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圆薄板非对称大变形弯曲问题*

王林祥　王新志　邱　平

（兰州 甘肃工业大学，基础课教学研究部）

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摘　要

本文首先导出圆薄板非轴对称大变形问题的位移基本方程及边界条件．利用变换和摄动法将非线性位移方程线性化，得到了近似边值问题．作为算例，文中研究了圆薄板在较复杂载荷作用下的非线性弯曲问题．

关键词　板　位移　非线性　变换　摄动法

一、引　言

圆薄板非轴对称大变形问题是一个难度较大的问题．1961年Fife开始研究这一问题[1]，1964年 Срубщик 也研究了此问题[2]．我国江福汝教授1982年研究了这方面的问题[3,4,5]，我们也曾用修正迭代法作了些工作[6]．以往的研究工作都是求解以挠度和应力函数描述的非线性耦联方程组，边界条件是大大简化了的，板面内位移的边界条件是以积分形式给出的．对于一般边界条件下的圆薄板非对称大变形问题，至今尚未见到有人作过研究．本文将位移视为基本未知量，导出了以位移描述的圆薄板非轴对称大变形弯曲问题的非线性方程和通常的边界条件，并具体地给出了求解问题的方法．

二、基本方程和边界条件

略去体力下的平衡方程

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}=0 \tag{2.1}$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+2\frac{\tau_{r\theta}}{r}=0 \tag{2.2}$$

$$D\Delta^2(W)=Q+h\frac{\partial^2 W}{\partial r^2}\sigma_r+2h\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)\tau_{r\theta}+h\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\sigma_\theta \tag{2.3}$$

中面变形几何方程

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$$\varepsilon_r=\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 \tag{2.4}$$

$$\varepsilon_\theta=\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{U}{r}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\theta}\right)^2 \tag{2.5}$$

$$\varepsilon_{r\theta}=\frac{1}{r}\frac{\partial U}{\partial\theta}-\frac{V}{r}+\frac{\partial V}{\partial r}+\frac{1}{r}\frac{W}{\partial\theta}\frac{W}{r} \tag{2.6}$$

物理方程

$$\sigma_r=\frac{E}{1-\mu^2}(\varepsilon_r+\mu\varepsilon_\theta) \tag{2.7}$$

$$\sigma_\theta=\frac{E}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_r) \tag{2.8}$$

$$\tau_{r\theta}=\frac{E}{2(1+\mu)}\varepsilon_{r\theta} \tag{2.9}$$

其中E为板的弹性模量，μ为Poisson比，U，V，W分别为板的径向、切向和横向位移，

$$D=\frac{Eh^3}{12(1-\mu^2)},\ \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

将方程(2.4)、(2.5)、(2.6)代入(2.7)、(2.8)、(2.9)，再代入(2.1)、(2.2)、(2.3)可得

$$\frac{r}{1-\mu}\left\{\frac{\partial^2 U}{\partial r^2}+\frac{\partial W}{\partial r}\frac{\partial^2 W}{\partial r^2}+\mu\frac{\partial}{\partial r}\left[\frac{U}{r}+\frac{1}{r}\frac{V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{W}{\partial\theta}\right)\right]^2\right\}$$
$$+\left\{\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2-\left[\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2\right]\right\}$$
$$+\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]=0 \tag{2.10}$$

$$\frac{r}{2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]+\frac{1}{1-\mu}\frac{\partial}{\partial\theta}\left\{\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2+\mu\left[\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2\right]\right\}+\left[\frac{1}{r}\frac{\partial U}{\partial\theta}\right.$$
$$\left.+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]=0 \tag{2.11}$$

$$D\Delta^2(W)=Q+\frac{Eh}{1-\mu^2}\left\{\frac{\partial^2 W}{\partial r^2}\left[\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2+\mu\left(\frac{U}{r}+\frac{1}{r}\cdot\frac{\partial V}{\partial\theta}\right.\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2\right)\right]+(1-\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)+\frac{1}{r}\frac{\partial W}{\partial\theta}\frac{\partial W}{\partial r}\right]$$
$$+\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\left[\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)^2\right.$$
$$\left.\left.+\mu\left(\frac{\partial U}{\partial r}+\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2\right)\right]\right\} \tag{2.12}$$

通常边界条件：

(一) 周边固定夹紧

$r=a$时，$W=\partial W/\partial r=U=V=0$ (2.13a,b,c,d)

$r=0$时，W，$\partial W/\partial r$，U，V有限 (2.14a,b,c,d)

（二） 周边铰支

$r=a$时，$W=M_r=U=V=0$ (2.15a,b,c,d)

$r=0$时，W，M_r，U，V有限 (2.16a,b,c,d)

（三） 周边可移夹紧

$r=a$时，$W=\partial W/\partial r=\sigma_r=\tau_{r\theta}=0$ (2.17a,b,c,d)

$r=0$时，W，$\partial W/\partial r$，σ_r，$\tau_{r\theta}$有限 (2.18a,b,c,d)

引入无量纲量

$$x=\frac{r}{a},\ u=\frac{aU}{h^2},\ v=\frac{aV}{h^2},\ w=\frac{W}{h},\ q=\frac{a^4}{Dh}Q$$

