非均匀变厚度梁的动力响应的一般解

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摘　要

在本文中提出一个新方法——阶梯折算法来研究在任意载荷下任意非均匀和任意变厚度伯努利-欧拉梁的动力响应问题. 研究了自由振动和强迫振动. 新方法需要将区间离散为一定数目的元素，每个元素可看作是均匀和等厚度的. 因此均匀、等厚度梁的一般解可在每个元素上应用. 然后用初参数表示的整个梁的一般解使之满足相邻二元素间的物理和几何连续条件，这样就可以得到解析形式的自由振动的频率方程和解析形式的强迫振动的最终解，它化为求解二元线性代数方程，与离散元素的数目无关. 现在的方法可推广应用至任意非均匀及任意变厚度有粘滞性和其他种类的梁以及其他结构元件问题上去.

关键词　非均匀　变厚度　伯努利-欧拉梁　离散　动力响应

一、引　言

在各种不同的工程领域中，许多具有非均匀和变厚度的结构元件（**梁、板和壳**）被应用着，由于它们有较好的结构特性和优化的形状使得工程师节约了材料. 但是很难得出具有任意非均匀和任意厚度梁的静力平衡和动力响应的一般解析解. 因此，这些问题虽已被研究过，但大部分是应用数值解法，例如转换矩阵法[1~2]，加权残数法[3]，变分法[4~5]，有限元法[6]，有限差分法[7]，边界元法[8]，动力刚度法[9]，联合应用模型分析的动力法[10~11] 以及伴同拉氏变换的动力刚度法[12]. 但是，这些方法的大部分需要过细的离散区间导出了数量庞大的方程组和繁复的计算，或则为解答的收敛率所局限.

1965年叶开沅[1]提出了一个非常有效用初参数表示的，亦即称为阶梯折算法研究了非均匀、变厚度梁的静力平衡，得出了一般解析解. 这个问题曾经为许多学者研究过[13~15]. 自此以后，一系列非均匀、变厚度实用问题用此方法得到解决. 例如，高速旋转圆盘[16~18]，结构元件的优化[19]，以及钢筋混凝土构件的计算[20]等等. 但是当时所提的方法仅适用于具有物理连续条件的构件的静力平衡问题，而不适用于同时具有物理和几何连续条件的动力问题和稳定性问题.

在本文中，我们提出了修正的阶梯折算法克服了前述的困难. 新方法同样要将区间的离

散为一定数量的元素，每个元素均当作均匀和等厚度。因此均匀、等厚度梁的一般解析解可用于各个元素。然后，用初参数表示的均匀、等厚度整个梁的一般解可以由满足相邻区间的物理和几何连续条件而得到。在自由振动情况，可以得出解析形式的频率方程。在强迫振动的情形，也可以得出呈解析形式的最终结果。问题化为求解一组仅具有二个独立未知数的代数方程组，此方程组的解与划分的元素数目无关。现在的方法可推广应用于任意非均匀和任意变厚度有粘滞性和其他种类的梁以及其他构件问题上去。

二、具有非均匀变厚度梁的自由振动

考虑如图1所示的均匀等厚度梁．梁的自由振动基本方程为

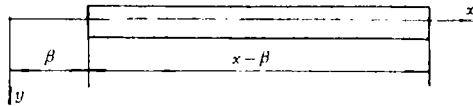$$D\frac{\partial^4 y_1}{\partial x_1^4} = -\frac{\gamma A}{g}\frac{\partial^2 y_1}{\partial t^2} \tag{2.1}$$

图　1

其中　$D=EI$ 为抗弯刚度，E 为杨氏模量，I 为截面对中性轴的慢性矩，t 为时间，A 为截面面积，γ 为密度，g 为重力加速度．弯矩、剪力和挠度间的关系式为

$$M_1 = -D\frac{\partial^2 y_1}{\partial x_1^2}, \quad Q_1 = -D\frac{\partial^3 y_1}{\partial x_1^3} \tag{2.2}$$

令

$$y_1(x_1,t) = T(t)X_1(x_1), \quad M_1 = M^*(x_1)T(t) \tag{2.3a}$$

$$Q_1(x_1,t) = Q^*(x_1)T(t) \tag{2.3b}$$

其中　$T(t)$ 为时间的周期函数，它具有圆频率 p，然后引进下列无量纲量：

$$\left.\begin{array}{l} X = \dfrac{X_1}{l}, \quad x = \dfrac{x_1}{l} \\[2mm] M(x) = M^*(lx), \quad Q(x) = Q^*(lx) \\[2mm] k^4 = \dfrac{p^2\gamma A l^4}{gD}, \quad \beta = \dfrac{l_0}{l} \end{array}\right\} \tag{2.4}$$

