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厚壳理论及其在圆柱壳中的应用

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摘　要

本文从Hellinger-Reissner广义变分原理出发，以位移和应力的假设为基础，建立了厚壳理论．文中把壳体的位移展开为其厚度方向的幂级数，对平行和垂直于中面的位移分别保留其级数的前四项和前三项，并假定壳体的法向挤压和横向剪切应力沿壳厚为三次曲线，使其满足上下壳面上的应力条件，利用变分原理推导出分析厚壳所需的物理方程，平衡方程和边界条件．文中对圆柱壳的情况作了实例计算，并作了光弹性实验，结果表明理论和实验符合良好．

关键词　厚壳　圆柱壳　法向挤压应力　变分原理　光弹性实验

一、引　　言

在各种工程领域中，经常遇到厚壳结构的分析问题．曾有不少学者提出过中厚度壳理论[1]~[4]，这些理论都是在Kirchhoff-Love直法线假设的基础上进行修正，在一定程度上考虑了挤压和剪切应力的影响，使之适用于较厚体的分析．对于更厚的壳体，这些理论仍不适用．诚然在处理某些问题时，可把它作为三维弹性体来进行分析[5]，但是这种方法只有在很特殊的情况下才能得到问题的解．对于一般的情况，问题相当复杂而难于求得问题的解．本文采用位移和应力为壳厚方向高次幂级数的假设，将三维的弹性厚壳近似地化为二维问题来处理，使厚壳分析工作得到大量的合理简化，较容易找到问题的解．数值例子表明，本文的理论与实验符合良好．

二、厚壳的基本理论

（1）　壳体中位移和应力的假设

图1为壳体中的一个单元，α，β，γ为互相垂直的曲线坐标，壳体表面上作用有载荷p_1^+，p_1^-，p_2^+，p_2^-，p_3^+，p_3^-．假定沿α，β，γ方向的位移u，v，w可展开为γ的幂级数，对u，v取至级数的前四项，w取至级数的前三项，即假定壳体的位移场为：

$$\left.\begin{aligned}u&=u_0(\alpha,\beta)+\gamma u_1(\alpha,\beta)+\gamma^2u_2(\alpha,\beta)+\gamma^3u_3(\alpha,\beta)\\v&=v_0(\alpha,\beta)+\gamma v_1(\alpha,\beta)+\gamma^2v_2(\alpha,\beta)+\gamma^3v_3(\alpha,\beta)\\w&=w_0(\alpha,\beta)+\gamma w_1(\alpha,\beta)+\gamma^2w_2(\alpha,\beta)\end{aligned}\right\}\tag{2·1}$$

式中　γ的各次幂的系数为互相独立的函数.

在厚壳中法向挤压和横向剪切应力的影响相当重要，要设法使它们满足上、下表面的应力条件.如图1所示，在壳体上、表面上的应力条件为：

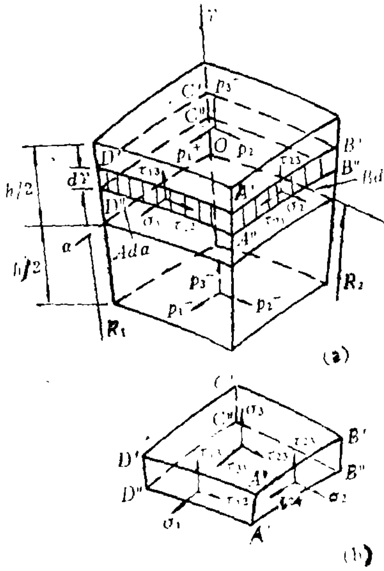

图1　壳体单元

$$\sigma_3(\alpha,\beta,\pm h/2)=\begin{cases}p_3^+\\p_3^-\end{cases},\quad \tau_{13}(\alpha,\beta,\pm h/2)=\begin{cases}p_1^+\\p_1^-\end{cases},$$

$$\tau_{23}(\alpha,\beta,\pm h/2)=\begin{cases}p_2^+\\p_2^-\end{cases}\tag{2·2}$$

我们假设

$$\left.\begin{aligned}(1+k_1\gamma)(1+k_2\gamma)\sigma_3&=\omega_0^*(\alpha,\beta)+\gamma\omega_1^*(\alpha,\beta)\\&\quad+\gamma^2\omega_2^*(\alpha,\beta)+\gamma^3\omega_3^*(\alpha,\beta)\\(1+k_2\gamma)\tau_{13}&=\varphi_0^*(\alpha,\beta)+\gamma\varphi_1^*(\alpha,\beta)+\gamma^2\varphi_2^*(\alpha,\beta)+\gamma^3\varphi_3^*(\alpha,\beta)\\(1+k_1\gamma)\tau_{23}&=\psi_0^*(\alpha,\beta)+\gamma\psi_1^*(\alpha,\beta)+\gamma^2\psi_2^*(\alpha,\beta)+\gamma^3\psi_3^*(\alpha,\beta)\end{aligned}\right\}\tag{2·3a}$$