则方程(2.10)、(2.11)、(2.12)可化为

$$\frac{x}{1-\mu}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial}{\partial x}\left[\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right]\right\}$$
$$+\left\{\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\left[\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right]\right\}$$
$$+\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]=0 \tag{2.19}$$

$$\frac{x}{2}\frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]+\frac{1}{1-\mu}\frac{\partial}{\partial\theta}\left\{\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2+\mu\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}+\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)\right.$$
$$\left.+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]=0 \tag{2.20}$$

$$\nabla^2(w)=q+12\left\{\frac{\partial^2 w}{\partial x^2}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\mu\left(\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right)\right]\right.$$
$$+(1-\mu)\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)\left[\frac{1}{x}\frac{\partial u}{\partial\theta}+x\frac{\partial}{\partial x}\left(\frac{v}{x}\right)+\frac{1}{x}\frac{\partial w}{\partial\theta}\frac{\partial w}{\partial x}\right]$$
$$+\left(\frac{1}{x}\frac{\partial w}{\partial x}+\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}\right)\left[\frac{u}{x}+\frac{1}{x}\frac{\partial v}{\partial\theta}+\frac{1}{2}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right)^2\right.$$
$$\left.\left.+\mu\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\right]\right\} \tag{2.21}$$

其中 $$\nabla=\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}$$

相应的无量纲量下的边界条件为：

周边夹紧固定

$x=1$时，$w=\partial w/\partial x=u=v=0$ (2.22a,b,c,d)

$x=0$时，w，$w/\partial x$，u，v有限 (2.23a,b,c,d)

周边铰支

$x=1$时，$w=M_r=u=v=0$ (2.24a,b,c,d)

$x=0$时，w，M_r，u，v有限 (2.25a,b,c,d)

周边可移夹紧

$x=1$时，$w=\partial w/\partial x=\sigma_r=\tau_{r\theta}=0$ (2.26a,b,c,d)

$x=0$时，w，$\partial w/\partial x$，σ_r，$\tau_{r\theta}$有限 (2.27a,b,c,d)

三、问题的求解

为了便于说明求解方法，本文将只讨论在周边夹紧固定边界条件下问题的求解.

将u，v，w，q展为Fourier级数，有

$$u(x,\ \theta)=\sum_{k=-\infty}^{+\infty}[u_{rk}(x)+iu_{ik}(x)]e^{ik\theta} \tag{3.1}$$

$$v(x,\ \theta)=\sum_{k=-\infty}^{+\infty}[v_{rk}(x)+iv_{ik}(x)]e^{ik\theta} \tag{3.2}$$

$$w(x,\ \theta)=\sum_{k=-\infty}^{+\infty}[w_{rk}(x)+iw_{ik}(x)]e^{ik\theta} \tag{3.3}$$

$$q(x,\ \theta)=\sum_{k=-\infty}^{+\infty}[q_{rk}(x)+iq_{ik}(x)]e^{ik\theta} \tag{3.4}$$

其中$u_{rk}(x)$，$u_{ik}(x)$，$v_{rk}(x)$，$v_{ik}(x)$，$w_{rk}(x)$，$w_{ik}(x)$，$q_{rk}(x)$，$q_{ik}(x)$为x的实值函数.将式(3.1)、(3.2)、(3.3)和(3.4)代入方程(2.19)、(2.20)、(2.21)、(2.22)和(2.23)，由$e^{ik\theta}$的正交性可得下列边值问题：

$$\begin{aligned}&xu''_{rk}+u'_{rk}-\frac{2+k^2(1-\mu)}{2x}u_{rk}-\frac{k}{2}\Big[(1+\mu)v'_{ik}-(3-\mu)v_{ik}\cdot\frac{1}{x}\Big]\\&=-\sum_{m=-\infty}^{+\infty}\Big[x(w''_{rm}w'_{rk-m}-w''_{im}w'_{ik-m})+\frac{1-\mu}{2}(w'_{rm}w'_{rk-m}-w'_{im}w'_{ik-m})\\&\quad-\frac{(k-m)(k-k\mu+2m\mu)}{2x}(w'_{rm}w_{rk-m}-w'_{im}w_{ik-m})\\&\quad+\frac{m(k-m)(1+\mu)}{2x^2}(w_{rm}w_{rk-m}-w_{im}w_{ik-m})\Big]\end{aligned} \tag{3.5}$$

$$\begin{aligned}&xv''_{ik}+v'_{ik}-\frac{1-\mu+2k^2}{(1-\mu)x}v_{ik}+\frac{k}{1-\mu}\Big[(1+\mu)u'_{rk}+\frac{3-\mu}{x}u_{rk}\Big]\\&=-\sum_{m=-\infty}^{+\infty}\Big[(k-m)(w''_{rm}w_{rk-m}-w''_{im}w_{ik-m})+\frac{m(1+\mu)}{1-\mu}(w'_{rm}w'_{rk-m}-w'_{im}w'_{ik-m})\\&\quad+\frac{k-m}{x}(w'_{rm}w_{rk-m}-w'_{im}w_{ik-m})-\frac{2m^2(k-m)}{(1-\mu)x^2}(w_{rm}w_{rk-m}-w_{im}w_{ik-m})\Big]\end{aligned} \tag{3.6}$$

$$xu''_{ik}+u'_{ik}-\frac{2+k^2(1-\mu)}{2x}u_{ik}+\frac{k}{2}\Big[(1+\mu)v'_{rk}-\frac{3-\mu}{x}v_{rk}\Big]$$
$$=-\sum_{m=-\infty}^{+\infty}\Big[x(w''_{im}w'_{rk-m}+w''_{rm}w'_{ik-m})+\frac{1-\mu}{2}(w'_{rm}w'_{ik-m}+w'_{im}w'_{rk-m})$$
$$-\frac{(k-m)(k-k\mu+2m\mu)}{2x}(w'_{rm}w_{ik-m}+w'_{im}w_{rk-m})$$
$$+\frac{m(k-m)(1+\mu)}{2x^2}(w_{rm}w_{ik-m}+w_{im}w_{rk-m})\Big] \tag{3.7}$$