我们可以将方程(2.1)和(2.2)变为下列无量纲形式：

$$\frac{d^4 X}{dx^4} - k^4 X = 0 \tag{2.5}$$

$$\frac{M(x)l}{D} = -\frac{d^2 X}{dx^2}, \quad \frac{Q(x)l^2}{D} = -\frac{d^3 X}{dx^3} \tag{2.6}$$

方程(2.5)的解为

$$X = X(\beta)f_1(\beta,x,k) + \frac{X'(\beta)}{k}f_2(\beta,x,k) + \frac{M(\beta)l}{Dk^2}f_3(\beta,x,k)$$

$$+ \frac{Q(\beta)l^2}{Dk^3}f_4(\beta,x,k) \tag{2.7}$$

其中

$$f_1 = \frac{1}{2}[\cos k(x-\beta) + \operatorname{ch} k(x-\beta)]$$

$$f_2 = \frac{1}{2}[\sin k(x-\beta) + \operatorname{sh} k(x-\beta)]$$

$$f_3 = \frac{1}{2}[\cos k(x-\beta) - \operatorname{ch} k(x-\beta)]$$

$$f_4 = \frac{1}{2}[\sin k(x-\beta) - \operatorname{sh} k(x-\beta)]$$

$$(2.8)$$

式中 $X(\beta), X'(\beta), M(\beta)l/D, Q(\beta)l^2/D$ 为四个初参数，亦即为梁的起始端的挠度、倾角、弯矩和剪力。解(2.7)称为梁的自由振动的初参数形式的解。它将作为求解非均匀变厚度梁的动力响应的基本解。函数 $f_i(\beta, x, k)$ $(i=1,2,3,4)$ 有如表1所示的循环微分关系

表 1   **函数 $f_i(\beta,x,k)$ $(i=1,2,3,4)$ 的循环微分关系**

i	$f_i(\beta,x,k)$	$\dfrac{d}{dx}f_i(\beta,x,k)$	$\dfrac{d^2}{dx^2}f_i(\beta,x,k)$	$\dfrac{d^3}{dx^3}f_i(\beta,x,k)$
1	$f_1(\beta,x,k)$	$-kf_4(\beta,x,k)$	$-k^2f_3(\beta,x,k)$	$k^3f_2(\beta,x,k)$
2	$f_2(\beta,x,k)$	$kf_1(\beta,x,k)$	$-k^2f_4(\beta,x,k)$	$-k^3f_3(\beta,x,k)$
3	$f_3(\beta,x,k)$	$-kf_2(\beta,x,k)$	$-k^2f_1(\beta,x,k)$	$k^3f_4(\beta,x,k)$
4	$f_4(\beta,x,k)$	$kf_3(\beta,x,k)$	$-k^2f_2(\beta,x,k)$	$-k^3f_1(\beta,x,k)$

当 $x=\beta$，这些函数有如表2所示的关系。

表 2   **函数 $f_i(\beta,x,k)$ $(i=1,2,3,4)$ 在 $x=\beta$ 时的性质**

i	$f_i(\beta,\beta,k)$	$\dfrac{d}{dx}f_i(\beta,\beta,k)$	$\dfrac{d^2}{dx^2}f_i(\beta,\beta,k)$	$\dfrac{d^3}{dx^3}f_i(\beta,\beta,k)$
1	1	0	0	0
2	0	k	0	0
3	0	0	$-k^2$	0
4	0	0	0	$-k^3$

$f_i(\beta, x, k)$ $(i=1,2,3,4)$ 为方程(2.5)的齐次解的组合。在 $x=\beta$ 时，$f_1(\beta,x,k)$ 有一挠度跳跃，$f_2(\beta,x,k)$ 有一倾角跳跃，$f_3(\beta,x,k)$ 有一弯矩跳跃，$f_4(\beta,x,k)$ 有一剪力跳跃。

现在让我们考虑一具有任意非均匀和变厚度的梁。离散以后，$D(x)$, $\gamma(x)$, $A(x)$, 和 $k(x)$ 定义如下 (图2)：

图 2

$$\begin{cases}
D_0 = D(0), \quad \gamma_0 = \gamma(0), \quad A_0 = A(0), \\
\quad k_0 = k(0), \qquad\qquad (0 \leqslant x < \beta_1) \\
D_1 = D(\beta_1), \quad \gamma_1 = \gamma(\beta_1), \quad A_1 = A(\beta_1), \\
\quad k_1 = k(\beta_1), \qquad\quad (\beta_1 \leqslant x < \beta_2) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_i = D(\beta_i), \quad \gamma_i = \gamma(\beta_i), \quad A_i = A(\beta_i), \quad k_i = k(\beta_i), \quad (\beta_i \leqslant x < \beta_{i+1}) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_n = D(\beta_n), \quad \gamma_n = \gamma(\beta_n), \quad A_n = A(\beta_n), \quad k_n = k(\beta_n), \quad (\beta_n \leqslant x < 1)
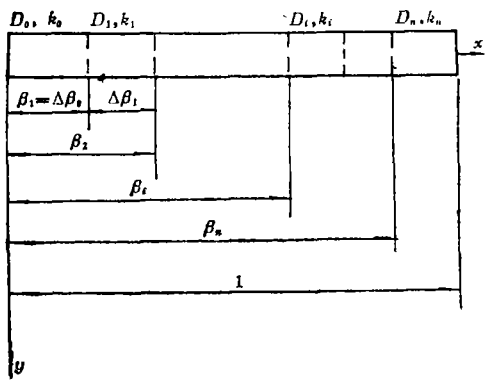\end{cases}$$