并使之满足条件(2·2)，便得到如下的应力表达式：

$$\left.\begin{aligned}(1+k_1\gamma)(1+k_2\gamma)\sigma_3&=[\omega_1(\alpha,\beta)+\gamma\omega_2(\alpha,\beta)]\left(1-\frac{4\gamma^2}{h^2}\right)\\&\quad+\frac{1}{2}(p_3^+H^+-p_3^-H^-)+\frac{\gamma}{h}(p_3^+H^++p_3^-H^-)\\(1+k_2\gamma)\tau_{13}&=[\varphi_1(\alpha,\beta)+\gamma\varphi_2(\alpha,\beta)]\left(1-\frac{4\gamma^2}{h^2}\right)\\&\quad+\frac{1}{2}(p_1^+H_1^+-p_1^-H_1^-)+\frac{\gamma}{h}(p_1^+H_1^++p_1^-H_1^-)\\(1+k_1\gamma)\tau_{23}&=[\psi_1(\alpha,\beta)+\gamma\psi_2(\alpha,\beta)]\left(1-\frac{4\gamma^2}{h^2}\right)\\&\quad+\frac{1}{2}(p_2^+H_2^+-p_2^-H_2^-)+\frac{\gamma}{h}(p_2^+H_2^++p_2^-H_2^-)\end{aligned}\right\}\tag{2·3b}$$

式中　φ_1，φ_2，ψ_1，ψ_2，ω_1，ω_2是互相独立的函数，k_1，k_2为中面曲率，而

$$H^\pm=(1\pm k_1h/2)(1\pm k_2h/2),\quad H_1^\pm=(1\pm k_2h/2),\quad H_2^\pm=(1\pm k_1h/2)\tag{2·3c}$$

上式中的因子$(1+k_1\gamma)$和$(1+k_2\gamma)$是为了运算方便而乘上去的.我们对位移和应力同时分别作了假设，但因各具有γ的高次幂，它们之间不会造成大的矛盾，实例计算表明了这一点.

（2）　壳体的几何方程

在正交曲线坐标中，弹性力学的几何方程为：

$$
\left.\begin{aligned}
e_1&=\frac{1}{1+k_1\gamma}\left(\frac{1}{A}\frac{\partial u}{\partial \alpha}+\frac{v}{AB}\frac{\partial A}{\partial B}+k_1 w\right)\\
e_2&=\frac{1}{1+k_2\gamma}\left(\frac{1}{B}\frac{\partial v}{\partial B}+\frac{u}{AB}\frac{\partial B}{\partial \alpha}+k_2 w\right)\\
e_3&=\frac{\partial w}{\partial \gamma},\quad e_{13}=\frac{1}{A(1+k_1\gamma)}\frac{\partial w}{\partial \alpha}+\frac{\partial u}{\partial \gamma}-\frac{k_1}{1+k_1\gamma}u\\
e_{23}&=\frac{1}{B(1+k_2\gamma)}\frac{\partial w}{\partial \beta}+\frac{\partial v}{\partial \gamma}-\frac{k_2}{1+k_2\gamma}v\\
e_{12}&=\frac{B(1+k_2\gamma)}{A(1+k_1\gamma)}\frac{\partial}{\partial \alpha}\left[\frac{v}{B(1+k_2\gamma)}\right]+\frac{A(1+k_1\gamma)}{B(1+k_2\gamma)}\frac{\partial}{\partial \beta}\left[\frac{u}{A(1+k_1\gamma)}\right]
\end{aligned}\right\}\qquad(2\cdot4)
$$

式中　e_1，e_2，e_3，e_{13}，e_{23}，e_{12}为壳体中的应变分量，A、B为中面的Lamé系数。

把位移(2·1)代入(2·4)式后，将其按γ的幂展开，取三次幂以下的各项，得到如下的壳体几何方程：

$$
\left.\begin{aligned}
&e_1=\varepsilon_1+\gamma\chi_{\alpha1}+\gamma^2\chi_{\alpha2}+\gamma^3\chi_{\alpha3},\quad e_2=\varepsilon_2+\gamma\chi_{\beta1}+\gamma^2\chi_{\beta2}+\gamma^3\chi_{\beta3}\\
&e_3=w_1+2\gamma w_2,\quad e_{13}=\gamma_{13}+\gamma\kappa_{13}+\gamma^2\psi_{13}+\gamma^3\lambda_{13}\\
&e_{23}=\gamma_{23}+\gamma\kappa_{23}+\gamma^2\psi_{23}+\gamma^3\lambda_{23},\quad e_{12}=\gamma_{12}+\gamma\kappa_{12}+\gamma^2\psi_{12}+\gamma^3\lambda_{12}
\end{aligned}\right\}\qquad(2\cdot5)
$$

式中ε_1，ε_2，…，λ_{12}列在附录中。

（3）　壳体的内力

把壳体横截面上的应力向中面简化，就得单位长度的内力，如图2所示：

$$
\left.\begin{aligned}
(N_{11},M_{11})&=\int_{-h/2}^{h/2}\sigma_1(1+k_2\gamma)(1,\gamma)d\gamma\\
(N_{22},M_{22})&=\int_{-h/2}^{h/2}\sigma_2(1+k_1\gamma)(1,\gamma)d\gamma\\
(N_{12},M_{12})&=\int_{-h/2}^{h/2}\tau_{12}(1+k_2\gamma)(1,\gamma)d\gamma\\
(N_{21},M_{21})&=\int_{-h/2}^{h/2}\tau_{21}(1+k_1\gamma)(1,\gamma)d\gamma\\
(N_{13},N_{23})&=\int_{-h/2}^{h/2}(\tau_{13}(1+k_2\gamma),\tau_{23}(1+k_1\gamma))d\gamma
\end{aligned}\right\}\qquad(2\cdot6\text{a})
$$