$$xv''_{rk}+v'_{rk}-\frac{1-\mu+2k^2}{(1-\mu)x}v_{rk}-\frac{k}{1-\mu}\Big[(1+\mu)u'_{ik}+\frac{3-\mu}{x}u_{ik}\Big]$$
$$=\sum_{m=-\infty}^{+\infty}\Big[(k-m)(w''_{rm}w_{ik-m}+w''_{im}w_{rk-m})+\frac{m(1+\mu)}{1-\mu}(w'_{rm}w'_{ik-m}+w'_{im}w'_{rk-m})$$
$$+\frac{k-m}{x}(w'_{rm}w_{ik-m}+w'_{im}w_{rk-m})-\frac{2m^2(k-m)}{(1-\mu)x^2}(w_{rm}w_{ik-m}+w_{im}w_{rk-m})\Big] \tag{3.8}$$

$$H_k(w_{rk})=q_{rk}+12\sum_{m=-\infty}^{+\infty}\Big\{w''_{rm}\Big[v'_{rk-m}+\frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{rk-m-l}-w'_{il}w'_{ik-m-l})$$
$$+\frac{\mu}{x}u_{rk-m}-\frac{\mu}{x}(k-m)v_{ik-m}-\frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{rl}w_{rk-m-l}-w_{il}w_{ik-m-l})\Big]$$
$$-w''_{im}\Big[u'_{ik-m}+\frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{ik-m-l}+w'_{il}w'_{rk-m-l})+\frac{\mu}{x}u_{ik-m}+\frac{\mu}{x}(k-m)v_{rk-m}$$
$$-\frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{rl}w_{ik-m-l}+w_{il}w_{rk-m-l})\Big]$$
$$-m(1-\mu)\frac{d}{dx}\Big(\frac{w_{im}}{x}\Big)\Big[-\frac{k-m}{x}u_{ik-m}+x\frac{d}{dx}\Big(\frac{v_{rk-m}}{x}\Big)$$
$$-\sum_{l=-\infty}^{+\infty}\frac{1}{x}(k-m-l)(w'_{rl}w_{ik-m-l}+w'_{il}w_{rk-m-l})\Big]$$
$$-m(1-\mu)\frac{d}{dx}\Big(\frac{w_{rm}}{x}\Big)\Big[\frac{k-m}{x}u_{rk-m}+x\frac{d}{dx}\Big(\frac{v_{ik-m}}{x}\Big)+\sum_{l=-\infty}^{+\infty}\frac{1}{x}(k-m$$
$$-l)(w'_{rl}w_{rk-m-l}-w'_{il}w_{ik-m-l})\Big]+\Big(\frac{1}{x}w'_{rm}-\frac{m^2}{x^2}w_{rm}\Big)\Big[\frac{1}{x}u_{rk-m}$$
$$-\frac{k-m}{x}v_{ik-m}-\frac{1}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{rl}w_{rk-m-l}-w_{il}w_{ik-m-l})+\mu u'_{rk-m}$$

$$+\frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{rk-m-l}-w'_{il}w'_{ik-m-l})\Big]-\Big(\frac{1}{x}w'_{im}-\frac{m^2}{x^2}w_{im}\Big)\Big[\frac{1}{x}u_{ik-m}$$

$$+\frac{k-m}{x}v_{rk-m}-\frac{1}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{rl}w_{ik-m-l}+w_{il}w_{rk-m-l})+\mu u'_{ik-m}$$

$$+\frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{ik-m-l}+w'_{il}w'_{rk-m-l})\Big]\Big\} \tag{3.9}$$

$$H_k(w_{ik})=q_{ik}+12\sum_{m=-\infty}^{+\infty}\Big\{w''_{rm}\Big[u'_{ik-m}+\frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{ik-m-l}+w'_{il}w'_{rk-m-l})$$

$$+\frac{\mu}{x}u_{ik-m}+\frac{\mu}{x}(k-m)v_{rk-m}-\frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{rl}w_{ik-m-l}+w_{il}w_{rk-m-l})\Big]$$

$$+w''_{im}\Big[u'_{rk-m}+\frac{1}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{rk-m-l}-w'_{il}w'_{ik-m-l})+\frac{\mu}{x}u_{rk-m}-\frac{\mu}{x}(k-m)v_{ik-m}$$

$$-\frac{\mu}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m-l)(w_{rl}w_{rk-m-l}-w_{il}w_{ik-m-l})\Big]$$

$$-m(1-\mu)\frac{d}{dx}\Big(\frac{w_{im}}{x}\Big)\Big[\frac{k-m}{x}u_{rk-m}+x\frac{d}{dx}\Big(\frac{v_{im}}{x}\Big)+\frac{1}{x}\sum_{l=-\infty}^{+\infty}(k-m$$

$$-l)(w'_{rl}w_{rk-m-l}-w'_{il}w_{ik-m-l})\Big]+m(1-\mu)\frac{d}{dx}\Big(\frac{w_{rm}}{x}\Big)\Big[-\frac{k-m}{x}u_{ik-m}$$

$$+x\frac{d}{dx}\Big(\frac{v_{rk-m}}{x}\Big)-\frac{1}{x}\sum_{l=-\infty}^{+\infty}(k-m-l)(w'_{rl}w_{ik-m-l}+w'_{il}w_{rk-m-l})\Big]$$

$$+\Big(\frac{1}{x}w'_{rm}-\frac{m^2}{x^2}w_{rm}\Big)\Big[\frac{1}{x}u_{ik-m}+\frac{k-m}{x}v_{rk-m}-\frac{1}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m$$

$$-l)(w_{rl}w_{ik-m-l}+w_{il}w_{rk-m-l})+\mu u'_{ik-m}+\frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{ik-m-l}+w'_{il}w'_{rk-m-l})\Big]$$

$$+\Big(\frac{1}{x}w'_{im}-\frac{m^2}{x^2}w_{im}\Big)\Big[\frac{1}{x}u_{rk-m}-\frac{k-m}{x}v_{ik-m}-\frac{1}{2x^2}\sum_{l=-\infty}^{+\infty}l(k-m$$

$$-l)(w_{rl}w_{rk-m-l}-w_{il}w_{ik-m-l})+\mu u'_{rk-m}+\frac{\mu}{2}\sum_{l=-\infty}^{+\infty}(w'_{rl}w'_{rk-m-l}$$

$$-w'_{il}w'_{ik-m-l})\Big]\Big\} \tag{3.10}$$

边界条件

$x=1$时，$u_{rk}=u_{ik}=v_{rk}=v_{ik}=0$ (3.11a,b,c,d)