$$(2.9)$$

任一元素的基本方程为：

$$\frac{d^4X}{dx^4} - k_i^4 X = 0 \tag{2.10}$$

其中

$$k_i = k_0 + \sum_{j=1}^{i} \{x - \beta_j\}^\circ (k_j - k_{j-1}) \tag{2.11}$$

$$D_i = D_0 + \sum_{j=1}^{i} \{x - \beta_j\}^\circ (D_j - D_{j-1}) \tag{2.12}$$

$$\{x - \beta_i\}^\circ = \begin{cases} 1 & (\text{当} x \geqslant \beta_i \text{时}) \\ 0 & (\text{当} x < \beta_i \text{时}) \end{cases} \tag{2.13}$$

为Heaviside函数。

在$x = \beta_i$的连续条件为：

$$\lim_{\varepsilon \to 0} X(\beta_i - \varepsilon) = X(\beta_i), \quad \lim_{\varepsilon \to 0} X'(\beta_i - \varepsilon) = X'(\beta_i) \tag{2.14a}$$

$$\lim_{\varepsilon \to 0} \frac{M(\beta_i - \varepsilon) l}{D_0} = \frac{M(\beta_i) l}{D_0}, \quad \lim_{\varepsilon \to 0} \frac{Q(\beta_i - \varepsilon) l^2}{D_0} = \frac{Q(\beta_i) l^2}{D_0} \tag{2.14b}$$

前面二个称为**几何连续条件**，后面二个称为**物理连续条件**。相应的边界条件为：

a) 端点夹紧

$$X = X' = 0, \quad (\text{当} x = 0 \text{或} 1 \text{时}) \tag{2.15}$$

b) 端点简支

$$X = M = 0, \quad (\text{当} x = 0 \text{或} 1 \text{时}) \tag{2.16}$$

c) 端点悬空

$$M = Q = 0, \quad (\text{当} x = 0 \text{或} 1 \text{时}) \tag{2.17}$$

现在具有任意非均匀和变厚度梁的自由振动问题转变为**边值问题**：它包含基本方程 (2.1)，连续条件(2.14)和边界条件(2.15)～(2.17)。

我们引入下列4×4矩阵$[A_{ij}(\beta_i, x, \delta_i)](i, j = 1, 2, 3, 4)$

$$[A_{ij}(\beta_i, x, k_i, \delta_i)]$$

$$= \begin{bmatrix} f_1(\beta_i, x, k_i) & f_2(\beta_i, x, k_i)/k_i & \delta_i f_3(\beta_i, x, k_i)/k_i^2 & \delta_i f_4(\beta_i, x, k_i)/k_i^3 \\ -k_i f_4(\beta_i, x, k_i) & f_1(\beta_i, x, k_i) & -\delta_i f_2(\beta_i, x, k_i)/k_i & \delta_i f_3(\beta_i, x, k_i)/k_i^2 \\ k_i^2 f_3(\beta_i, x, k_i)/\delta_i & k_i f_4(\beta_i, x, k_i)/\delta_i & f_1(\beta_i, x, k_i) & f_2(\beta_i, x, k_i)/k_i \\ -k_i^3 f_2(\beta_i, x, k_i)/\delta_i & k_i^2 f_3(\beta_i, x, k_i)/\delta_i & -k_i f_4(\beta_i, x, k_i) & f_1(\beta_i, x, k_i) \end{bmatrix}$$

$$\tag{2.18}$$

不难证明

$$A_{ij}(\beta_i, x, k_i, \delta_i) = A_{ij}(0, x - \beta_i, k_i, \delta_i) \tag{2.19}$$

令矢量为

$$\{\delta(x)\} = \left\{ X(x) \quad X'(x) \quad \frac{M(x) l}{D_0} \quad \frac{Q(x) l^2}{D_0} \right\} \tag{2.20}$$

然后我们有

$$\{\delta(x)\} = [A_{ij}(\beta, x, k_i, \delta_i)]\{\delta(\beta_i)\} \tag{2.21}$$

其中

$$\delta_i = 1 + \sum_{j=1}^{i} \{x - \beta_j\}^\circ \left(\frac{D_0}{D_j} - \frac{D_0}{D_{j-1}} \right) \tag{2.22}$$

