a_{4m}^{(5)}=[\xi_{1y}Y_{1m}(b)+\xi_{2y}Y_{2m}(b)](mb/a)^3\\
&b_{1n}^{(5)}=(\eta_{1x}S_m-\eta_{2x}T_m)n^3,\quad b_{2n}^{(5)}=(\eta_{1x}T_m+\eta_{2x}S_m)n^3\\
&b_{3n}^{(5)}=(\eta_{1x}Q_m+\eta_{2x}R_m)n^3,\quad b_{4n}^{(5)}=(\eta_{1x}R_m-\eta_{2x}Q_m)n^3
\end{aligned}\right\}\tag{3.10e}$$

式中

$$\left.\begin{aligned}
&\xi_{1y}=r_{1y}[(r_{1y}^2-3r_{2y}^2)D_y-(H+2D_{xy})]\\
&\xi_{2y}=r_{2y}[(r_{2y}^2-3r_{1y}^2)D_y+(H+2D_{xy})]\\
&\eta_{1x}=(H+2D_{xy})(r_{1x}^2-r_{2x}^2)-D_y\\
&\eta_{2x}=2(H+2D_{xy})r_{1x}r_{2x}\\
&S_m=\Psi_m[\Omega_m X_{1n}(a)+8\alpha_{1n}\alpha_{2n}X_{2n}(a)](\cos m\pi-1)\\
&T_m=\Psi_m[\Omega_m X_{2n}(a)-8\alpha_{1n}\alpha_{2n}X_{1n}(a)](\cos m\pi-1)\\
&R_m=\Psi_m[\Omega_m X_{3n}(a)-8\alpha_{1n}\alpha_{2n}X_{4n}(a)](\cos m\pi+1)\\
&Q_m=\Psi_m[\Omega_m X_{4n}(a)+8\alpha_{1n}\alpha_{2n}X_{3n}(a)](\cos m\pi+1)
\end{aligned}\right\}\qquad(3.10\text{f})$$

式中

$$\Psi_m=-\frac{2m\pi}{\Omega_m^2+(8\alpha_{1n}\alpha_{2n})^2},\quad \Omega_m=4(\alpha_{1n}^2-\alpha_{2n}^2)+(m\pi)^2\qquad(3.10\text{g})$$

在(3.10e,f,g)诸式中，分别用 $a_{jm}^{(7)}$, $b_{jn}^{(7)}$, η_{iy}, ξ_{ix}, S_n, T_n, R_n, Q_n, Ψ_n, Ω_n, $X_{jn}(a)$, $Y_{jm}(b)$, β_{im}, r_{ix}, r_{iy}, D_x, a/b, n和m代替 $b_{jn}^{(6)}$, $a_{jm}^{(6)}$, η_{ix}, ξ_{iy}, S_m, T_m, R_m, Q_m, Ψ_m, Ω_m, $Y_{jm}(b)$, $X_{jn}(a)$, α_{in}, r_{iy}, r_{ix}, D_y, b/a, m和n，即得到$a_{jm}^{(7)}$, $b_{jn}^{(7)}$, η_{iy}, ξ_{ix}, S_n, T_n, R_n, Q_n, Ψ_n, Ω_n, $j=1,2,3,4$, $i=1,2$。

值得指出的是(1)(3.8g～j)四式的形式同样适用于$H^2=D_xD_y$, $H^2<D_xD_y$两种情形；(2) $p_{km}^{(5)}, U_m^{(5)}$, $p_{km}^{(6)}, U_m^{(6)}$, $p_{kn}^{(7)}$, $U_n^{(7)}$, $p_{kn}^{(8)}, U_n^{(8)}$, $p_k^{(s)}, U^{(s)}$, $p_k^{(t)}$, $U^{(t)}$在三种情形下是相同的。

四、数值结果与讨论

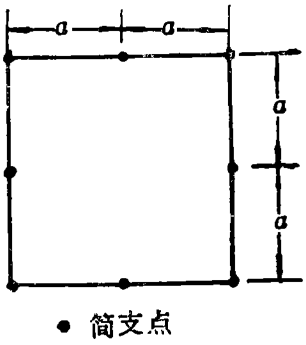
本文对图3所示的两种支承的方板进行了计算；材料是各向同性的。计算结果列于表1，2，3中，表中结果均以无量纲形式给出（对均布荷载q为$10Dw/qa^4$，对中心集中荷载为

表1　图3(a)所示八点支承正方形板的中点挠度($\mu=0.2$)

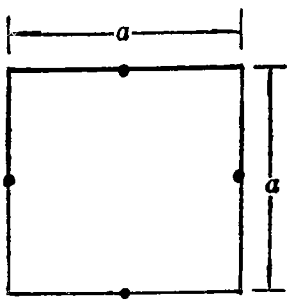
方法	均布荷载q	中心集中荷载P
A.S.	0.7768	0.5004
C.S.	0.7814	0.5007
F.S.S.	0.7783	0.5001

表2　图3(b)所示仅在四边中点支承正方形板的中点和角点挠度($\mu=0.2$)

方法＼载荷	均布荷载q		中心集中荷载P	
	w_1	w_2	w_1	w_2
A.S.	0.05258	0.01536	0.15456	−0.04727
F.S.S.	0.05267	0.01526	0.15447	−0.04726



(a)



(b)

图 3

表3 逐渐增加每边支承点时正方形板中点的挠度($\mu=0.2$)

载荷	支点数N_s	3	4	5	6	7	8	9	S.S.
均布载荷q	A.S.	0.7768	0.6806	0.6617	0.6556	0.6528	0.6517	0.6499	0.6499
	C.S.	0.7814		0.6625		0.6534			0.6499
中心集中载荷P	A.S.	0.5004	0.4745	0.4680	0.4659	0.4651	0.4646	0.4644	0.4640
	C.S.	0.5007		0.4681					0.4640

* A.S.为本文的解；C.S.为文献[3]的配位解；F.S.S.为文献[4]的样条有限条解；S.S.为四边完全简支.

$10Dw/Pa^2$)。表1，2表明用本文方法得到的中点及角点挠度与文献[3]、[4]的解是十分一致的。从表3中可以看出，对图3(a)所示支承的方板，随着每边点支承数的增加，其中点挠度逐渐趋于四边简支方板的中点挠度；在每边有$N_s=8$个等距离支承点（包括角点）时，两者之差不足0.015%。

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An Exact Solution for the Bending of Point-Supported Orthotropic Rectangular Thin Plates

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Abstract

A closed series solution is proposed for the bending of point-supported orthotropic rectangular thin plates. The positions of support points and the distribution of transverse load are arbitrary. If the number of simply supported points gradually increases the solution can infinitely approach to Navier's solution. For the square plate simply supported on the middle of each edge and free at each corner, the results are very close to the numerical solutions in the past.

Key words exact solution, point supports, orthotropic rectangular thin plate