$w_{rk}=w_{ik}=w'_{rk}=w'_{ik}=0$ (3.12a,b,c,d)

$x=0$时，u_{rk}，u_{ik}，v_{rk}，v_{ik}，w'_{rk}，w'_{ik}有限 (3.13a,b,c,d,e,f)

其中 $'=\frac{d}{dx}$，$''=\frac{d^2}{dx^2}$，$H_k=\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{k^2}{x^2}\right)^2$，$(k=0, \pm1, \pm2, \cdots)$

采用修正迭代法[6]即可求得以上边值问题的各次近似解析解.

四、算　　例

设有一半径为a，厚度为h的圆薄板，周边固定夹紧，承受横向载荷$Q(r,\theta)=2Q_0\cos n\theta$，其中$n\geqslant1$，$n\neq2$，4，材料的泊松比$\mu=0.3$. 此时

$$q_{rk}(x)=\begin{cases} q, & k=\pm n \\ 0, & \text{其他} \end{cases};\quad q_{ik}=0 \quad (k=0, \pm1, \pm2, \cdots)$$

w的一次近似边值问题：

$$H_k(w_{rk}^{①})=q_{rk},\quad H_k(w_{ik}^{①})=q_{ik}$$

边界条件

$x=1$ 时，$w_{rk}^{①}=w_{ik}^{①}=w_{rk}^{①\prime}=w_{ik}^{①\prime}=0$

$x=0$ 时，$w_{rk}^{①}$，$w_{ik}^{①}$，$w_{rk}^{①\prime}$，$w_{ik}^{①\prime}$ 有限

其中$k=0$，±1，±2，$\cdots$.

求解上述边值问题，可得

$$w_{rn}^{①}(x)=w_{r-n}^{①}(x)=(y_0/\alpha)[2x^4+(2-n)x^n+(n-4)x^{n+2}]$$

$$w_{rk}^{①}(x)=0 \qquad (k=0, \pm1, \pm2, \cdots; k\neq\pm n)$$

$$w_{ik}^{①}(x)=0 \qquad (k=0, \pm1, \pm2, \cdots)$$

$$w^{①}(x, \theta)=(2y_0/\alpha)[2x^4+(2-n)x^n+(n-4)x^{n+2}]\cos n\theta \tag{4.1}$$

其中

$$y_0=w^{①}(x_0, \theta_0),\quad 0<x_0<1,\quad 0\leqslant\theta_0\leqslant2\pi,\quad \theta_0\neq\pi/2, 3\pi/2$$

$$\alpha=2[2x_0^4+(2-n)x_0^n+(n-4)x_0^{n+2}]\cos n\theta_0$$

u，v的一次近似边值问题为

$$xu_{r0}^{①\prime\prime}+u_{r0}^{①\prime}-\frac{u_{r0}^{①}}{x}=-\left[2xw_{rn}^{①\prime\prime}w_{rn}^{①\prime}+(1-\mu)(w_{rn}^{①\prime})^2+\frac{2n^2\mu}{x}w_{rn}^{①\prime}w_{rn}^{①}-\frac{n^2(1+\mu)}{x^2}(w_{rn}^{①})^2\right]$$

$$xu_{r2m}^{①\prime\prime}+u_{r2m}^{①\prime}-\frac{2+(2m)^2(1-\mu)}{2x}u_{r2m}^{①}-m\left[(1+\mu)v_{i2m}^{①\prime}-(3-\mu)\frac{v_{i2m}^{①}}{x}\right]$$

$$=-\left[xw_{rm}^{①\prime\prime}w_{rm}^{①\prime}+\frac{1-\mu}{2}(w_{rm}^{①\prime})^2-\frac{m^2}{x}w_{rm}^{①\prime}w_{rm}^{①}+\frac{m^2(1+\mu)}{2x^2}(w_{rm}^{①})^2\right]$$

$$xv_{i2m}^{①''}+v_{i2m}^{①'}-\frac{1-\mu+8m^2}{(1-\mu)x}v_{i2m}^{①}+\frac{2m}{1-\mu}\left[(1+\mu)u_{r2m}^{①'}+\frac{3-\mu}{x}u_{r2m}^{①}\right]$$

$$=-\left[mw_{rm}^{①''}w_{rm}^{①}+\frac{m(1+\mu)}{1-\mu}(w_{rm}^{①'})^2+\frac{m}{x}w_{rm}^{①'}w_{rm}^{①}-\frac{2m^3}{(1-\mu)x^2}(w_{rm}^{①})^2\right]$$

边界条件

$x=1$时，$u_{rk}^{①}=u_{ik}^{①}=v_{rk}^{①}=v_{ik}^{①}=0$

$x=0$时，$u_{rk}^{①}$，$u_{ik}^{①}$，$v_{rk}^{①}$，$v_{ik}^{①}$ 有限

其中$m=\pm n$，$k=0$，±1，±2，…．关于其他的$u_{rk}^{①}$，$u_{ik}^{①}$，$v_{rk}^{①}$和$v_{ik}^{①}$，方程(3.5)～(3.8)给出的是线性齐次方程．求解以上边值问题，可得

$$u_{r0}^{①}(x)=(y_0^2/a^2)(d_1x^7+d_2x^{n+5}+d_3x^{2n+3}+d_4x^{n+3}+d_5x^{2n+1}+d_6x+d_7x^{2n-1})$$

$$u_{r2n}^{①}(x)=u_{r-2n}^{①}(x)=(y_0^2/a^2)(e_1x^7+e_2x^{n+5}+e_3x^{2n+3}+e_4x^{n+3}+e_5x^{2n+1}+e_6x^{2n-1})$$

$$v_{i2n}^{①}(x)=-v_{i-2n}^{①}(x)=(y_0^2/a^2)(e_7x^7+e_8x^{n+5}+e_9x^{2n+3}+e_{10}x^{n+3}+e_{11}x^{2n+1}+e_{12}x^{2n-1})$$