现在，具有任意非均匀和变厚度梁的自由振动的解可写成如下初参数形式的解

$$\{\delta(x)\} = [b_{lm}(x)]\{\delta(0)\} \tag{2.23}$$

其中

$$b_{lm}(x) = A_{lm}(\beta_{i-1}, x, k_{i-1}, \delta_{i-1}) + \sum_{j=1}^{i} \sum_{s=1}^{4} \{x - \beta_j\}^\circ f_{mj}^{(s)} A_{ls}(\beta_i, x, k_i, \delta_i)) \tag{2.24}$$

以及 $f_{1i}^{(s)}, f_{2i}^{(s)}, f_{3i}^{(s)}, f_{4i}^{(s)}(s=1,2,3,4)$ 为待定常数由满足连续条件决定，它们是

$$f_{mi}^{(l)} = \Delta A_{lm} + \sum_{j=1}^{i-1} \{x - \beta_j\}^\circ \sum_{s=1}^{4} f_{mj}^{(s)} \Delta A_{ls} \tag{2.25}$$

其中

$$\Delta A_{lm} = A_{lm}(\beta_{j-1}, \beta_j, k_{j-1}, \delta_{j-1}) - A_{lm}(\beta_j, \beta_j, k_j, \delta_j) \tag{2.26}$$

在 $x=1$

$$\{\delta(1)\} = [b_{lm}]\{\delta(0)\} \tag{2.27}$$

它也就是

$$\left.\begin{aligned}
X(1) &= b_{11}X(0) + b_{12}X'(0) + b_{13}\frac{M(0)l}{D_0} + b_{14}\frac{Q(0)l^2}{D_0} \\[6pt]
X'(1) &= b_{21}X(0) + b_{22}X'(0) + b_{23}\frac{M(0)l}{D_0} + b_{24}\frac{Q(0)l^2}{D_0} \\[6pt]
\frac{M(1)l}{D_0} &= b_{31}X(0) + b_{32}X'(0) + b_{33}\frac{M(0)l}{D_0} + b_{34}\frac{Q(0)l^2}{D_0} \\[6pt]
\frac{Q(1)l^2}{D_0} &= b_{41}X(0) + b_{42}X'(0) + b_{43}\frac{M(0)l}{D_0} + b_{44}\frac{Q(0)l^2}{D_0}
\end{aligned}\right\} \tag{2.28}$$

以上述方程为基础，我们可以找出如表3所示的在各种不同边界条件下的频率方程。

现在我们用上述结果来计算一个例子. 考虑图3所示的均匀阶梯形梁，它的尺寸，惯性矩，截面面积均如图3所示.

令

$$I_1 = \nu I_0, \quad z = \frac{k_1}{k_0}, \quad \delta_1 = \frac{I_0}{I_1} = \frac{1}{\nu}$$

由(2.25)，我们有

$$f_{11}^{(1)} = f_1(0, \beta_1, k_0) - 1, \quad f_{12}^{(1)} = \frac{1}{k_0} f_2(0, \beta_1, k_0)$$

$$f_{31}^{(1)} = \frac{1}{k_0^2} f_3(0, \beta_1, k_0), \quad f_{41}^{(1)} = \frac{1}{k_0^3} f_4(0, \beta_1, k_0);$$

$$f_{11}^{(2)} = -k_0 f_4(0, \beta_1, k_0), \quad f_{21}^{(2)} = f_1(0, \beta_1, k_0) - 1$$

$$f_{31}^{(2)} = -\frac{1}{k_0} f_2(0, \beta_1, k_0), \quad f_{41}^{(2)} = \frac{1}{k_0^2} f_3(0, \beta_1, k_0);$$

$$f_{11}^{(3)} = k_0^2 f_3(0, \beta_1, k_0), \qquad f_{21}^{(3)} = k_0 f_4(0, \beta_1, k_0)$$

$$f_{31}^{(3)} = f_1(0, \beta_1, k_0) - 1, \qquad f_{41}^{(3)} = \frac{1}{k_0} f_2(0, \beta_1, k_0);$$

$$f_{11}^{(4)} = -k_0^3 f_2(0, \beta_1, k_0), \qquad f_{21}^{(4)} = k_0^2 f_3(0, \beta_1, k_0)$$

$$f_{31}^{(4)} = -k_0 f_4(0, \beta_1, k_0), \qquad f_{41}^{(4)} = f_1(0, \beta_1, k_0) - 1;$$