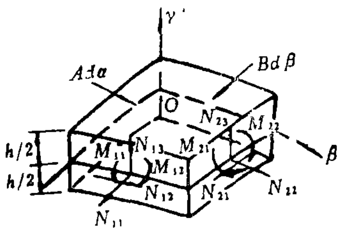

图2　壳体的内力

为了方便起见并引入如下的记号：

$$
\left.\begin{aligned}
(R_{11},P_{11})&=\int_{-h/2}^{h/2}\sigma_1(1+k_2\gamma)(\gamma^2,\gamma^3)d\gamma\\
(R_{22},P_{22})&=\int_{-h/2}^{h/2}\sigma_2(1+k_1\gamma)(\gamma^2,\gamma^3)d\gamma\\
(M_{13},R_{13},P_{13})&=\int_{-h/2}^{h/2}\tau_{13}(1+k_2\gamma)(\gamma,\gamma^2,\gamma^3)d\gamma\\
(M_{23},R_{23},P_{23})&=\int_{-h/2}^{h/2}\tau_{23}(1+k_1\gamma)(\gamma,\gamma^2,\gamma^3)d\gamma\\
(R_{21},P_{21})&=\int_{-h/2}^{h/2}\tau_{21}(1+k_1\gamma)(\gamma^2,\gamma^3)d\gamma\\
(R_{12},P_{12})&=\int_{-h/2}^{h/2}\tau_{12}(1+k_2\gamma)(\gamma^2,\gamma^3)d\gamma\\
(S_0,S_1,S_2,S_3)&=\int_{-h/2}^{h/2}\sigma_3(1+k_1\gamma)(1+k_2\gamma)(1,\gamma,\gamma^2,\gamma^3)d\gamma
\end{aligned}\right\}\qquad(2\cdot6\text{b})
$$

（4） 壳体基本方程的推导

求解壳体的位移和应力，除几何方程外，还需要平衡方程，物理方程和边界条件．我们应用Hellinger-Reissner广义变分原理[6]来推导这些方程．本文限于讨论各向同性线性材料．对于壳体，Hellinger-Reissner变分原理形式为：

$$\begin{aligned}\delta\Gamma=\delta\Big[&\iiint_V\Big\{\sigma_1e_1+\sigma_2e_2+\sigma_3e_3+\tau_{12}e_{12}+\tau_{13}e_{13}\\&+\tau_{23}e_{23}-\frac{1}{2E}[\sigma_1^2+\sigma_2^2+\sigma_3^2\\&-2\mu(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)+2(1+\mu)(\tau_{12}^2+\tau_{13}^2\\&+\tau_{23}^2)]\Big\}(1+k_1\gamma)(1+k_2\gamma)ABd\alpha d\beta d\gamma\\&-\iint_S[(p_1^+u^++p_2^+v^++p_3^+w^+)H^++(p_1^-u_1^-+p_2^-v^-\\&+p_3^-w^-)H^-]ABd\alpha d\beta\\&-\int_l\Big[\int_{-h/2}^{h/2}(\sigma_nu_n+\tau_{nt}u_t+\tau_{n\gamma}w)(1+k_t\gamma)d\gamma\Big]dl\Big]=0\end{aligned}\tag{2·7}$$

其中 E是杨氏模量，μ为Poisson比，V为壳体体积，S为中面面积，u^+、u^-等分别表示上下顶面处的位移，σ_n，τ_{nt}，$\tau_{n\gamma}$为壳体边界的法向和切向应力，k_t为沿边界的中面曲率，l为边界长度．把式(2·1)、(2·3b)和(2·4)代入(2·7)，然后对γ进行分部积分，利用位移和应力变分的任意性可得到：

壳体物理方程：

$$\left.\begin{aligned}&e_1=\frac{1}{E}[\sigma_1-\mu(\sigma_2+\sigma_3)],\ e_2=\frac{1}{E}[\sigma_2-\mu(\sigma_1+\sigma_3)],\\&e_{12}=\frac{2(1+\mu)}{E}\tau_{12}\\&\frac{2h}{3}w_1+\frac{1}{E}\Big\{-S_0+(k_1+k_2)S_1-\Big(k_1^2+k_2^2+k_1k_2-\frac{4}{h^2}\Big)S_2\\&\qquad+\Big[k_1^3+k_1^2k_2+k_1k_2^2+k_2^3-\frac{4}{h^2}(k_1+k_2)\Big]S_3\\&\qquad+\mu\Big[N_{11}-k_2M_{11}+\Big(k_2^2-\frac{4}{h^2}\Big)R_{11}-k_2\Big(k_2^2-\frac{4}{h^2}\Big)P_{11}+N_{22}\\&\qquad-k_1M_{22}+\Big(k_1^2-\frac{4}{h^2}\Big)R_{22}-k_1\Big(k_1^2-\frac{4}{h^2}\Big)P_{22}\Big]\Big\}=0\\&\frac{h^3w_2}{15}+\frac{1}{E}\Big\{-S_1+(k_1+k_2)S_2-\Big(k_1^2+k_2^2+k_1k_2-\frac{4}{h^2}\Big)S_3\\&\qquad+\mu\Big[M_{11}-k_2R_{11}+\Big(k_2^2-\frac{4}{h^2}\Big)P_{11}\\&\qquad+M_{22}-k_1R_{22}+\Big(k_1^2-\frac{4}{h^2}\Big)P_{22}\Big\}=0\\&\frac{2h}{3}\gamma_{13}+\frac{k_1h^3}{30}\kappa_{13}+\frac{h^3}{30}\psi_{13}+\frac{k_1h^5}{280}\lambda_{13}-\frac{2(1+\mu)}{E}\Big\{N_{13}\end{aligned}\right\}\tag{2·8}$$