其中

$d_1=(-11n^2-1072)/120$

$d_2=2(-3n^4-33n^3-88n^2+696n+1504)/[5(n^2+10n+24)]$

$d_3=(-13n^5+51n^4+122n^3-150n^2-1072n-864)/[20(n^2+3n+2)]$

$d_4=6n(n^3+11n^2+10n-72)/[5(n^2+6n+8)]$

$d_5=(13n^4-58n^3-9n^2+118n+56)/[10(n+1)]$

$d_6=(11n^7-144n^6+705n^5-1536n^4+1140n^3+912n^2-1856n+768)/[120(n^5+12n^4+43n^3+36n^2-44n-48)]$

$d_7=13n(-n^2+4n-4)$

$e_1=-4(25n^4-601n^2+3216)/[20(n^4-25n^2+144)]$

$e_2=(-43n^6+133n^5+1898n^4-7448n^3-8272n^2+31744n+36096)/[15(3n^4-20n^3-20n^2+160n+192)]$

$e_3=(-27n^5+68n^4+577n^3-1022n^2-2368n-864)/[80(2n^2+3n+1)]$

$e_4=n(43n^5-145n^4-624n^3+2428n^2-1024n-1728)/[5(9n^4-36n^3-28n^2+96n+64)]$

$e_5=(19683n^{14}-152815n^{13}-609774n^{12}+5782598n^{11}+7778073n^{10}-81603595n^9-79453626n^8+559060888n^7+608258112n^6-182631550n^5-2118150048n^4+1525037824n^3+3027443712n^2+1512751104n+249274368)/[3240(18n^{11}-45n^{10}-821n^9+873n^8+12819n^7+1500n^6-76248n^5-78096n^4+112688n^3+237120n^2+140544n+27648)]$

$e_6=(-19683n^{14}+177326n^{13}+536016n^{12}-5739082n^{11}-4870329n^{10}+62980436n^9+33958596n^8-317292272n^7-232489776n^6+887413696n^5+524303808n^4-752105216n^3-611023872n^2-173887488n-8847360)/[6480(18n^{11}-45n^{10}-821n^9+873n^8+12819n^7+1500n^6-76248n^5-78096n^4+112688n^3+237120n^2+140544n+27648)]$

$e_7=n(-20n^4+441n^2-1936)/[20(n^4-25n^2+144)]$

$e_8=2n(-28n^5+241n^4-388n^3-1040n^2+1088n+4096)/[15(3n^4-20n^3-20n^2+160n+192)]$

$$e_9=n(-27n^4+202n^3-307n^2-328n+208)/[80(2n^2+3n+1)]$$

$$e_{10}=2n(28n^5-157n^4+104n^3+252n^2+528n-1280)/[5(9n^4-36n^3-28n^2+96n+64)]$$

$$\begin{aligned}e_{11}=&(255879n^{15}-2759335n^{14}-1477202n^{13}+92821934n^{12}-126449531n^{11}\\&-1094284195n^{10}+1746320122n^9+6109412224n^8-6666083104n^7\\&-18592653872n^6+5521793696n^5+26256476032n^4+8420074496n^3\\&-6418710528n^2-2575097856n-30965760)/[3240(234n^{12}-711n^{11}-10358n^{10}\\&+17096n^9+160536n^8-70233n^7-1001724n^6-481512n^5+2011616n^4\\&+2293744n^3+167232n^2-624384n-193536)]\end{aligned}$$

$$\begin{aligned}e_{12}=&(-255879n^{15}+2443019n^{14}+5726926n^{13}-78360178n^{12}-23140703n^{11}\\&+852837971n^{10}+598696n^9-4362509708n^8-801321184n^7+13163806480n^6\\&+604053632n^5-13447494464n^4-2678573824n^3+2016629760n^2\\&+1102196736n+61931520)/[6480(234n^{12}-711n^{11}-10358n^{10}+17096n^9\\&+160536n^8-70233n^7-1001724n^6-481512n^5+2011616n^4+2293744n^3\\&+167232n^2-624384n-193536)]\end{aligned}$$

其他的$u^{①}_{rk}$，$u^{①}_{ik}$，$v^{①}_{rk}$和$v^{①}_{ik}$均为零.

w的二次近似边值问题：

$$\begin{aligned}H_n(w^{②}_{rn})=H_{-n}(w^{②}_{r-n})=&q+12\Big\{w^{①''}_{rn}\Big[u^{①'}_{r0}+\frac{\mu}{x}u^{①}_{r0}\Big]+w^{①''}_{rn}\Big[u^{①'}_{r2n}+\frac{\mu}{x}u^{①}_{r2n}\\&-\frac{2n\mu}{x}v^{①}_{i2n}\Big]+n(1-\mu)\frac{d}{dx}\Big(\frac{w^{①}_{rn}}{x}\Big)\Big[\frac{2n}{x}u^{①}_{r2n}+x\frac{d}{dx}\Big(\frac{v^{①}_{i2n}}{x}\Big)\Big]\\&+\Big(\frac{1}{x}w^{①'}_{rn}-\frac{n^2}{x^2}w^{①}_{rn}\Big)\Big[\frac{1}{x}u^{①}_{r0}+\mu u^{①'}_{r0}+\frac{1}{x}u^{①}_{r2n}-\frac{2n}{x}v^{①}_{i2n}\\&+\mu u^{①'}_{r2n}\Big]\Big\}+12\Big\{w^{①''}_{rn}\Big[\frac{3}{2}(w^{①'}_{rn})^2+\frac{\mu n^2}{2x^2}(w^{①}_{rn})^2\Big]\\&+n^2(1-\mu)\frac{d}{dx}\Big(\frac{w^{①}_{rn}}{x}\Big)\Big[\frac{1}{x}w^{①'}_{rn}w^{①}_{rn}\Big]+\Big(\frac{1}{x}w^{①'}_{rn}-\frac{n^2}{x^2}w^{①}_{rn}\Big)\Big[\frac{n^2}{2x^2}(w^{①}_{rn})^2\\&+\frac{3\mu}{2}(w^{①'}_{rn})^2\Big]\Big\}\end{aligned}$$