表3　各种不同边界条件下的非均匀变厚度的频率方程

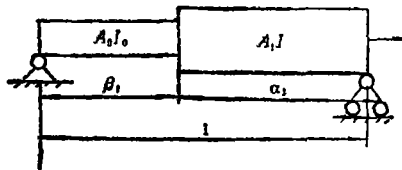| 种　类	杆件简图	边界条件	频率方程
两端夹紧		$x = 0, \ X = X' = 0$ $x = 1, \ X = X' = 0$	$b_{13}b_{24} - b_{23}b_{14} = 0$
两端简支		$x = 0, \ X = M = 0$ $x = 1, \ X = M = 0$	$b_{12}b_{34} - b_{32}b_{14} = 0$
两端悬空		$x = 0, \ M = Q = 0$ $x = 1, \ M = Q = 0$	$b_{31}b_{42} - b_{32}b_{41} = 0$
一端简支 一端夹紧		$x = 0, \ X = M = 0$ $x = 1, \ X = X' = 0$	$b_{12}b_{24} - b_{22}b_{14} = 0$
一端简支 一端悬空		$x = 0, \ X = M = 0$ $x = 1, \ M = Q = 0$	$b_{11}b_{32} - b_{31}b_{12} = 0$
一端夹紧 一端悬空		$x = 0, \ X = X' = 0$ $x = 1, \ M = Q = 0$	$b_{33}b_{44} - b_{34}b_{43} = 0$

图 3

由(2.23)在$x=1$，我们得到

$$b_{12} = A_{12}(\beta_1, \alpha_1, k_1) + f_{21}^{(1)} A_{11}(0, \alpha_1, k_1) + f_{21}^{(2)} A_{12}(0, \alpha_1, k_1)$$

$$+ f_{21}^{(3)} A_{13}(0, \alpha_1, k_1) + f_{21}^{(4)} A_{14}(0, \alpha_1, k_1)$$

因此

$$b_{12} = \frac{1}{k_1} f_2(0, \alpha_1, k_1) + \frac{1}{k_1} f_2(0, \beta_1, k_0) f_1(0, \alpha_1, k_0)$$

$$+ (f_1(0, \beta_1, k_0) - 1) \frac{1}{k_1} f_2(0, \alpha_1, k_1) + k_0 f_4(0, \beta_1, k_0) \frac{\delta_1}{k_1^2} f_3(0, \alpha_1, k_1)$$

$$+ k_0^2 f_3(0, \beta_1, k_0) \frac{\delta_1}{k_1^3} f_4(0, \alpha_1, k_1)$$

$$= \frac{1}{k_0}f_1(0,\alpha_1,k_1)f_2(0,\beta_1,k_0) + \frac{1}{k_1}f_2(0,\alpha_1,k_1)f_1(0,\beta_1,k_0)$$

$$+ \frac{k_0}{\nu k_1^2}f_3(0,\alpha_1,k_1)f_4(0,\beta_1,k_0) + \frac{k_0^2}{\nu k_1^3}f_4(0,\alpha_1,k_1)f_3(0,\beta_1,k_0)$$

同样

$$b_{34} = \frac{\nu k_1^2}{k_0^3}f_1(0,\alpha_1,k_1)f_4(0,\beta_1,k_0) + \frac{\nu k_1}{k_0^2}f_4(0,\beta_1,k_0)f_1(0,\beta_1,k_0)$$

$$+ \frac{1}{k_0}f_1(0,\alpha_1 k_1)f_2(0,\beta_1,k_0) + \frac{1}{k_1}f_2(0,\alpha_1,k_1)f_1(0,\beta_1,k_0),$$

$$b_{32} = \frac{\nu k_1^2}{k_0}f_3(0,\alpha_1,k_1)f_2(0,\beta_1,k_0) + \nu k_1 f_4(0,\alpha_1,k_1)f_1(0,\alpha_1,k_1)$$

$$+ k_0 f_1(0,\alpha_1,k_1)f_4(0,\beta_1,k_0) + \frac{k_0^2}{k_1}f_2(0,\alpha_1,k_1)f_3(0,\beta_1,k_0),$$

$$b_{14} = \frac{1}{k_0^3}f_1(0,\alpha_1,k_1)f_4(0,\beta_1,k_0) + \frac{1}{k_1 k_0^2}f_2(0,\alpha_1,k_1)f_3(0,\beta_1,k_0)$$

$$+ \frac{1}{\nu k_0 k_1^2}f_3(0,\alpha_1,k_1)f_2(0,\beta_1,k_0) + \frac{1}{\nu k_1^3}f_4(0,\alpha_1,k_1)f_1(0,\beta_1,k_0);$$

将这些结果代入频率方程

$$b_{12}b_{34} - b_{32}b_{14} = 0$$

经过简化，我得到

$$(\nu z^2 - 1)^2(\sin k_0\beta_1 \operatorname{ch}k_0\beta_1 \sin k_1\alpha_1 \operatorname{ch}k_1\alpha_1 + \cos k_0\beta_1 \operatorname{sh}k_0\beta_1 \cos k_1\alpha_1 \operatorname{sh}k_1\alpha_1)$$

$$- (\nu z^2 + 1)^2(\sin k_0\beta_1 \operatorname{ch}k_0\beta_1 \cos k_1\alpha_1 \operatorname{sh}k_1\alpha_1 + \cos k_0\beta_1 \operatorname{sh}k_0\beta_1 \sin k_1\alpha_1 \operatorname{ch}k_1\alpha_1)$$

$$- 4\nu z^3 \sin k_0\beta_1 \operatorname{sh}k_0\beta_1 \cos k_1\alpha_1 \operatorname{ch}k_1\alpha_1 - 4\nu z \cos k_0\beta_1 \operatorname{ch}k_0\beta_1 \sin k_1\alpha_1 \operatorname{sh}k_1\alpha_1 = 0$$