$$-(k_2-k_1)M_{13}+\left(k_2^2-k_1k_2-\frac{4}{h^2}\right)R_{13}$$

$$-\left[k_2^3-k_1k_2^2+\frac{4}{h^2}(k_1-k_2)\right]P_{13}\Big\}=0$$

$$\frac{k_1h^2}{30}\gamma_{13}+\frac{h^3}{30}\kappa_{13}+\frac{k_1h^5}{280}\psi_{13}+\frac{h^5}{280}\lambda_{13}$$

$$-\frac{2(1+\mu)}{E}\left\{M_{13}-(k_2-k_1)R_{13}+\left(k_2^2-k_1k_2-\frac{4}{h^2}\right)P_{13}\right\}=0$$

$$\frac{2h}{3}\gamma_{23}+\frac{k_2h^3}{30}\kappa_{23}+\frac{h^3}{30}\psi_{23}+\frac{k_2h^5}{280}\lambda_{23}$$

$$-\frac{2(1+\mu)}{E}\left\{N_{23}-(k_1-k_2)M_{23}+\left(k_1^2-k_1k_2-\frac{4}{h^2}\right)R_{23}\right.$$

$$\left.-\left[k_1^3-k_1^2k_2+(k_2-k_1)\frac{4}{h^2}\right]P_{23}\right\}=0$$

$$\frac{k_2h^3}{30}\gamma_{23}+\frac{h^3}{30}\kappa_{23}+\frac{k_2h^5}{280}\psi_{23}+\frac{h^5}{280}\lambda_{23}$$

$$-\frac{2(1+\mu)}{E}\left\{M_{23}-(k_1-k_2)R_{23}+\left(k_1^2-k_1k_2-\frac{4}{h^2}\right)P_{23}\right\}=0$$

壳体的平衡方程：

$$\frac{\partial}{\partial\alpha}(BN_{11})-N_{22}\frac{\partial B}{\partial\alpha}+N_{12}\frac{\partial A}{\partial\beta}+\frac{\partial}{\partial\beta}(AN_{21})$$
$$+AB(k_1N_{13}+p_1^+H^++p_1^-H^-)=0$$

$$\frac{\partial}{\partial\beta}(AN_{22})-N_{11}\frac{\partial A}{\partial\beta}+N_{21}\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\alpha}(BN_{12})$$
$$+AB(k_2N_{23}+p_2^+H^++p_2^-H^-)=0$$

$$\frac{\partial}{\partial\alpha}(BN_{13})+\frac{\partial}{\partial\beta}(AN_{23})-AB(k_1N_{11}$$
$$+k_2N_{22}-p_3^+H^+-p_3^-H^-)=0$$

$$\frac{\partial}{\partial\beta}(AM_{21})+M_{12}\frac{\partial A}{\partial\beta}-M_{22}\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\alpha}(BM_{11})$$
$$-AB\left[N_{13}+\frac{h}{2}(p_1^-H^--p_1^+H^+)\right]=0$$

$$\frac{\partial}{\partial\alpha}(BM_{12})+M_{21}\frac{\partial B}{\partial\alpha}-M_{11}\frac{\partial A}{\partial\beta}+\frac{\partial}{\partial\beta}(AM_{22})$$
$$-AB\left[N_{23}+\frac{h}{2}(p_2^-H^--p_2^+H^+)\right]=0$$

$$\frac{\partial}{\partial\beta}(AM_{23})+\frac{\partial}{\partial\alpha}(BM_{13})-AB\left[k_1M_{11}+k_2M_{22}\right.$$
$$\left.+S_0-\frac{h}{2}(p_3^+H^+-p_3^-H^-)\right]=0$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial\alpha}(BR_{11})-R_{22}\frac{\partial B}{\partial\alpha}+R_{12}\frac{\partial A}{\partial\beta}+\frac{\partial}{\partial\beta}(AR_{21})\\
&\qquad -AB\left[2M_{13}+k_1R_{13}-\frac{h^2}{4}(p_1^+H^++p_1^-H^-)\right]=0\\
&\frac{\partial}{\partial\beta}(AR_{22})-R_{11}\frac{\partial A}{\partial\beta}+R_{21}\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\alpha}(BR_{12})\\
&\qquad -AB\left[2M_{23}+k_2R_{23}-\frac{h^2}{4}(p_2^+H^++p_2^-H^-)\right]=0\\
&\frac{\partial}{\partial\alpha}(BR_{13})+\frac{\partial}{\partial\beta}(AR_{23})-AB\left[k_1R_{11}+k_2R_{22}\right.\\
&\qquad \left.+2S_1-\frac{h^2}{4}(p_3^+H^++p_3^-H^-)\right]=0\\
&\frac{\partial}{\partial\alpha}(BP_{11})-P_{22}\frac{\partial B}{\partial\alpha}+\frac{\partial}{\partial\beta}(AP_{21})+P_{12}\frac{\partial A}{\partial\beta}\\
&\qquad -AB\left[2k_1P_{13}+3R_{13}-\frac{h^3}{8}(p_1^+H^+-p_1^-H^-)\right]=0\\
&\frac{\partial}{\partial\beta}(AP_{22})-P_{11}\frac{\partial A}{\partial\beta}+\frac{\partial}{\partial\alpha}(BP_{12})+P_{21}\frac{\partial B}{\partial\alpha}\\
&\qquad -AB\left[2k_2P_{23}+3R_{23}-\frac{h^3}{8}(p_2^+H^+-p_2^-H^-)\right]=0
\end{aligned}\right\}\tag{2·9}$$