$$\begin{aligned}H_{3n}(w^{②}_{r3n})=H_{-3n}(w^{②}_{r-3n})=&12\Big\{w^{①''}_{rn}\Big[u^{①'}_{r2n}+\frac{\mu}{x}u^{①}_{r2n}-\frac{2n\mu}{x}v^{①}_{i2n}\Big]\\&-n(1-\mu)\frac{d}{dx}\Big(\frac{w^{①}_{rn}}{x}\Big)\Big[\frac{2n}{x}u^{①}_{r2n}+x\frac{d}{dx}\Big(\frac{v^{①}_{i2n}}{x}\Big)\Big]\\&+\Big(\frac{1}{x}w^{①'}_{rn}-\frac{n^2}{x^2}w^{①}_{rn}\Big)\Big[\frac{1}{x}u^{①}_{r2n}-\frac{2n}{x}v^{①}_{i2n}+\mu u^{①'}_{r2n}\Big]\Big\}\\&+12\Big\{w^{①''}_{rn}\Big[\frac{1}{2}(w^{①'}_{rn})^2-\frac{n^2\mu}{2x^2}(w^{①}_{rn})^2\Big]\\&-n(1-\mu)\frac{d}{dx}\Big(\frac{w^{①}_{rn}}{x}\Big)\Big[\frac{n}{x}w^{①'}_{rn}w^{①}_{rn}\Big]+\Big(\frac{1}{x}w^{①'}_{rn}\end{aligned}$$

$$-\frac{n^2}{x^2}w_{rn}^{①}\Big)\Big[-\frac{n^2}{2x^2}(w_{rn}^{①})^2+\frac{\mu}{2}(w_{rn}^{①\prime})^2\Big]\Big\}$$

$$H_k(w_{rk}^{②})=0 \qquad (k=0,\ \pm1,\ \pm2,\ \cdots;\ k\neq\pm n,\ \pm3n)$$

$$H_k(w_{ik}^{②})=0 \qquad (k=0,\ \pm1,\ \pm2,\ \cdots)$$

边界条件

$x=1$ 时，$w_{rk}^{②}=w_{ik}^{②}=w_{rk}^{②\prime}=w_{ik}^{②\prime}=0$

$x=0$ 时，$w_{rk}^{②}$，$w_{ik}^{②}$，$w_{rk}^{②\prime}$，$w_{ik}^{②\prime}$ 有限　　$(k=0,\ \pm1,\ \pm2,\ \cdots)$

$x=x_0$，$\theta=\theta_0$ 时，$\sum\limits_{k=-\infty}^{+\infty}[w_{rk}^{②}(x)+iw_{ik}^{②}(x)]e^{ik\theta}=y_0$

求解以上边值问题，可得w的二次近似解析解

$$\begin{aligned}w^{②}(x,\ \theta)&=\sum_{k=-\infty}^{\infty}[w_{rk}^{②}(x)+iw_{ik}^{②}(x)]e^{ik\theta}\\&=\frac{q}{(n^4-20n^2+64)}[2x^4+(n-4)x^{n+2}-(n-2)x^n]\cos(n\theta)\\&\quad+\frac{2y_0^3}{a^3}[g_1x^{12}+g_2x^{n+10}+g_3x^{2n+8}+g_4x^{n+8}+g_5x^{3n+6}+g_6x^{2n+6}\\&\quad+g_7x^6+g_8x^{3n+4}+g_9x^{2n+4}+g_{10}x^{n+4}+g_{11}x^{3n+2}+g_{13}x^{n+2}+g_{12}x^{3n}\\&\quad+g_{14}x^n]\cos(n\theta)+\frac{2y_0^3}{a^3}[h_1x^{12}+h_2x^{n+10}+h_3x^{2n+8}+h_4x^{n+8}\\&\quad+h_6x^{3n+6}+h_6x^{2n+6}+h_7x^{3n+4}+h_8x^{2n+4}+h_9x^{3n+2}+h_{10}x^{3n}]\cos(3n\theta)\end{aligned}\tag{4.2}$$