这个结果与文献[1]中用转换矩阵法和文献[8]中用寻常的方法求得的完全一致。

三、在任意谐载荷下的强迫振动

在任意谐载荷下的非均匀变厚度梁的强迫振动问题也可以转化为下列边值问题：

1) 基本方程

$$\frac{d^4 X}{dx^4} - k_1^4 X = f(x) \tag{3.1}$$

2) 与(2.14)相同的连续条件。

3) 与(2.15)～(2.17)相同的边界条件。

此一边值问题的解可写成如下形式：

$$\{\delta(x)\} = [b_{lm}(x)]\{\delta(0)\} + \{p(x)\} \tag{3.2}$$

其中 $[b_{lm}(x)\{\delta(0)\}$ 为方程(3.1)的齐次解，如方程(2.23)所示，并且

$$\{p(x)\} = \left[X^*(x), X^{*\prime}(x), \frac{M^*(x)l}{D_0}, \frac{Q^*(x)l^2}{D_0} \right]^T = [p_m]^T \quad (m=1,2,3,4) \tag{3.3}$$

其中

$$p_m = I_m + \sum_{j=1}^{i-1} \{x - \beta_j\}^\circ \sum_{s=1}^{4} g_j^{(s)} A_{ms}(\beta_i, x, k_i, \delta_i) \tag{3.4}$$

以及

$$I_1 = \frac{1}{2k_i^3} \int_0^x [\mathrm{sh}k_i(x-\xi) - \sin k_i(x-\xi)] f(\xi) d\xi \tag{3.5a}$$

$$I_2 = \frac{1}{2k_i^2} \int_0^x [\mathrm{ch}k_i(x-\xi) - \cos k_i(x-\xi)] f(\xi) d\xi \tag{3.5b}$$

$$I_3 = -\frac{1}{2k_i\delta_i} \int_0^x [\mathrm{sh}k_i(x-\xi) + \sin k_i(x-\xi)] f(\xi) d\xi \tag{3.5c}$$

$$I_4 = -\frac{1}{2\delta_i} \int_0^x [\mathrm{ch}k_i(x-\xi) + \cos k_i(x-\xi)] f(\xi) d\xi \tag{3.5d}$$

(3.3)式是方程(3.1)的特解.在方程(3.4)中 $g_j^{(s)}$ 是由满足连续条件(2.14)得到的任意待定常数.

$$g_i^{(m)} = \Delta I_m + \sum_{j=1}^{i-1} \{x - \beta_j\}^\circ \sum_{s=1}^{4} g_j^{(s)} \Delta A_{ms} \tag{3.6}$$

其中 $\Delta I_1 = \dfrac{1}{2k_{i-1}^3} \displaystyle\int_{\beta_{i-1}}^{\beta_i - \beta_{i-1}} [\mathrm{sh}k_{i-1}(\beta_i - \beta_{i-1} - \xi) - \sin k_{i-1}(\beta_i - \beta_{i-1} - \xi)] f(\xi) d\xi$

$$+ \frac{1}{2k_i^3} \int_{\beta_i}^0 [\mathrm{sh}k_i\xi - \sin k_i\xi] f(\xi) d\xi \tag{3.7a}$$

$$\Delta I_2 = \frac{1}{2k_{i-1}^2} \int_{\beta_{i-1}}^{\beta_i - \beta_{i-1}} [\mathrm{ch}k_{i-1}(\beta_i - \beta_{i-1} - \xi) - \cos k_{i-1}(\beta_i - \beta_{i-1} - \xi)] f(\xi) d\xi$$

$$- \frac{1}{2k_i^2} \int_{\beta_i}^0 [\mathrm{ch}k_i\xi - \cos k_i\xi] f(\xi) d\xi \tag{3.7b}$$

$$\Delta I_3 = -\frac{1}{2\delta_{i-1}k_{i-1}} \int_{\beta_{i-1}}^{\beta_i - \beta_{i-1}} [\mathrm{sh}k_{i-1}(\beta_i - \beta_{i-1} - \xi) + \sin k_{i-1}(\beta_i - \beta_{i-1} - \xi)] f(\xi) d\xi$$

$$+ \frac{1}{2\delta_i k_i} \int_{\beta_i}^0 [\mathrm{sh}k_i\xi + \sin k_i\xi] f(\xi) d\xi \tag{3.7c}$$

$$\Delta I_4 = -\frac{1}{2\delta_{i-1}} \int_{\beta_{i-1}}^{\beta_i - \beta_{i-1}} [\mathrm{ch}k_{i-1}(\beta_i - \beta_{i-1} - \xi) + \cos k_{i-1}(\beta_i - \beta_{i-1} - \xi)] f(\xi) d\xi$$

$$+ \frac{1}{2\delta_i} \int_{\beta_i}^0 [\mathrm{ch}k_i\xi - \cos k_i\xi] f(\xi) d\xi \tag{3.7d}$$