应力边界条件：

$$\left.\begin{aligned}
&N_n-N_{11}\cos^2\theta_1-N_{22}\cos^2\theta_2-(N_{12}+N_{21})\cos\theta_1\cos\theta_2=0\\
&N_{nt}+(N_{11}-N_{22})\cos\theta_1\cos\theta_2-N_{12}\cos^2\theta_1+N_{21}\cos^2\theta_2=0\\
&N_{nr}-N_{13}\cos\theta_1-N_{23}\cos\theta_2=0\\
&M_n-M_{11}\cos^2\theta_1-M_{22}\cos^2\theta_2-(M_{12}+M_{21})\cos\theta_1\cos\theta_2=0\\
&M_{nt}+(M_{11}-M_{22})\cos\theta_1\cos\theta_2-M_{12}\cos^2\theta_1+M_{21}\cos^2\theta_2=0\\
&M_{nr}-M_{23}\cos\theta_2-M_{13}\cos\theta_1=0\\
&R_n-R_{11}\cos^2\theta_1-R_{22}\cos^2\theta_2-(R_{12}+R_{21})\cos\theta_1\cos\theta_2=0\\
&R_{nt}+(R_{11}-R_{22})\cos\theta_1\cos\theta_2-R_{12}\cos^2\theta_1+R_{21}\cos^2\theta_2=0\\
&R_{nr}-R_{23}\cos\theta_2-R_{13}\cos\theta_1=0\\
&P_n-P_{11}\cos^2\theta_1-P_{22}\cos^2\theta_2-(P_{12}+P_{21})\cos\theta_1\cos\theta_2=0\\
&P_{nt}+(P_{11}-P_{22})\cos\theta_1\cos\theta_2-P_{12}\cos^2\theta_1+P_{21}\cos^2\theta_2=0
\end{aligned}\right\}\tag{2·10a}$$

式中θ_1，θ_2分别是壳体边界法线与α和β的夹角，而

$$\left.\begin{aligned}
&(N_n,M_n,R_n,P_n)=\int_{-h/2}^{h/2}\sigma_n(1+k_t\gamma)(1,\gamma,\gamma^2,\gamma^3)d\gamma\\
&(N_{nt},M_{nt},R_{nt},P_{nt})=\int_{-h/2}^{h/2}\tau_{nt}(1+k_t\gamma)(1,\gamma,\gamma^2,\gamma^3)d\gamma\\
&(N_{nr},M_{nr},R_{nr})=\int_{-h/2}^{h/2}\tau_{nr}(1+k_t\gamma)(1,\gamma,\gamma^2)d\gamma
\end{aligned}\right\}\tag{2·10b}$$

上面推得的式(2·8)，(2·9)是位移和应力的形式分别受(2·1)和(2·3)的约束时的物理方

程和平衡方程. (2·8)的前三式即为Hooke定律中的三个关系式，其余各式是由于本文作了应力假设而得到的补充物理方程，从中可求得函数φ_1，φ_2，ψ_1，ψ_2，ω_1，ω_2. (2·9)中的前五式与薄壳中的平衡方程相同，其余各式是由于增加了变形的自由度而增加的补充平衡方程. 本文讨论的壳体理论中共有二十个未知函数，其中有十一个位移参数(u_0，v_0，…，w_2)，三个应力(σ_1，σ_2，τ_{12})和六个应力参数(φ_1，φ_2，ψ_1，ψ_2，ω_1，ω_2). 共有二十个方程，其中有九个物理方程，十一个平衡方程，刚好求解二十个未知函数. 求解时可采用位移法，把应力，应力参数及各内力素都用位移参数表示，然后由平衡方程和边界条件求出位移参数.

三、圆柱壳实例和光弹实验

我们来考察两端简支的圆柱厚壳的轴对称问题（图3). 把满足下列条件的支承边界称为简支边：

$$(w_0, w_1, w_2, N_{11}, M_{11}, R_{11}, P_{11})|_{\alpha=0,a}=0 \tag{3·1}$$

在轴对称情况下$p_2^{\pm}=0$，$v=0$，p_1^-，p_3^+和位移只是α的函数，p_1^+和p_1^-的合力为零. 由(2·9)可得由位移参数表达的平衡方程：

$$[L_{ij}]_{7\times7}\{U\}=\{P\} \tag{3·2}$$

其中 $\{U\}=(u_0, w_0, u_1, w_1, u_2, w_2, u_3)^T$为位移列阵. $\{P\}=(p_1, p_2, \cdots, p_7)^T$为载荷列阵. p_1，p_3，p_5，p_7和p_2，p_4，p_6分别只与$p_1^{\pm}$，$dp_3^{\pm}/d\alpha$和$p_3^{\pm}$，$dp_1^{\pm}/d\alpha$有关. 在算子矩阵$[L_{ij}]_{7\times7}$中，当$i=1$，3，5，7，$j=2$，4，6和$i=2$，4，6，$j=1$，3，5，7时，算子L_{ij}只与$d/d\alpha$有关，而其余的算子只与$d^2/d\alpha^2$有关.