其中

$$\begin{aligned}g_1=&-(12/5)[20n^4-20n^3e_7+n^2(31d_1-11e_1-200)+26ne_7-1000d_1-1000e_1\\&-12672]/(n^4-244n^2+14400)\end{aligned}$$

$$\begin{aligned}g_2=&-(12/5)[27n^5-n^4(7e_1+7e_7+153)+n^3(-21d_1+3d_2+3e_2+26e_7\\&-20e_8-498)+n^2(-41d_1-13e_1+25d_2-17e_2+13e_7-21e_8-2040)\\&+n(396d_1+396e_1-132d_2-132e_2-20e_7+68e_8+11424)+416d_1+416e_1\\&-736d_2-736e_2+30336]/[320(n^2+9n+20)]\end{aligned}$$

$$\begin{aligned}g_3=&-(2/5)[19n^6-462n^5-n^4(7d_2+21e_2+21e_8-1000)+n^3(-33d_2+9e_2+12d_3\\&+12e_8+87e_8-40e_9+5520)+n^2(34d_2+90e_2+38d_3-46e_3+12e_8-84e_9\\&-1040)+n(684d_2+684e_2-528d_3-528e_3-96e_8+220e_9-38016)+624d_2\\&+624e_2-944d_3-944e_3-45312]/(3n^4+56n^3+356n^2+896n+768)\end{aligned}$$

$$\begin{aligned}g_4=&(1/80)[27n^5-n^4(7e_1+7e_7+87)+n^3(20e_{10}-21d_1-3d_4-3e_4-534)\\&+n^2(21e_{10}+63d_1+49e_1-19d_4+23e_4+49e_7-1632)+n(-110e_{10}-42d_1\\&-42e_1+132d_4+132e_4-42e_7+5664)+472(d_4+e_4)]/(n^2+7n+12)\end{aligned}$$

$$\begin{aligned}g_6=&(3/40)[3n^7-23n^6-243n^5+n^4(7d_3+14e_3+14e_9+821)+n^3(12d_3-9e_3-61e_9\\&+1394)+n^2(-75d_3-103e_3+e_9-2232)+n(-288d_3-288e_3+76e_9-7584)\\&-208(d_3+e_3+24)]/(2n^4+15n^3+40n^2+45n+18)\end{aligned}$$

$$g_8=(2/5)[38n^6-804n^5+7n^4(3e_{10}-d_2-3e_2+d_4+3e_4-3e_8)+1748n^4$$

$$+n^3(-101e_{10}+40e_{11}-7d_2+35e_2+19d_4-23e_4-12d_5-12e_5+35e_8+5592)$$
$$+n^2(30e_{10}+84e_{11}+70d_2+42e_2-44d_4-100e_4-26d_5+58e_5+42e_8-4864)$$
$$+8n(19e_{10}-38e_{11}-7d_2-7e_2-59d_4-59e_4+66d_5+66e_5-7e_8-1872)$$
$$-416(d_4+e_4-d_5-e_5)]/(3n^4+40n^3+180n^2+320n+192)$$

$$g_7=-(156/5)d_6(n^2-16)/(n^4-52n^2+576)$$

$$g_8=-(3/40)[21n^7-95n^6-327n^5+7n^4(-2e_{11}+d_3+2e_3-d_5-2e_5+2e_9)+1555n^4$$
$$+n^3(68e_{11}-14d_3-35e_3-5d_5+16e_5-35e_9+582)+n^2(-22e_{11}-7d_3$$
$$+7e_3+80d_5+108e_5+7e_9-3008)+n(-104e_{11}+14d_3+14e_3+182d_5$$
$$+182e_5+14e_9-1824)+104(d_5+e_5)]/(2n^4+9n^3+14n^2+9n+2)$$

$$g_9=(-6/5)[19n^6-342n^5+n^4(21e_{10}+7d_4+21e_4+892)+n^3(-49e_{10}-40e_{12}$$
$$-7d_4-49e_4+12d_7+12e_6-72)+n^2(-84e_{12}-28d_4+14d_7-70e_6-992)$$
$$+4n(7e_{10}+97e_{12}+7d_4+7e_4-132d_7-132e_6)+112(d_7+e_6)]/(9n^4$$
$$+72n^3+188n^2+192n+64)$$

$$g_{10}=(312/5)(n^2-3n-4)d_6/(32n^2+96n+64)$$

$$g_{11}=(3/40)[21n^6-121n^5+42n^4+n^3(-14e_{11}+14e_{12}-7d_5-14e_5+7d_7+14e_6+620)$$
$$+n^2(42e_{11}-75e_{12}+21d_5+42e_5-2d_7-23e_6-792)+n(-28e_{11}+43e_{12}$$
$$-14d_5-28e_5-85d_7-113e_6+32)+132e_{12}-76d_7-76e_6]/(2n^4+3n^3+n^2)$$

$$g_{12}=-(21/40)n[n^5-6n^4+12n^3+n^2(2e_{12}+d_7-8)-n(5e_{12}+3d_7+7e_6)+2(e_{12}$$
$$+d_7+e_6)]/[n^2(2n-1)]$$

$$g_{13}=0.5[(n-12)g_1-10g_2-(n+8)g_3-8g_4-(2n+6)g_5-(n+6)g_6$$
$$+(n-6)g_7-2(n+2)g_8-(n+4)g_9-4g_{10}-2(n+1)g_{11}-2ng_{12}]$$

$$g_{14}=0.5[(10-n)g_1+8g_2+(n+6)g_3+6g_4+2(n+2)g_5+(n+4)g_6-(n-4)g_7$$
$$+2(n+1)g_8+(n+2)g_9+2g_{10}+2ng_{11}+2(n-1)g_{12}]$$

$$h_1=(4/15)[20n^4+20n^3e_7-73n^2(e_1+8)-278ne_7+1000e_1+4224]/(9n^2$$
$$-244n^2+1600)$$

$$h_2=(3/40)[27n^5+n^4(-7e_1+7e_7-249)+n^3(42e_1-3e_2-68e_7+20e_8+142)$$
$$+n^2(69e_1-67e_2+113e_7-21e_8+3272)+n(-396e_1+132e_2+188e_7-236e_8$$
$$-3808)-416e_1+736e_2-10112]/(2n^4-9n^3-31n^2+90n+200)$$

$$h_3=(6/5)[7n^6-190n^5+n^4(-7e_2+7e_8+1384)+n^3(75e_2-12e_3-101e_8+40e_9$$
$$-2480)+n^2(22e_2-122e_3+212e_8-84e_9-5968)+n(-684e_2+528e_3$$
$$+320e_8-388e_9+12672)-624e_2+944e_3+15104]/(25n^4-280n^3$$
$$+268n^2+2688n+2304)$$