和 $\quad \Delta A_{ms} = A_{ms}(\beta_{i-1}, \beta_i, k_{i-1}, \delta_{i-1}) - A_{ms}(\beta_i, \beta_i, k_i, \delta_i) \qquad (m, s = 1, 2, 3, 4) \tag{3.8}$

四、在初始条件下的强迫振动

我们考虑非均匀、变厚度梁在初始条件下的强迫振动.基本方程为:

$$\frac{\partial^2}{\partial x^2} D \frac{\partial^2 y}{\partial x^2} + \frac{\gamma A}{g} \frac{\partial^2 y}{\partial t^2} = F(x, t) \tag{4.1}$$

边界条件为：

a) 端点夹紧

$$y = y' = 0 \quad (x = 0 \text{或} 1) \tag{4.2a}$$

b) 端点简支

$$y = M = 0 \quad (x = 0 \text{或} 1) \tag{4.2b}$$

c) 端点悬空

$$M = Q = 0 \quad (x = 0 \text{或} 1) \tag{4.2c}$$

以及连续条件为：

$$\lim_{\varepsilon \to 0} X(\beta_i - \varepsilon) = X(\beta_i); \quad \lim_{\varepsilon \to 0} X'(\beta_i - \varepsilon) = X'(\beta_i) \tag{4.3a}$$

$$\lim_{\varepsilon \to 0} \frac{M(\beta_i - \varepsilon)l}{D_0} = \frac{M(\beta_i)l}{D_0}, \quad \lim_{\varepsilon \to 0} \frac{Q(\beta_i - \varepsilon)l^2}{D_0} = \frac{Q(\beta_i)l^2}{D_0} \tag{4.3b}$$

初始条件为：

$$y(x, 0) = y_0(x), \quad y(x, 0) = y_0'(x) \quad (t = 0) \tag{4.4}$$

令方程(4.1)的解为

$$y(x, t) = \sum_{n=1}^{\infty} Y_n(x) q_n(t) \tag{4.5}$$

其中 $Y_n(x)$ 为在已给边界条件(4.2)下和连续条件(4.3)下具有自然频率 p_n 的简正坐标。由第二节，我们知道当阶梯数 n 趋近于无穷大，所得的非均匀、变厚度梁的自由振动的解便趋近于精确解。我们由(2.3)便可得到相应的边界条件下的 $Y_n(x)$ 和自然频率。我们得到的简正型函数有正交性。简正坐标满足下面的方程：

$$q_n'' + p_n^2 = Q_n \tag{4.6}$$

其中

$$Q_n = \int_0^1 F(x, t) Y_n(x) dx \tag{4.7}$$

因此，方程(4.1)的解可写成如下的形式

$$y(x, t) = \sum_{n=1}^{\infty} Y_n(x) \left[q_{n0} \cos p_n t + \frac{q_{n0}'}{p_n} \sin p_n t \right.$$

$$\left. + \frac{1}{p_n} \int_0^t q_n(t) \sin p_n(t - \tau) d\tau \right] \tag{4.8}$$

其中

$$q_{n0} = \int_0^1 \frac{\gamma A}{g} y_0 Y_n(x) dx, \quad q_{n0}' = \int_0^1 \frac{\gamma A}{g} y_0' Y_n(x) dx \tag{4.9a,b}$$

五、讨 论

1. 函数集 $\{Y_n(x)\}$ 为正交，可以证明如下：不难证出下列具有任意非均匀和任意变厚度梁在下列方程所表示的

$$\frac{d^2}{dx^2} D(x) \frac{d^2 Y}{dx^2} + \frac{\gamma(x) A(x)}{g} Y = 0 \tag{5.1}$$

$\{Y_n(x)\}$ 是正交的．在我们的方法中，我们有如下形式的基本方程

$$\frac{d^2}{dx^2}D_i(x)\frac{d^2Y}{dx^2}+\frac{\gamma_i(x)A_i(x)}{g}Y=0 \tag{5.2}$$