在一般的边界下，可用欧拉待定指数法求解平衡方程(3.2)在相应边界条件下的解. 对于简支边界条件也可用富氏级数求解. 这里采用后一种方法. 设：

$$\begin{Bmatrix} u_i \\ w_i \end{Bmatrix}=\sum_{m=0}^{\infty}\begin{Bmatrix} A_m^{(i)}\cos\dfrac{m\pi\alpha}{a} \\ C_m^{(i)}\sin\dfrac{m\pi\alpha}{a} \end{Bmatrix} \quad \begin{pmatrix} i=0,1,2,3 \\ j=0,1,2 \end{pmatrix} \tag{3·3}$$

把载荷展开为相应的级数：

$$\begin{Bmatrix} p_1^{\pm} \\ p_3^{\pm} \end{Bmatrix}=\sum_{m=0}^{\infty}\begin{Bmatrix} k_m^{\pm}\cos\dfrac{m\pi\alpha}{a} \\ q_m^{\pm}\sin\dfrac{m\pi\alpha}{a} \end{Bmatrix} \tag{3·4}$$

(3·3)，(3·4)可满足边界条件(3·1). 把(3·3)，(3·4)代入(3·2)可确定系数$A_m^{(i)}$，$C_m^{(j)}$，从而求得位移和应力. 对图4所示的局部载荷可求得：

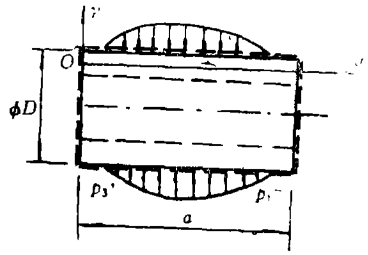

图3 简支轴对称圆柱壳

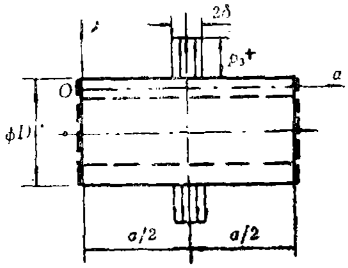

图4 局部载荷圆柱壳

$$k_m^{\pm}=0,\quad q_m^{-}=0,\quad q_m^{+}=\begin{cases}(-1)^{\frac{m+1}{2}}\dfrac{4p_3^{+}}{m\pi}\sin\dfrac{m\pi\delta}{a} & (m=2n+1)\\ 0 & (m=2n)\end{cases}\tag{3·5}$$

其中 $n=0,\ 1,\ 2,\ \cdots$.

下列各图线是对局部载荷情况的理论计算和光弹实验的结果. 数据取 $a=40\text{cm}$, $\delta=0.4\text{cm}$, $D=11.2\text{cm}$, $\mu=0.25$, $\lambda=D/2h=1.0$, 而 $\alpha'=(a-a/2)/\delta$.