$$h_4=-(3/40)[27n^5-7n^4e_1+7n^4e_7-183n^4+n^3(-20e_{10}+42e_1+3e_4-42e_7$$
$$+106)+n^2(21e_{10}-77e_1+61e_4+77e_7+1248)+n(194e_{10}+42e_1$$
$$-132e_4-42e_7-1888)-472e_4]/(2n^4-7n^3-19n^2+42n+72)$$

$$h_5=(1/120)[33n^5-337n^4+n^3(33e_3-33e_9+902)+n^2(-47e_3+99e_9+408)$$
$$+n(-288e_3+132e_9-2528)-208e_3-1664]/(3n^2+5n+2)$$

$$h_6=-(6/5)[14n^6-260n^5+n^4(-7e_{10}-7e_2+7e_4+7e_8+1492)+n^3(87e_{10}-40e_{11}$$
$$+49e_2-61e_4+12e_5-49e_8-2728)+n^2(-170e_{10}+84e_{11}-98e_2-12e_4$$
$$+110e_5+98e_8-1152)+2n(-132e_{10}+152e_{11}+28e_2+236e_4-264e_5$$

$$-28e_8+2496)+416(e_4-e_5)]/(25n^4-200n^3+124n^2+960n+576)$$

$$h_7=-(3/40)[33n^5-311n^4+n^3(26e_{11}+7e_3-26e_5-7e_9+866)+n^2(-78e_{11}-21e_3$$
$$+52e_5+21e_9-448)+n(-104e_{11}+14e_{13}+182e_5-14e_9-608)$$
$$+104e_5]/(9n^2+9n+2)$$

$$h_8=(42/5)[n^6-10n^5-n^4(e_{10}-4e_4-36)+n^3(5e_{10}-5e_4+4e_6-56)-2n^2(4e_{10}$$
$$-4e_4+13e_6-16)+4n(e_{10}-e_4+11e_6)-16e_6]/(25n^4-120n^3+28n^2+192n+64)$$

$$h_9=0.5[3(n-6)h_1+2(n-5)h_2+(n-8)h_3+2(n-4)h_4-6h_5+(n-6)h_6-4h_7$$
$$+(n-4)h_8]$$

$$h_{10}=0.5[-(3n-10)h_1+2(4-n)h_2-(n-6)h_3-2(n-3)h_4+4h_5-(n-4)h_6$$
$$+2h_7-(n-2)h_8]$$

式(4.2)中取$x=x_0$，$\theta=\theta_0$，可得

$$q=C_1y_0+C_2y_0^3 \tag{4.3}$$

其中

$$C_1=2(n^4-20n^2+64)/\alpha$$

$$C_2=-4(n^4-20n^2+64)\alpha^{-4}[(g_1x_0^{12}+g_2x_0^{n+10}+g_3x_0^{3n+8}+g_4x_0^{n+8}+g_5x_0^{3n+6}$$
$$+g_6x_0^{2n+6}+g_7x_0^6+g_8x_0^{3n+4}+g_9x_0^{2n+4}+g_{10}x_0^{n+4}+g_{11}x_0^{3n+2}+g_{13}x_0^{n+2}+g_{12}x_0^{3n}$$
$$+g_{14}x_0^n)\cos(n\theta_0)+(h_1x_0^{12}+h_2x_0^{n+10}+h_3x_0^{2n+8}+h_4x_0^{n+8}+h_5x_0^{3n+6}+h_6x_0^{2n+6}$$
$$+h_7x_0^{3n+4}+h_8x_0^{2n+4}+h_9x_0^{3n+2}+h_{10}x_0^{3n})\cos(3n\theta_0)]$$

给定n，x_0和θ_0，由式(4.3)可绘出相应载荷与点(x_0，θ_0)的挠度的特征曲线。

$$n=1,\quad x_0=\frac{3}{8}\left[2\cos\left(\frac{5\pi}{3}-\frac{1}{3}\mathrm{arc\,tg}\frac{8\sqrt{11}}{5}\right)+1\right],\quad \theta_0=0$$

时的特征曲线如图1所示，这里的点(x_0，θ_0)对应于线性结果中挠度最大的点。

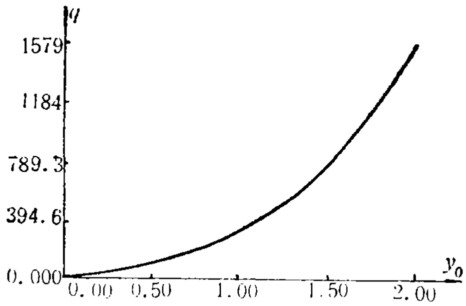

图1 载荷与挠度特征曲线

五、结 论

由以上讨论可看出，板在沿θ方向按余弦规律变化的载荷作用下，挠度w的展开式中一定只含余弦项，不含正弦项。将本文所选坐标旋转90°即可推出，板在沿θ方向按正弦规律变化的载荷作用下，挠度w的展开式中一定只含正弦项，不含余弦项。利用本文的结果进行实算可得出，板内最大挠度所处的位置是随载荷大小的变化而变化的。

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Nonsymmetrical Large Deformation Bending Problem of Circular Thin Plates

Wang Lin-xiang　Wang Xin-zhi　Qiu Ping

(*Gansu University of Technology, Lanzhou*)

Abstract

To begin with, in this paper, the displacement governing equations and the boundary conditions of nonsymmetrical large deflection problem of circular thin plates are derived. By using the transformation and the perturbation method, the nonlinear displacement equations are linearized, and the approximate boundary value problems are obtained. As an example, the nonlinear bending problem of circular thin plates subjected to comparatively complex loads is studied.

Key words plate, displacement, nonlinear, transformation, perturbation method