由于 $D_i(x)\subset D(x)$．由方程(5.2)得到的函集 $\{Y_n(x)\}$ 依旧是正交的．

2．我们用转换矩阵法[1]得到与表3中同样的频率方程．

3．我们可以将表3中的频率方程变为如下的幂级数方程

$$1-a_1x-a_2x^2-a_3x^3+\cdots=0 \tag{5.3}$$

其中　$a_1,a_2,a_3,\cdots$ 为实数．用文献[21]附录3中的方法，我们可得一阶近似的根

$$x_1^{(1)}=1/a_1 \tag{5.4}$$

二阶近似

$$x_1^{(2)}=\sqrt{1/(2a_2+a_1^2)} \tag{5.5}$$

三阶近似

$$x_1^{(3)}=\sqrt[3]{1/(3a_3+3a_1a_2+a_1^3)} \tag{5.6}$$

以下各阶近似依此类推．我们可以得到解析形的频率方程的根．

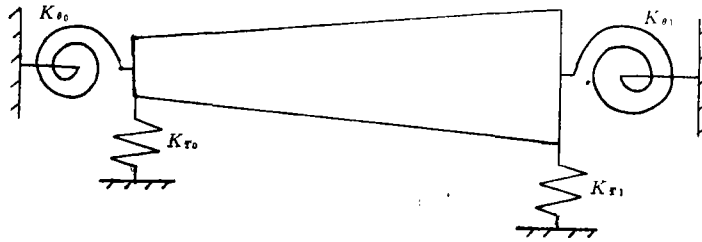4．本文提出的方法可推广对具有任意非均匀和任意变厚度的其他结构元件，如杆、板和壳的研究．

5．假如所考虑的梁有如图4所示的弹性约束边界条件：

图 4

在 $x=0$

$$\frac{Q(0)l^2}{D_0}=K_{T0}X(0),\qquad \frac{M(0)l}{D_0}=K_{\theta0}X'(0) \tag{5.7a}$$

在 $x=1$

$$\frac{Q(1)l^2}{D_0}=K_{T1}X(1),\qquad \frac{M(1)l}{D_0}=K_{\theta1}X'(1) \tag{5.7b}$$

将(5.7)代入(2.23)，我们有

$$\left.\begin{array}{l}X(1)=(b_{11}+b_{14}K_{T0})X(0)+(b_{12}+b_{13}K_{\theta0})X'(0)\\ X'(1)=(b_{21}+b_{24}K_{T0})X(0)+(b_{22}+b_{23}K_{\theta0})X'(0)\\ K_{\theta0}X'(1)=(b_{31}+b_{34}K_{T0})X(0)+(b_{32}+b_{33}K_{\theta0})X'(0)\\ K_{T1}X(1)=(b_{41}+b_{44}K_{T0})X(0)+(b_{42}+b_{43}K_{\theta0})X'(0)\end{array}\right\} \tag{5.8}$$

由这些方程，我们可得到所求的频率方程

$$\begin{aligned}&(b_{31}+b_{34}K_{T0}-b_{21}K_{\theta1}-b_{24}K_{T0}K_{\theta1})(b_{42}+b_{43}+b_{43}K_{\theta0}-b_{12}K_{T1}\\ &\quad-b_{13}K_{T1}K_{\theta0})-(b_{32}+b_{33}K_{\theta0}-b_{22}K_{\theta1}-b_{23}K_{\theta0}K_{\theta1})\\ &\quad\times(b_{41}+b_{44}K_{T0}-b_{11}K_{T1}-b_{14}K_{T0}K_{T1})=0\end{aligned} \tag{5.9}$$

表3中不同边界条件下的频率方程为方程(5.9)的特例. 例如在 $x=0$，夹紧的端点条件为：$K_{T0}=K_{\theta0}=\infty$ 简支的端点条件为：$K_{T0}=\infty$，$K_{\theta0}=0$；悬空的端点条件为：$K_{T0}=K_{\theta0}=0$.

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General Analytic Solution of Dynamic Response of Beams with Nonhomogeneity and Variable Cross Section

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Abstract

In this paper, a new method, the step-reduction method, is proposed to investigate the dynamic response of the Bernoulli-Euler beams with arbitrary nonhomogeneity and arbitrary variable cross section under arbitrary loads. Both free vibration and forced vibration of such beams are studied. The new method requires to discretize the space domain into a number of elements. Each element can be treated as a homogeneous one with uniform thickness. Therefore, the general analytical solution of homogeneous beams with uniform cross section can be used in each element. Then, the general analytic solution of the whole beam in terms of initial parameters can be obtained by satisfying the physical and geometric continuity conditions at the adjacent elements. In the case of free vibration, the frequency equation in analytic form can be obtained, and in the case of forced vibration, a final solution in analytical form can also be obtained which is involved in solving a set of simultaneous algebraic equations with only two unknowns which are independent of the numbers of elements divided. The present analysis can also be extended to the study of the vibration of such beams with viscous and hysteretic damping and other kinds of beams and other structural elements with arbitrary nonhomogeneity and arbitrary variable thickness.

Key words nonhomogeneity, variable thickness, Bernoulli-Euler beam, discretization, dynamic response