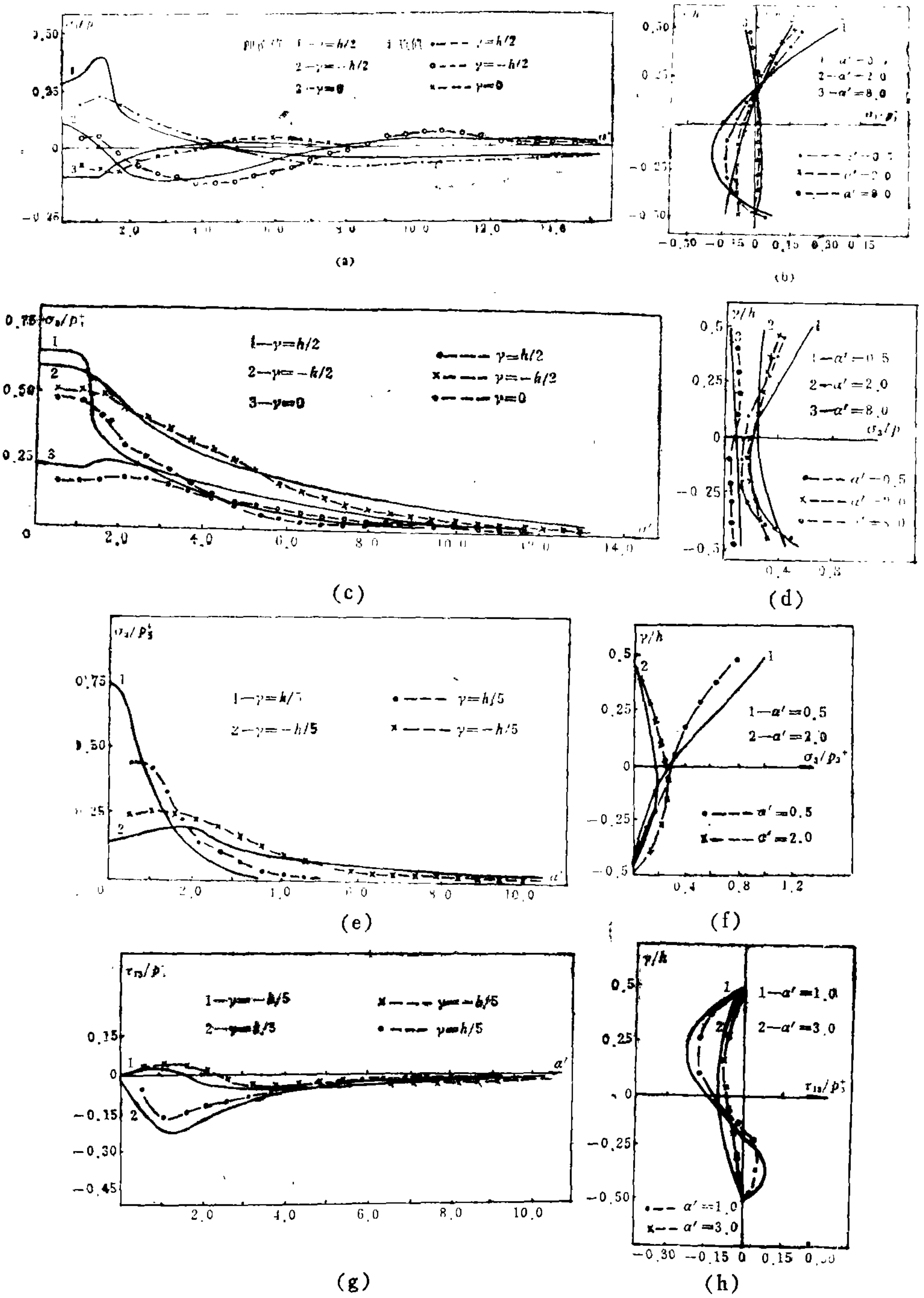

图 5　圆柱壳应力的理论值与实验值

四、结　　论

从本文的分析可得出如下结论：

（1）　本文提出的厚壳理论比较全面地考虑了法向挤压和横向剪切应力的影响，可用于厚壳结构的分析。

（2）　文中例子的理论结果与实验符合良好，所给出的σ_3，τ_{13}与中厚壳理论的直线分布和抛物线分布的假设相差较大。

附　　录

$$e_1=\frac{1}{A}\frac{\partial u_0}{\partial \alpha}+\frac{v_0}{AB}\frac{\partial A}{\partial \beta}+k_1w_0$$

$$\chi_{\alpha 1}=\frac{1}{A}\left(\frac{\partial u_1}{\partial \alpha}-k_1\frac{\partial u_0}{\partial \alpha}\right)+\frac{1}{AB}(v_1-k_1v_0)\frac{\partial A}{\partial \beta}+k_1(w_1-k_1w_0)$$

$$\chi_{\alpha 2}=\frac{1}{A}\left(\frac{\partial u_2}{\partial \alpha}-k_1\frac{\partial u_1}{\partial \alpha}+k_1^2\frac{\partial u_0}{\partial \alpha}\right)+\frac{1}{AB}(v_2-k_1v_1-k_1^2v_0)\frac{\partial A}{\partial \beta}+k_1(w_2-k_1w_1+k_1^2w_0)$$

$$\chi_{\alpha 3}=\frac{1}{A}\left(\frac{\partial u_3}{\partial \alpha}-k_1\frac{\partial u_2}{\partial \alpha}+k_1^2\frac{\partial u_1}{\partial \alpha}-k_1^3\frac{\partial u_0}{\partial \alpha}\right)+\frac{1}{AB}(v_3-k_1v_2+k_1^2v_1-k_1^3v_0)\frac{\partial A}{\partial \beta}$$
$$+k_1^2(-w_2+k_2w_1-k_1^2w_0)$$

$$e_2=\frac{1}{B}\frac{\partial v_0}{\partial \beta}+\frac{u_0}{AB}\frac{\partial B}{\partial \alpha}+k_2w_0$$

$$\chi_{\beta 1}=\frac{1}{B}\left(\frac{\partial v_1}{\partial \beta}-k_2\frac{\partial v_0}{\partial \beta}\right)+\frac{1}{AB}(u_1-k_2u_0)\frac{\partial B}{\partial \alpha}+k_2(w_1-k_2w_0)$$

$$\chi_{\beta 2}=\frac{1}{B}\left(\frac{\partial v_2}{\partial \beta}-k_2\frac{\partial v_1}{\partial \beta}+k_2^2\frac{\partial v_0}{\partial \beta}\right)+\frac{1}{AB}(u_2-k_2u_1+k_2^2u_0)\frac{\partial B}{\partial \alpha}+k_2(w_2-k_2w_1+k_2^2w_0)$$

$$\chi_{\beta 3}=\frac{1}{B}\left(\frac{\partial v_3}{\partial \beta}-k_2\frac{\partial v_2}{\partial \beta}+k_2^2\frac{\partial v_1}{\partial \beta}-k_2^3\frac{\partial v_0}{\partial \beta}\right)+\frac{1}{AB}(u_3-k_2u_2+k_2^2u_1-k_2^3u_0)\frac{\partial B}{\partial \alpha}$$
$$+k_2^2(-w_2+k_2w_1-k_2^2w_0)$$

$$\gamma_{23}=\frac{1}{B}\frac{\partial w_0}{\partial \beta}+v_1-k_2v_0$$

$$\kappa_{23}=\frac{1}{B}\left(\frac{\partial w_1}{\partial \beta}-k_2\frac{\partial w_0}{\partial \beta}\right)+(2v_2-k_2v_1+k_2^2v_0)$$

$$\psi_{23}=\frac{1}{B}\left(\frac{\partial w_2}{\partial \beta}-k_2\frac{\partial w_1}{\partial \beta}+k_2^2\frac{\partial w_0}{\partial \beta}\right)+(3v_3-k_2v_2+k_2^2v_1-k_2^3v_0)$$

$$\lambda_{23}=\frac{1}{B}\left(-k_2\frac{\partial w_2}{\partial \beta}+k_2^2\frac{\partial w_1}{\partial \beta}-k_2^3\frac{\partial w_0}{\partial \beta}\right)-k_2(v_3-k_2v_2+k_2^2v_1-k_2^3v_0)$$

$$\gamma_{13}=\frac{1}{A}\frac{\partial w_0}{\partial \alpha}+u_1-k_1u_0$$

$$\kappa_{13}=\frac{1}{A}\left(\frac{\partial w_1}{\partial \alpha}-k_1\frac{\partial w_0}{\partial \alpha}\right)+(2u_2-k_1u_1+k_1^2u_0)$$

$$\psi_{13}=\frac{1}{A}\left(\frac{\partial w_2}{\partial \alpha}-k_1\frac{\partial w_1}{\partial \alpha}+k_1^2\frac{\partial w_0}{\partial \alpha}\right)+(3u_3-k_1u_2+k_1^2u_1+k_1^3u_0)$$

$$\lambda_{13}=\frac{1}{A}\left(-k_1\frac{\partial w_2}{\partial \alpha}+k_1^2\frac{\partial w_1}{\partial \alpha}-k_1^3\frac{\partial w_0}{\partial \alpha}\right)-k_1(u_3-k_1u_2+k_1^2u_1-k_1^3u_0)$$

$$\gamma_{12}=\frac{1}{A}\left(\frac{\partial v_0}{\partial \alpha}-\frac{v_0}{B}\frac{\partial B}{\partial \alpha}\right)+\frac{1}{B}\left(\frac{\partial u_0}{\partial \beta}-\frac{u_0}{A}\frac{\partial A}{\partial \beta}\right)$$

$$\kappa_{12}=\frac{1}{A}\left(\frac{\partial v_1}{\partial \alpha}-k_1\frac{\partial v_0}{\partial \alpha}\right)-\frac{1}{AB}(v_1-k_2v_0)\frac{\partial B}{\partial \alpha}+\frac{1}{B}\left(\frac{\partial u_1}{\partial \alpha}-k_2\frac{\partial u_0}{\partial \alpha}\right)-\frac{1}{AB}(u_1-k_1u_0)\frac{\partial A}{\partial \beta}$$

$$\psi_{12}=\frac{1}{A}\left(\frac{\partial v_2}{\partial \alpha}-k_1\frac{\partial v_1}{\partial \alpha}+k_1^2\frac{\partial v_0}{\partial \alpha}\right)-\frac{1}{AB}(v_2-k_2v_1+k_2^2v_0)\frac{\partial B}{\partial \alpha}+\frac{1}{B}\left(\frac{\partial u_2}{\partial \beta}-k_2\frac{\partial u_1}{\partial \beta}\right.$$

$$\left.+k_2^2\frac{\partial u_0}{\partial \beta}\right)-\frac{1}{AB}(u_2-k_1u_1+k_1^2u_0)\frac{\partial A}{\partial \beta}$$

$$\lambda_{12}=\frac{1}{A}\left(\frac{\partial v_3}{\partial \alpha}-k_1\frac{\partial v_2}{\partial \alpha}+k_1^2\frac{\partial v_1}{\partial \alpha}-k_1^3\frac{\partial v_0}{\partial \alpha}\right)-\frac{1}{AB}(v_3-k_2v_2+k_2^2v_1-k_2^3v_0)\frac{\partial B}{\partial \alpha}$$

$$+\frac{1}{B}\left(\frac{\partial u_3}{\partial \beta}-k_2\frac{\partial u_2}{\partial \beta}+k_2^2\frac{\partial u_1}{\partial \beta}-k_2^3\frac{\partial u_0}{\partial \beta}\right)-\frac{1}{AB}(u_3-k_1u_2+k_1^2u_1-k_1^3u_0)\frac{\partial A}{\partial \beta}$$

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Theory of Thick-Walled Shells and Its Application in Cylindrical Shell

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Abstract

In this paper, a theory of thick-walled shells is established by means of Hellinger-Reissner's variational principle, with displacement and stress assumptions. The displacements are expanded into power series of the thickness coordinate. Only the first four and the first three terms are used for the displacements parallel and normal to middle surface, respectively. The normal extruding and transverse shear stresses are assumed to be cubic polynomials and to satisfy the boundary stress conditions on the outer and inner surfaces of the shell. The governing equations and boundary conditions are derived by means of variational principle. As an example, a thick-walled cylindrical shell is disscussed with the theory proposed. Furthermore, a photoelastic experiment has been carried out, and the results are in fair agreement with the computations.

Key words thick-walled shell, cylindrical shell, normal extruding stress, variational principle, photoelastic experiment